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On nonlinear matrix equations $X \pm \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I$

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Abstract

We study the nonlinear matrix equations $X + \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I$ and $X - \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I$, where n_i are positive integers for $i = 1, 2, \ldots, m$. The iterative algorithms for obtaining positive definite solutions for these equations are proposed. The necessary and sufficient conditions for the existence of positive definite solutions of these equations are derived. Moreover, the rate of convergence of the sequences generated from the algorithms is studied. The efficiency of proposed algorithms is illustrated by numerical examples.

Keywords: matrix equation; positive definite solution; iterative algorithm

1 Introduction

Consider the nonlinear matrix equations:

$$X + \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I \tag{1.1}$$

and

$$X - \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I, \tag{1.2}$$

where X is an unknown square matrix, I is the identity matrix, A_i are square complex matrices and n_i are positive integers for $i = 1, 2, \ldots, m$.

Nonlinear matrix equations of type (1.1) and (1.2) have many applications in engineering, control theory, dynamic programming, stochastic filtering, ladder networks, statistics, *etc.*; see [1–4] and the references therein. When $m = 1$ and $n_1 = 1$, (1.2) arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [3]. When $m > 1$ and $n_i = 1$ for $i = 1, 2, \ldots, m$, (1.1) arises in solving a large-scale system of linear equations in many physical calculations [5] and (1.2) is recognized as playing an important role in modeling certain optimal interpolation problems [6, 7].

In the last few years, many authors have been greatly interested in developing the theory and numerical approaches for positive definite solutions to the nonlinear matrix equations of the form (1.1) and (1.2). Similar types of (1.1) and (1.2) have been investigated [8–12]. The matrix equations $X \pm A^* X^{-1} A = Q$ have been studied by several authors [1–4, 13, 14] and different iterative algorithms for computing the positive definite solutions with linear and quadratic rate of convergence are proposed. Ivanov *et al.* [15] derived sufficient conditions for the existence of positive definite solutions for the matrix equations $X \pm A^*X^{-2}A = I$ and they proposed iterative algorithms for obtaining positive definite solutions of these equations. El-Sayed [16] presented two iterative methods for calculating the positive definite solutions of the matrix equation $X - A^*X^{-n}A = Q$, for the integer $n \geq 1$, the first method is derived for a normal matrix A and for the second method a sufficient condition for convergence is given for $n = 2^k$. El-Sayed and Ran [17] studied the general matrix equation $X + A^*F(X)A = Q$ where F maps positive definite matrices either into positive definite matrices or into negative definite matrices and satisfies some monotonicity property. Hasanov and Ivanov [18] considered the matrix equations $X \pm A^*X^{-n}A = Q$, they studied the solutions and perturbation analysis of these solutions. They also derived a sufficient condition for the existence of a unique positive definite solution of the equation $X - A^*X^{-n}A = Q$. Hasanov [19] established and proved theorems for the necessary and sufficient conditions of the existence of positive definite solutions for the matrix equations $X \pm A^*X^{-q}A = Q$ with $0 < q \leq 1$, he showed that the equation $X - A^*X^{-q}A = Q$ has a unique positive definite solution by using the properties of matrix sequence in Banach space. Also, in [5] some conditions for the existence of positive definite solution of the equation $X + \sum_{i=1}^{m} A_i^*X^{-1}A_i = I$ have been obtained and two iterative algorithms to find the maximal positive definite solution of this equation have been presented. Duan *et al.* [20] gave two perturbation estimates for the positive definite solution of the equation $X - \sum_{i=1}^{m} A_i^*X^{\delta_i}A_i = Q$ with $0 < |\delta_i| < 1$. Duan *et al.* [6] studied the equation $X - \sum_{i=1}^{m} N_i^*X^{-1}N_i = I$, they used the Thompson metric to prove that the matrix equation always has a unique positive definite solution and they derived a precise perturbation bound for the unique positive definite solution. In addition, other nonlinear matrix equations such as $X^s \pm A^TX^{-t}A = I_n$ [21], $AX^2 + BX + C = 0$ [22], and $X = Q + A^H(I \otimes X - C)^{-\delta}A^*$ [23] have been investigated.

In this paper, we study the positive definite solutions of (1.1) and (1.2). We derive the necessary and sufficient conditions for the existence of positive definite solutions. We suggest iterative algorithms for obtaining positive definite solutions of these equations. Moreover, under some conditions we obtain the rates of convergence of the iterative sequences of approximate solutions and the stopping criterions. Finally, we give some numerical examples to ensure the performance and the effectiveness of the suggested iterative algorithms.

The following notations will be used in this paper. $A^\star$ denotes the complex conjugate transpose of A. We write $A > 0$ ($A \geq 0$), if matrix A is positive definite (positive semidefinite). If $A - B$ is positive definite (positive semidefinite), then we write $A > B$ ($A \geq B$). Moreover, we denote $\rho(A)$ by the spectral radius of A. We use $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ to denote the maximal and minimal eigenvalues of A. $\|\cdot\|$ and $\|\cdot\|_\infty$ denote the spectral and infinity norm, respectively.

Lemma 1.1 [24] *If $A \geq B > 0$, then $A^{-1} \leq B^{-1}$.*

Lemma 1.2 [24] *If A and B are positive definite matrices for which $A - B > 0$ and $AB = BA$ are satisfied, then $A^n - B^n > 0$.*

Lemma 1.3 [25] *If $A > B > 0$ (or $A \geq B > 0$), then $A^\alpha > B^\alpha > 0$ (or $A^\alpha \geq B^\alpha > 0$), for all $\alpha \in (0,1]$, and $0 < A^\alpha < B^\alpha$ (or $0 < A^\alpha \leq B^\alpha$), for all $\alpha \in [-1,0)$.*

2 The matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I$

In this section, we give some necessary and sufficient conditions for the existence of positive definite solutions of (1.1). We present the following iterative algorithm to compute the positive definite solution of (1.1).

Algorithm 2.1

$$\begin{cases} X_0 = I, \\ X_{s+1} = I - \sum_{i=1}^{m} A_i^* X_s^{-n_i} A_i, \quad \text{for } s = 0, 1, 2, \ldots. \end{cases}$$

Remark 2.1 Letting $m = 1$ in Algorithm 2.1, we get Algorithm (2.2) in [10] which is proposed for obtaining the positive definite solutions of the matrix equation $X + A^* X^{-n} A = I$. Also, letting $n_i = 1$, $\forall i = 1, 2, \ldots, m$, in Algorithm 2.1, we get Algorithm 2.7 in [5], which is proposed for obtaining the positive definite solutions of the matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$.

The following theorem provides the necessary condition for the existence of positive definite solutions of (1.1).

Theorem 2.1 *If* (1.1) *has a positive definite solution* X, *then*

$$\left(A_i A_i^*\right)^{\frac{1}{n_i}} < X \leq I - \sum_{i=1}^{m} A_i^* A_i, \quad i = 1, 2, \ldots, m. \tag{2.1}$$

Proof Since X is a positive definite solution of (1.1), then $X \leq I$ and $\sum_{i=1}^{m} A_i^* X^{-n_i} A_i < I$. Using the inequality $X \leq I$ and Lemmas 1.1, 1.2, we have

$$X = I - \sum_{i=1}^{m} A_i^* X^{-n_i} A_i \leq I - \sum_{i=1}^{m} A_i^* A_i.$$

Also, from the inequality $\sum_{i=1}^{m} A_i^* X^{-n_i} A_i < I$, we have $A_i^* X^{-n_i} A_i < I$. Then

$$A_i^* X^{-n_i/2} X^{-n_i/2} A_i < I,$$

which implies that

$$X^{-n_i/2} A_i A_i^* X^{-n_i/2} < I.$$

Using Lemma 1.3, we obtain

$$\left(A_i A_i^*\right)^{1/n_i} < X.$$

This completes the proof. □

Remark 2.2 Letting $n_i = 1$, $\forall i = 1, 2, \ldots, m$, in (2.1) we get the condition $A_i A_i^* < X \leq I - \sum_{i=1}^{m} A_i^* A_i$, which is necessary for the existence of positive definite solutions of the matrix equation $X + \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$ ([5], Theorem 2.1).

Lemma 2.1 *If A_i, $i = 1,2,\dots,m$, are hermitian matrices and $A_iA_j = A_jA_i$, for all $i,j = 1,2,\dots,m$, then*

$$A_jX_s = X_sA_j, \quad j = 1,2,3,\dots,m, \tag{2.2}$$

where the sequence $\{X_s\}$, $s = 0,1,2,\dots$, is determined by Algorithm 2.1.

Proof Since $X_0 = I$, $A_jX_0 = X_0A_j$. By using the condition $A_iA_j = A_jA_i$, we have

$$A_jX_1 = A_j\left(I - \sum_{i=1}^{m} A_i^2\right) = A_j - \sum_{i=1}^{m} A_jA_i^2 = A_j - \sum_{i=1}^{m} A_i^2A_j = \left(I - \sum_{i=1}^{m} A_i^2\right)A_j = X_1A_j.$$

We suppose that $A_jX_s = X_sA_j$. Then for X_{s+1}, we have

$$\begin{aligned}
A_jX_{s+1} &= A_j\left(I - \sum_{i=1}^{m} A_iX_s^{-n_i}A_i\right) \\
&= A_j - \sum_{i=1}^{m} A_jA_iX_s^{-n_i}A_i \\
&= A_j - \sum_{i=1}^{m} A_iA_jX_s^{-n_i}A_i \\
&= A_j - \sum_{i=1}^{m} A_iX_s^{-n_i}A_jA_i \\
&= A_j - \sum_{i=1}^{m} A_iX_s^{-n_i}A_iA_j \\
&= \left(I - \sum_{i=1}^{m} A_iX_s^{-n_i}A_i\right)A_j \\
&= X_{s+1}A_j.
\end{aligned}$$

Hence, the equalities (2.2) are true, for all $s = 0,1,2,\dots$. □

Lemma 2.2 *If A_i, $i = 1,2,\dots,m$, are hermitian matrices and $A_iA_j = A_jA_i$, for all $i,j = 1,2,\dots,m$, then*

$$X_sX_r = X_rX_s. \tag{2.3}$$

Here the sequences $\{X_s\}$, $\{X_r\}$, $s,r = 0,1,2,\dots$, are determined by Algorithm 2.1.

Proof Since $X_0 = I$, $X_0X_r = X_rX_0$, $\forall r = 0,1,2,\dots$. According to Lemma 2.1, we have

$$\begin{aligned}
X_1X_r &= \left(I - \sum_{i=1}^{m} A_i^2\right)\left(I - \sum_{i=1}^{m} A_iX_{r-1}^{-n_i}A_i\right) \\
&= I - \sum_{i=1}^{m} A_i^2 - \sum_{i=1}^{m} A_iX_{r-1}^{-n_i}A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_j^2A_iX_{r-1}^{-n_i}A_i
\end{aligned}$$

$$
\begin{aligned}
&= I - \sum_{i=1}^{m} A_i^2 - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i A_j^2 \\
&= I - \sum_{i=1}^{m} A_i^2 - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{i=1}^{m}\sum_{j=1}^{m} A_i X_{r-1}^{-n_i} A_i A_j^2 \\
&= \left(I - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i\right)\left(I - \sum_{i=1}^{m} A_i^2\right) \\
&= X_r X_1.
\end{aligned}
$$

That is, $X_1 X_r = X_r X_1$, $\forall r = 0,1,2,\ldots$. We suppose that $X_s X_r = X_r X_s$, $\forall r = 0,1,2,\ldots$. Then for X_{s+1}, we have

$$
\begin{aligned}
X_{s+1} X_r &= \left(I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i\right)\left(I - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i\right) \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m} A_j X_s^{-n_j} A_j \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_j X_s^{-n_j} A_j A_i X_{r-1}^{-n_i} A_i \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_i X_s^{-n_j} A_i A_j X_{r-1}^{-n_i} A_j \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_i A_i X_s^{-n_j} X_{r-1}^{-n_i} A_j A_j \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_i A_i X_{r-1}^{-n_i} X_s^{-n_j} A_j A_j \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{j=1}^{m}\sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i A_j X_s^{-n_j} A_j \\
&= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i + \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i \sum_{j=1}^{m} A_j X_s^{-n_j} A_j \\
&= \left(I - \sum_{i=1}^{m} A_i X_{r-1}^{-n_i} A_i\right)\left(I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i\right) \\
&= X_r X_{s+1}.
\end{aligned}
$$

Therefore, the equality (2.3) is true, for all $s, r = 0,1,2,\ldots$. □

Remark 2.3 When we compare Lemmas 2.1 and 2.2 by Lemmas 4 and 5 in [10], we note that the sequence $\{X_s\}$, $s = 0,1,2,\ldots$ (which is defined by Algorithm 2.1) satisfies the same properties of the sequence $\{X_s\}$, $s = 0,1,2,\ldots$ (which is defined by Algorithm (2.2) in [10]).

The following theorem provides the sufficient condition for the existence of positive definite solutions of (1.1).

Theorem 2.2 *Let A_i, $i = 1,2,\dots,m$, be hermitian matrices and $A_iA_j = A_jA_i$, for all $i,j = 1,2,\dots,m$. If $A_i^2 \le \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}I$, where $\alpha > 1$ and $n = \max_{1\le i\le m}\{n_i\}$, then* (1.1) *has a positive definite solution.*

Proof We consider the sequence $\{X_s\}$ generated from Algorithm 2.1. For X_0, we have $X_0 = I > \frac{1}{\alpha}I$. For X_1, we have

$$X_1 = I - \sum_{i=1}^{m} A_i^2 \ge I - \sum_{i=1}^{m} \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}I > I - \sum_{i=1}^{m} \frac{(\alpha-1)}{nm\alpha}I \ge I - \frac{(\alpha-1)}{\alpha}I = \frac{1}{\alpha}I,$$

$$X_1 = I - \sum_{i=1}^{m} A_i^2 \le I = X_0.$$

That is,

$$X_0 \ge X_1 > \frac{1}{\alpha}I.$$

We suppose that

$$X_{s-1} \ge X_s > \frac{1}{\alpha}I. \tag{2.4}$$

Using the inequalities (2.4) and Lemmas 1.1, 1.2, and 2.2, we obtain

$$\begin{aligned} X_{s+1} &= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i \\ &\le I - \sum_{i=1}^{m} A_i X_{s-1}^{-n_i} A_i \\ &= X_s. \end{aligned}$$

Also

$$\begin{aligned} X_{s+1} &= I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i \\ &> I - \sum_{i=1}^{m} \alpha^{n_i} A_i^2 \\ &\ge I - \sum_{i=1}^{m} \alpha^{n_i} \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}I \\ &= I - \sum_{i=1}^{m} \frac{1}{m}\frac{(\alpha-1)}{n\alpha^{(n_i+1)}}I \\ &> I - \frac{(\alpha-1)}{\alpha}I \\ &= \frac{1}{\alpha}I. \end{aligned}$$

Therefore the inequalities (2.4) are true, for all $s = 1, 2, \ldots$. That is, the sequence $\{X_s\}$ is monotonically decreasing and bounded below by $\frac{1}{\alpha}I$. Hence, the sequence $\{X_s\}$ converges to a positive definite solution X of (1.1). □

Theorem 2.3 *Let A_i, $i = 1, 2, \ldots, m$, be hermitian matrices and $A_iA_j = A_jA_i$, for all $i, j = 1, 2, \ldots, m$. If $A_i^2 \leq \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}I$, then* (1.1) *has a positive definite solution X which satisfies*

$$\|X_{s+1} - X\| < \left(\frac{\alpha - 1}{\alpha}\right)\|X_s - X\|, \tag{2.5}$$

where $\alpha > 1$, $n = \max_{1 \leq i \leq m}\{n_i\}$, and $\{X_s\}$, $s = 0, 1, 2, \ldots$, is the sequence determined by Algorithm 2.1.

Proof By Theorem 2.2, we know that the sequence $\{X_s\}$, $s = 0, 1, 2, \ldots$, is convergent to a positive definite solution X of (1.1). We consider the spectral norm of the matrix $X_{s+1} - X$. We have

$$\begin{aligned}
\|X_{s+1} - X\| &= \left\| I - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - I + \sum_{i=1}^{m} A_i X^{-n_i} A_i \right\| \\
&= \left\| \sum_{i=1}^{m} A_i \left(X^{-n_i} - X_s^{-n_i}\right) A_i \right\| \\
&\leq \sum_{i=1}^{m} \left\| A_i \left(X^{-n_i} - X_s^{-n_i}\right) A_i \right\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \left\| X^{-n_i} \left(X_s^{n_i} - X^{n_i}\right) X_s^{-n_i} \right\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \left\| X^{-n_i} \right\| \left\| X_s^{-n_i} \right\| \left\| X_s^{n_i} - X^{n_i} \right\| \\
&= \sum_{i=1}^{m} \|A_i\|^2 \left\| X^{-n_i} \right\| \left\| X_s^{-n_i} \right\| \left\| (X_s - X) \sum_{r=1}^{n_i} X_s^{n_i - r} X^{r-1} \right\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \left\| X^{-n_i} \right\| \left\| X_s^{-n_i} \right\| \|X_s - X\| \left(\sum_{r=1}^{n_i} \|X_s\|^{n_i - r} \|X\|^{r-1} \right).
\end{aligned}$$

From the proof of Theorem 2.2, we obtain $X_s^{-n_i} < \alpha^{n_i} I$, $X^{-n_i} \leq \alpha^{n_i} I$, and $X \leq X_s \leq I$. Then we have

$$\begin{aligned}
\|X_{s+1} - X\| &\leq \sum_{i=1}^{m} \|A_i\|^2 \left\| X^{-n_i} \right\| \left\| X_s^{-n_i} \right\| \|X_s - X\| \left(\sum_{r=1}^{n_i} \|X_s\|^{n_i - r} \|X\|^{r-1} \right) \\
&< \sum_{i=1}^{m} n_i \alpha^{2n_i} \|A_i\|^2 \|X_s - X\| \\
&\leq \sum_{i=1}^{m} n_i \alpha^{2n_i} \frac{(\alpha - 1)}{nm\alpha^{(2n_i+1)}} \|X_s - X\|
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{i=1}^{m} n\frac{(\alpha-1)}{nm\alpha}\|X_s-X\| \\ &= \frac{(\alpha-1)}{\alpha}\|X_s-X\|. \qquad \square\end{aligned}$$

Theorem 2.4 *Let A_i, $i=1,2,\ldots,m$, be hermitian matrices and $A_iA_j=A_jA_i$, for all $i,j=1,2,\ldots,m$. If $A_i^2 \leq \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}I$, where $\alpha>1$, $n=\max_{1\leq i\leq m}\{n_i\}$, and after s iterative steps of Algorithm* 2.1, *we have $\|I-X_s^{n_i}X_{s-1}^{-n_i}\|<\varepsilon$, then*

$$\left\|X_s+\sum_{i=1}^{m} A_iX_s^{-n_i}A_i-I\right\| < \frac{(\alpha-1)}{\alpha}\varepsilon. \tag{2.6}$$

Proof From Algorithm 2.1, we have

$$\begin{aligned}X_s+\sum_{i=1}^{m} A_iX_s^{-n_i}A_i-I &= X_s+\sum_{i=1}^{m} A_iX_s^{-n_i}A_i-X_s-\sum_{i=1}^{m} A_iX_{s-1}^{-n_i}A_i \\ &= \sum_{i=1}^{m} A_i\left(X_s^{-n_i}-X_{s-1}^{-n_i}\right)A_i.\end{aligned}$$

By taking the norm on both sides of the above equation, we have

$$\begin{aligned}\left\|X_s+\sum_{i=1}^{m} A_iX_s^{-n_i}A_i-I\right\| &= \left\|\sum_{i=1}^{m} A_i\left(X_s^{-n_i}-X_{s-1}^{-n_i}\right)A_i\right\| \\ &\leq \sum_{i=1}^{m} \|A_i\|^2\left\|X_s^{-n_i}-X_{s-1}^{-n_i}\right\| \\ &\leq \sum_{i=1}^{m} \|A_i\|^2\left\|X_s^{-n_i}\right\|\left\|I-X_s^{n_i}X_{s-1}^{-n_i}\right\|.\end{aligned}$$

From the proof of Theorem 2.2, we have $X_s^{-n_i}<\alpha^{n_i}I$, then

$$\begin{aligned}\left\|X_s+\sum_{i=1}^{m} A_iX_s^{-n_i}A_i-I\right\| &< \sum_{i=1}^{m} \frac{(\alpha-1)}{nm\alpha^{(2n_i+1)}}\alpha^{n_i}\left\|I-X_s^{n_i}X_{s-1}^{-n_i}\right\| \\ &< \sum_{i=1}^{m} \frac{(\alpha-1)}{m\alpha}\left\|I-X_s^{n_i}X_{s-1}^{-n_i}\right\| \\ &< \frac{(\alpha-1)}{\alpha}\varepsilon. \qquad \square\end{aligned}$$

3 The matrix equation $X-\sum_{i=1}^{m} A_i^*X^{-n_i}A_i=I$

In this section, we give some necessary and sufficient conditions for the existence of positive definite solutions of (1.2). We present the following iterative algorithm to compute the positive definite solution of (1.2).

Algorithm 3.1

$$\begin{cases} X_0=I, \\ X_{s+1}=I+\sum_{i=1}^{m} A_i^*X_s^{-n_i}A_i, \quad \text{for } s=0,1,2,\ldots. \end{cases}$$

Remark 3.1 Letting $m = 1$ in Algorithm 3.1, we get Algorithm (2.1) in [16] which is proposed for obtaining the positive definite solutions of the matrix equation $X - A^*X^{-n}A = I$. Also, letting $n_i = 1$, $\forall i = 1, 2, \dots, m$, in Algorithm 3.1, we get Algorithm (2.1) in [6] which is proposed for obtaining the positive definite solutions of the matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$.

The following theorem provides the necessary condition for the existence of positive definite solutions of (1.2).

Theorem 3.1 *If* (1.2) *has a positive definite solution* X, *then*

$$I \le X \le I + \sum_{i=1}^{m} A_i^* A_i. \tag{3.1}$$

Proof Since X is a positive definite solution of (1.2), then $\sum_{i=1}^{m} A_i^* X^{-n_i} A_i \ge 0$. Thus we get

$$X = I + \sum_{i=1}^{m} A_i^* X^{-n_i} A_i \ge I.$$

Also, from the inequality $X \ge I$ and Lemmas 1.1 and 1.2, we have

$$X = I + \sum_{i=1}^{m} A_i^* X^{-n_i} A_i \le I + \sum_{i=1}^{m} A_i^* A_i.$$

This completes the proof. □

Remark 3.2 The condition (3.1) is the same necessary condition for the existence of positive definite solutions of the matrix equation $X - \sum_{i=1}^{m} A_i^* X^{-1} A_i = I$ ([6], Remark 2.1). Also, if A is an invertible matrix and $m = 1$ in condition (3.1), then we get the condition $I < X < I + A^*A$, which is necessary for the existence of positive definite solutions of the matrix equation $X - A^*X^{-n}A = I$ ([16], Corollary 2.1).

Lemma 3.1 *If* A_i, $i = 1, 2, \dots, m$, *are hermitian matrices and* $A_iA_j = A_jA_i$, *for all* $i, j = 1, 2, \dots, m$, *then*

$$A_j X_s = X_s A_j, \quad j = 1, 2, 3, \dots, m, \tag{3.2}$$

where the sequence $\{X_s\}$, $s = 0, 1, 2, \dots$, *is determined by Algorithm* 3.1.

Proof The proof is similar to the proof of Lemma 2.1. □

Lemma 3.2 *If* A_i, $i = 1, 2, \dots, m$, *are hermitian matrices and* $A_iA_j = A_jA_i$, *for all* $i, j = 1, 2, \dots, m$, *then*

$$X_s X_r = X_r X_s. \tag{3.3}$$

Here the sequences $\{X_s\}$, $\{X_r\}$, $s, r = 0, 1, 2, \dots$, *are determined by Algorithm* 3.1.

Proof The proof is similar to the proof of Lemma 2.2. □

Remark 3.3 When we compare Lemmas 3.1 and 3.2 by Lemmas 2.3 and 2.4 in [16], we note that the sequence $\{X_s\}$, $s = 0, 1, 2, \dots$ (which is defined by Algorithm 3.1) satisfies the same properties of the sequence $\{X_s\}$, $s = 0, 1, 2, \dots$ (which is defined by Algorithm (2.1) in [16]).

Theorem 3.2 *Let A_i, $i = 1, 2, \dots, m$, be hermitian matrices and $A_iA_j = A_jA_i$, for all $i, j = 1, 2, \dots, m$. If $q = \sum_{i=1}^{m} n_i\|A_i\|^2(1 + \sum_{i=1}^{m}\|A_i\|^2)^{n_i-1} < 1$, then* (1.2) *has a positive definite solution X which satisfies*

$$X_{2s} \le X \le X_{2s+1} \tag{3.4}$$

and

$$\|X_{2s+1} - X_{2s}\| \le q^{2s}\sum_{i=1}^{m}\|A_i\|^2, \tag{3.5}$$

where the sequence $\{X_s\}$, $s = 0, 1, 2, \dots$, is determined by Algorithm 3.1.

Proof We consider the matrix sequence $\{X_s\}$ generated from Algorithm 3.1 and using Lemmas 1.1, 1.2 and 3.2. Since $A_i^2 \ge 0$, then

$$X_1 = I + \sum_{i=1}^{m} A_i^2 \ge I = X_0$$

and

$$X_2 = I + \sum_{i=1}^{m} A_iX_1^{-n_i}A_i \le I + \sum_{i=1}^{m} A_i^2 = X_1.$$

Consequently

$$X_0 \le X_2 \le X_1.$$

We find the relation between X_2, X_3, X_4, X_5. Using $X_0 \le X_2 \le X_1$, we obtain

$$X_3 = I + \sum_{i=1}^{m} A_iX_2^{-n_i}A_i \le I + \sum_{i=1}^{m} A_i^2 = X_1$$

and

$$X_3 = I + \sum_{i=1}^{m} A_iX_2^{-n_i}A_i \ge I + \sum_{i=1}^{m} A_iX_1^{-n_i}A_i = X_2.$$

Hence $X_2 \le X_3 \le X_1$. In the same way we can prove that

$$X_0 \le X_2 \le X_4 \le X_5 \le X_3 \le X_1.$$

We suppose that

$$X_0 \leq X_{2s} \leq X_{2s+2} \leq X_{2s+3} \leq X_{2s+1} \leq X_1. \tag{3.6}$$

Using the inequalities (3.6), we have

$$X_{2s+4} = I + \sum_{i=1}^{m} A_i X_{2s+3}^{-n_i} A_i \leq I + \sum_{i=1}^{m} A_i X_{2s+2}^{-n_i} A_i = X_{2s+3},$$
$$X_{2s+4} = I + \sum_{i=1}^{m} A_i X_{2s+3}^{-n_i} A_i \geq I + \sum_{i=1}^{m} A_i X_{2s+1}^{-n_i} A_i = X_{2s+2}.$$

Similarly

$$X_{2s+5} = I + \sum_{i=1}^{m} A_i X_{2s+4}^{-n_i} A_i \leq I + \sum_{i=1}^{m} A_i X_{2s+2}^{-n_i} A_i = X_{2s+3},$$
$$X_{2s+5} = I + \sum_{i=1}^{m} A_i X_{2s+4}^{-n_i} A_i \geq I + \sum_{i=1}^{m} A_i X_{2s+3}^{-n_i} A_i = X_{2s+4}.$$

Therefore, the inequalities (3.6) are true, for all $s = 0, 1, 2, \ldots$. Consequently the subsequences $\{X_{2s}\}$ and $\{X_{2s+1}\}$ are convergent to positive definite matrices. To prove that these sequences have a common limit, we consider

$$\begin{aligned}
\|X_{2s+1} - X_{2s}\| &= \left\| \sum_{i=1}^{m} A_i \bigl(X_{2s}^{-n_i} - X_{2s-1}^{-n_i}\bigr) A_i \right\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{2s}^{-n_i} - X_{2s-1}^{-n_i} \bigr\| \\
&= \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{2s}^{-n_i} \bigl(X_{2s-1}^{n_i} - X_{2s}^{n_i}\bigr) X_{2s-1}^{-n_i} \bigr\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{2s}^{-n_i} \bigr\| \bigl\| X_{2s-1}^{-n_i} \bigr\| \bigl\| X_{2s-1}^{n_i} - X_{2s}^{n_i} \bigr\| \\
&= \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{2s}^{-n_i} \bigr\| \bigl\| X_{2s-1}^{-n_i} \bigr\| \left\| (X_{2s-1} - X_{2s}) \sum_{r=1}^{n_i} X_{2s-1}^{n_i - r} X_{2s}^{r-1} \right\| \\
&\leq \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{2s}^{-n_i} \bigr\| \bigl\| X_{2s-1}^{-n_i} \bigr\| \|X_{2s-1} - X_{2s}\| \\
&\quad \times \left(\sum_{r=1}^{n_i} \|X_{2s-1}\|^{n_i - r} \|X_{2s}\|^{r-1} \right).
\end{aligned}$$

From the inequalities (3.6), we have $X_{2s}, X_{2s-1} \geq I$, and $X_{2s}, X_{2s-1} \leq I + \sum_{i=1}^{m} A_i^2$, for all $s = 1, 2, 3, \ldots$. Then we have

$$\|X_{2s+1} - X_{2s}\| \leq \sum_{i=1}^{m} n_i \|A_i\|^2 \|X_{2s-1} - X_{2s}\| \left(1 + \sum_{i=1}^{m} \|A_i\|^2 \right)^{n_i - 1}.$$

Hence

$$\|X_{2s+1} - X_{2s}\| \le q\|X_{2s} - X_{2s-1}\| \le \cdots \le q^{2s} \sum_{i=1}^{m} \|A_i\|^2,$$

that is,

$$\|X_{2s+1} - X_{2s}\| \to 0, \quad \text{as } s \to \infty.$$

Hence, the two subsequences $\{X_{2s}\}$ and $\{X_{2s+1}\}$ have the same limit X, which is a positive definite solution of (1.2). □

From Theorem 3.2, we can deduce the following corollary.

Corollary 3.1 *From inequality* (3.5), *we have the following upper bound*:

$$\max\bigl(\|X_{2s+1} - X\|, \|X - X_{2s}\|\bigr) \le q^{2s} \sum_{i=1}^{m} \|A_i\|^2. \tag{3.7}$$

Remark 3.4 Theorem 3.2 provides the sufficient condition $q = \sum_{i=1}^{m} n_i \|A_i\|^2 (1 + \sum_{i=1}^{m} \|A_i\|^2)^{n_i-1} < 1$ for the existence of positive definite solutions of (1.2), we note that when $m = 1$ we have the condition $\|A\|^2(1 + \|A\|^2)^{n-1} < \frac{1}{n}$, which is sufficient for the existence of positive definite solutions of the matrix equation $X - A^*X^{-n}A = I$ ([16], Theorem 2.1).

Theorem 3.3 *Let A_i, $i = 1, 2, \ldots, m$, be hermitian matrices and $A_iA_j = A_jA_i$, for all $i, j = 1, 2, \ldots, m$. If $q = \sum_{i=1}^{m} n_i \|A_i\|^2 (1 + \sum_{i=1}^{m} \|A_i\|^2)^{n_i-1} < 1$, and after s iterative steps of Algorithm* 3.1, *we have $\|I - X_{s-1}^{n_i} X_s^{-n_i}\| < \varepsilon$, then*

$$\left\| X_s - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - I \right\| < \varepsilon \sum_{i=1}^{m} \|A_i\|^2. \tag{3.8}$$

Proof From Algorithm 3.1, we have

$$\begin{aligned} X_s - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - I &= X_s - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - X_s + \sum_{i=1}^{m} A_i X_{s-1}^{-n_i} A_i \\ &= \sum_{i=1}^{m} A_i \bigl(X_{s-1}^{-n_i} - X_s^{-n_i}\bigr) A_i. \end{aligned}$$

By taking the norm on both sides of the above equation, we have

$$\begin{aligned} \left\| X_s - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - I \right\| &= \left\| \sum_{i=1}^{m} A_i \bigl(X_{s-1}^{-n_i} - X_s^{-n_i}\bigr) A_i \right\| \\ &\le \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{s-1}^{-n_i} - X_s^{-n_i} \bigr\| \\ &\le \sum_{i=1}^{m} \|A_i\|^2 \bigl\| X_{s-1}^{-n_i} \bigr\| \bigl\| I - X_{s-1}^{n_i} X_s^{-n_i} \bigr\|. \end{aligned}$$

From the proof of Theorem 3.2, we have $X_{s-1}^{-n_i} \leq I$, then

$$\left\| X_s - \sum_{i=1}^{m} A_i X_s^{-n_i} A_i - I \right\| \leq \sum_{i=1}^{m} \|A_i\|^2 \left\| I - X_{s-1}^{n_i} X_s^{-n_i} \right\| < \varepsilon \sum_{i=1}^{m} \|A_i\|^2. \qquad \square$$

4 Numerical examples

In this section, we use the iterative Algorithms 2.1 and 3.1 to compute the positive definite solutions of (1.1) and (1.2), respectively. The solutions are computed for different matrices A_i, $i = 1, 2, \ldots, m$, with different orders. We denote X, the solution obtained by Algorithms 2.1 and 3.1 and $\epsilon(X_s) = \|X - X_s\|_\infty$, $R_1(X_s) = \|X_s + \sum_{i=1}^{m} A_i^* X_s^{-n_i} A_i - I\|_\infty$, $Y_s = I - \sum_{i=1}^{m} A_i^* A_i - X_s$, $Z_{i,s} = X_s^{n_i} - A_i A_i^*$ $(i = 1, 2, \ldots, m)$, $R_2(X_s) = \|X_s - \sum_{i=1}^{m} A_i^* X_s^{-n_i} A_i - I\|_\infty$.

Example 4.1 Consider the matrix equation

$$X + A_1^* X^{-4} A_1 + A_2^* X^{-5} A_2 + A_3^* X^{-3} A_3 = I, \tag{4.1}$$

where A_1, A_2, and A_3 are given by

$$A_1 = \begin{pmatrix} 0.091 & 0.015 & 0.004 \\ 0.014 & 0.029 & 0.045 \\ -0.043 & 0.071 & 0.015 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.098 & 0.014 & 0.05 \\ 0.034 & 0.025 & 0.07 \\ 0.051 & 0.04 & -0.001 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.09 & -0.008 & 0.025 \\ 0.034 & -0.087 & 0.015 \\ 0.02 & -0.02 & 0.044 \end{pmatrix}.$$

We use Algorithm 2.1 to solve (4.1). After 22 iterations, we get the positive definite solution

$$X \approx X_{22} = \begin{pmatrix} 0.960979 & 0.0000874449 & -0.0134898 \\ 0.0000874449 & 0.982002 & -0.00301522 \\ -0.0134898 & -0.00301522 & 0.986046 \end{pmatrix}$$

and $R_1(X_{22}) = 1.11022 \times 10^{-16}$, $\lambda_{\min}(Y_{22}) = 0.000488382$, $\lambda_{\min}(Z_{1,22}) = 0.825427$, $\lambda_{\min}(Z_{2,22}) = 0.778478$, $\lambda_{\min}(Z_{3,22}) = 0.860326$.

The other results are listed in Table 1.

Table 1 Error analysis for Example 4.1

s	$\epsilon(X_s)$	$R_1(X_s)$	$\lambda_{\min}(Y_s)$	$\lambda_{\min}(Z_{1,s})$	$\lambda_{\min}(Z_{2,s})$	$\lambda_{\min}(Z_{3,s})$
0	3.90209×10^{-2}	3.33430×10^{-2}	-3.82258×10^{-2}	0.989301	0.980411	0.985409
1	5.67787×10^{-3}	4.70245×10^{-3}	1.11022×10^{-16}	0.849272	0.807341	0.879195
2	9.75412×10^{-4}	8.02922×10^{-4}	4.48982×10^{-4}	0.829614	0.783528	0.863646
3	1.72490×10^{-4}	1.41756×10^{-4}	4.83255×10^{-4}	0.826170	0.779373	0.860915
4	3.07337×10^{-5}	2.52483×10^{-5}	4.87548×10^{-4}	0.825560	0.778638	0.860431
5	5.48539×10^{-6}	4.50591×10^{-6}	4.88236×10^{-4}	0.825451	0.778506	0.860345
6	9.79481×10^{-7}	8.04563×10^{-7}	4.88356×10^{-4}	0.825431	0.778483	0.860329
7	1.74918×10^{-7}	1.43680×10^{-7}	4.88377×10^{-4}	0.825428	0.778479	0.860327
8	3.12382×10^{-8}	2.56594×10^{-8}	4.88381×10^{-4}	0.825427	0.778478	0.860326

Example 4.2 Consider the matrix equation

$$X + A_1^* X^{-7} A_1 + A_2^* X^{-2} A_2 + A_3^* X^{-11} A_3 + A_4^* X^{-4} A_4 = I, \tag{4.2}$$

where A_1, A_2, A_3, and A_4 are given by

$$A_1 = \begin{pmatrix} 0.023 & 0.015 & 0.014 & 0.001 \\ 0.014 & 0.011 & 0.029 & 0.045 \\ 0.01 & -0.043 & 0.071 & 0.015 \\ 0.011 & -0.043 & 0.071 & 0.015 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.019 & 0.014 & 0.085 & -0.01 \\ 0.034 & 0.062 & 0.035 & 0.07 \\ 0.029 & 0.051 & 0.04 & -0.001 \\ 0.026 & 0.01 & -0.043 & 0.015 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.002 & 0.014 & 0.03 & 0.025 \\ 0.034 & 0.087 & 0.015 & 0.067 \\ -0.022 & 0.012 & -0.02 & 0.044 \\ -0.041 & -0.034 & 0.071 & 0.015 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} 0.017 & 0.045 & -0.002 & 0.025 \\ -0.003 & 0.034 & 0.087 & 0.012 \\ 0.04 & 0.027 & -0.02 & 0.044 \\ 0.051 & -0.083 & 0.019 & 0.001 \end{pmatrix}.$$

We use Algorithm 2.1 to solve (4.2). After 29 iterations, we get the positive definite solution

$$X \approx X_{29} = \begin{pmatrix} 0.985316 & -0.00602454 & -0.00452105 & -0.00717771 \\ -0.00602454 & 0.963749 & 0.00114964 & -0.0154917 \\ -0.00452105 & 0.00114964 & 0.951857 & -0.0113565 \\ -0.00717771 & -0.0154917 & -0.0113565 & 0.97492 \end{pmatrix}$$

and $R_1(X_{29}) = 1.11022 \times 10^{-16}$, $\lambda_{\min}(Y_{29}) = 0.00144519$, $\lambda_{\min}(Z_{1,29}) = 0.649354$, $\lambda_{\min}(Z_{2,29}) = 0.878163$, $\lambda_{\min}(Z_{3,29}) = 0.517277$, $\lambda_{\min}(Z_{4,29}) = 0.785219$.

The other results are listed in Table 2.

Table 2 Error analysis for Example 4.2

s	$\epsilon(X_s)$	$R_1(X_s)$	$\lambda_{\min}(Y_s)$	$\lambda_{\min}(Z_{1,s})$	$\lambda_{\min}(Z_{2,s})$	$\lambda_{\min}(Z_{3,s})$	$\lambda_{\min}(Z_{4,s})$
0	4.81432×10^{-2}	3.79180×10^{-2}	-4.33175×10^{-2}	0.984520	0.982228	0.984476	0.987805
1	1.02252×10^{-2}	7.49107×10^{-3}	-1.78032×10^{-18}	0.719649	0.902084	0.604645	0.831201
2	2.73413×10^{-3}	2.08835×10^{-3}	1.05313×10^{-3}	0.664911	0.883407	0.535912	0.795502
3	6.45772×10^{-4}	5.05650×10^{-4}	1.33072×10^{-3}	0.652517	0.879228	0.521034	0.787317
4	1.40122×10^{-4}	1.08374×10^{-4}	1.41687×10^{-3}	0.650050	0.878406	0.518114	0.785684
5	3.17484×10^{-5}	2.37848×10^{-5}	1.43915×10^{-3}	0.649533	0.878227	0.517495	0.785339
6	7.96360×10^{-6}	5.93391×10^{-6}	1.44386×10^{-3}	0.649401	0.878180	0.517334	0.785250
7	2.02969×10^{-6}	1.53853×10^{-6}	1.44486×10^{-3}	0.649366	0.878167	0.517291	0.785226
8	4.91154×10^{-7}	3.76520×10^{-7}	1.44510×10^{-3}	0.649357	0.878164	0.517280	0.785220
9	1.14634×10^{-7}	8.75007×10^{-8}	1.44517×10^{-3}	0.649355	0.878164	0.517278	0.785219
10	2.71333×10^{-8}	2.05092×10^{-8}	1.44518×10^{-3}	0.649354	0.878164	0.517277	0.785219

Example 4.3 Consider the matrix equation

$$X + A_1^* X^{-2} A_1 + A_2^* X^{-2} A_2 = I, \tag{4.3}$$

where A_1 and A_2 are given as in Example 3.1 from [5]:

$$A_1 = \begin{pmatrix} 0.010 & -0.150 & -0.259 \\ 0.015 & 0.212 & -0.064 \\ 0.025 & -0.069 & 0.138 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.160 & -0.025 & 0.020 \\ -0.025 & -0.288 & -0.060 \\ 0.004 & -0.016 & -0.120 \end{pmatrix}.$$

We use Algorithm 2.1 to solve (4.3). After 78 iterations, we get the positive definite solution

$$X \approx X_{78} = \begin{pmatrix} 0.970376 & -0.0101782 & -0.00533509 \\ -0.0101782 & 0.733948 & -0.0493223 \\ -0.00533509 & -0.0493223 & 0.869915 \end{pmatrix}$$

and $R_1(X_{78}) = 1.11022 \times 10^{-16}$, $\lambda_{\min}(Y_{78}) = 0.001965$, $\lambda_{\min}(Z_{1,78}) = 0.479668$, $\lambda_{\min}(Z_{2,78}) = 0.427158$.

The other results are listed in Table 3.

Example 4.4 Consider the matrix equation

$$X - A_1^* X^{-2} A_1 - A_2^* X^{-6} A_2 - A_3^* X^{-10} A_3 - A_4^* X^{-3} A_4 = I, \tag{4.4}$$

where A_1, A_2, A_3, and A_4 are given by

$$A_1 = \begin{pmatrix} 0.1 & -0.1 & 0.4 \\ -0.1 & 0.2 & 0.5 \\ 0.4 & 0.2 & 0.1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0.2 & -0.4 & 0.5 \\ 0.9 & 0.2 & 0 \\ 0.1 & -0.4 & 0.3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -0.2 & 0.2 & 0.1 \\ -0.8 & 0.1 & 0.5 \\ 0.8 & -0.2 & -0.4 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 0 & 0.3 & 0.1 \\ 0.5 & 0.3 & 0.5 \\ 0.2 & -0.2 & -0.7 \end{pmatrix}.$$

We use Algorithm 3.1 to solve (4.4). After 133 iterations, we get the positive definite solution

$$X \approx X_{133} = \begin{pmatrix} 1.85505 & 0.204339 & -0.139713 \\ 0.204339 & 1.21553 & 0.144921 \\ -0.139713 & 0.144921 & 1.6854 \end{pmatrix}$$

and $R_2(X_{133}) = 2.22045 \times 10^{-16}$, $\lambda_{\min}(I + \sum_{i=1}^{4} A_i^* A_i - X_{133}) = 0.490411$, $\lambda_{\min}(X_{133} - I) = 0.100459$.

The other results are listed in Table 4.

Table 3 Error analysis for Example 4.3

s	$\epsilon(X_s)$	$R_1(X_s)$	$\lambda_{\min}(Y_s)$	$\lambda_{\min}(Z_{1,s})$	$\lambda_{\min}(Z_{2,s})$
0	2.66052×10^{-1}	1.56030×10^{-1}	-1.74299×10^{-1}	0.900634	0.910937
4	1.95763×10^{-2}	7.54210×10^{-3}	1.94201×10^{-3}	0.508120	0.455158
8	2.99945×10^{-3}	1.09180×10^{-3}	1.96189×10^{-3}	0.483981	0.431403
12	4.96010×10^{-4}	1.78894×10^{-4}	1.96449×10^{-3}	0.480380	0.427859
16	8.30170×10^{-5}	2.98954×10^{-5}	1.96492×10^{-3}	0.479787	0.427276
20	1.39223×10^{-5}	5.01230×10^{-6}	1.96499×10^{-3}	0.479688	0.427178
24	2.33562×10^{-6}	8.40831×10^{-7}	1.96500×10^{-3}	0.479671	0.427162
28	3.91847×10^{-7}	1.41065×10^{-7}	1.96500×10^{-3}	0.479669	0.427159
32	6.57408×10^{-8}	2.36667×10^{-8}	1.96500×10^{-3}	0.479668	0.427158

Table 4 Error analysis for Example 4.4

s	$\epsilon(X_s)$	$R_2(X_s)$	$\lambda_{\min}(I + \sum_{i=1}^{4} A_i^* A_i - X_s)$	$\lambda_{\min}(X_s - I)$
0	8.55046×10^{-1}	2.65	0.715946	0
10	1.83959×10^{-1}	3.26477×10^{-1}	0.544118	0.076489
20	1.06950×10^{-2}	1.87085×10^{-2}	0.493691	0.098958
30	5.94976×10^{-4}	1.04066×10^{-3}	0.490594	0.100375
40	3.30916×10^{-5}	5.78794×10^{-5}	0.490422	0.100455
50	1.84048×10^{-6}	3.21913×10^{-6}	0.490412	0.100459
60	1.02364×10^{-7}	1.79042×10^{-7}	0.490411	0.100459
70	5.69327×10^{-9}	9.95792×10^{-9}	0.490411	0.100459

Example 4.5 Consider the matrix equation

$$X - A_1^* X^{-5} A_1 - A_2^* X^{-9} A_2 - A_3^* X^{-14} A_3 = I, \tag{4.5}$$

where A_1, A_2, and A_3 are given by

$$A_1 = \begin{pmatrix} 0.01 & 0 & -0.01 & 0.04 \\ 0.11 & 0 & 0.12 & 0.05 \\ 0.06 & 0.05 & 0.12 & 0.01 \\ 0.04 & -0.09 & 0.02 & 0.03 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & -0.14 & 0.05 & 0.03 \\ 0.03 & 0.12 & 0 & 0.01 \\ 0.01 & -0.04 & 0.09 & 0.03 \\ 0.05 & 0.03 & 0 & 0.32 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -0.05 & 0.01 & 0.29 & 0.01 \\ -0.52 & 0 & 0.11 & -0.05 \\ 0 & -0.02 & 0 & -0.14 \\ 0.04 & -0.04 & 0.37 & 0.01 \end{pmatrix}.$$

We use Algorithm 3.1 to solve (4.5). After 78 iterations, we get the positive definite solution

$$X \approx X_{78} = \begin{pmatrix} 1.21033 & 0.0022825 & -0.00724383 & 0.036655 \\ 0.0022825 & 1.02578 & -0.00516744 & 0.00742036 \\ -0.00724383 & -0.00516744 & 1.07848 & 0.00429966 \\ 0.036655 & 0.00742036 & 0.00429966 & 1.069 \end{pmatrix}$$

and $R_2(X_{78}) = 2.22045 \times 10^{-16}$, $\lambda_{\min}(I + \sum_{i=1}^{3} A_i^* A_i - X_{78}) = 0.0223096$, $\lambda_{\min}(X_{78} - I) = 0.0237773$.

The other results are listed in Table 5.

Table 5 Error analysis for Example 4.5

s	$\epsilon(X_s)$	$R_2(X_s)$	$\lambda_{\min}(I + \sum_{i=1}^{3} A_i^* A_i - X_s)$	$\lambda_{\min}(X_s - I)$
0	2.10332×10^{-1}	2.95400×10^{-1}	0.047059	0
3	3.11244×10^{-2}	4.52684×10^{-2}	0.017503	0.028153
6	5.17330×10^{-3}	8.36352×10^{-3}	0.023078	0.023109
9	1.10730×10^{-3}	1.75717×10^{-3}	0.022155	0.023907
12	2.28175×10^{-4}	3.63481×10^{-4}	0.022341	0.023751
15	4.73953×10^{-5}	7.54424×10^{-5}	0.022303	0.023783
18	9.83002×10^{-6}	1.56497×10^{-5}	0.022311	0.023776
21	2.03950×10^{-6}	3.24685×10^{-6}	0.022309	0.023778
24	4.23122×10^{-7}	6.73607×10^{-7}	0.022310	0.023777
27	8.77834×10^{-8}	1.39750×10^{-7}	0.022310	0.023777
30	1.82120×10^{-8}	2.89934×10^{-8}	0.022310	0.023777

Table 6 Error analysis for Example 4.6

s	$\epsilon(X_s)$	$R_2(X_s)$	$\lambda_{\min}(I + \sum_{i=1}^{2} A_i^* A_i - X_s)$	$\lambda_{\min}(X_s - I)$
0	8.94901×10^{-1}	3.72279	0.501528	0
40	8.72900×10^{-2}	1.79569×10^{-1}	0.195273	0.120468
80	1.70413×10^{-2}	3.36946×10^{-2}	0.185914	0.131596
120	3.26922×10^{-3}	6.41673×10^{-3}	0.184203	0.133762
160	6.23654×10^{-4}	1.22239×10^{-3}	0.183879	0.134178
200	1.18834×10^{-4}	2.32858×10^{-4}	0.183817	0.134257
240	2.26383×10^{-5}	4.43579×10^{-5}	0.183805	0.134272
280	4.31248×10^{-6}	8.44986×10^{-6}	0.183803	0.134275
320	8.21498×10^{-7}	1.60964×10^{-6}	0.183803	0.134276
360	1.56490×10^{-7}	3.06625×10^{-7}	0.183803	0.134276
400	2.98101×10^{-8}	5.84098×10^{-8}	0.183803	0.134276

Example 4.6 Consider the matrix equation

$$X - A_1^* X^{-2} A_1 - A_2^* X^{-2} A_2 = I, \tag{4.6}$$

where A_1 and A_2 are given as in Example 4.1 from [6]:

$$A_1 = \begin{pmatrix} 0.3060 & 0.6894 & 0.6093 \\ 0.2514 & 0.4285 & 0.7642 \\ 0.0222 & 0.0987 & 0.8519 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.9529 & 0.6450 & 0.4801 \\ 0.4410 & 0.1993 & 0.9823 \\ 0.9712 & 0.0052 & 0.9200 \end{pmatrix}.$$

We use Algorithm 3.1 to solve (4.6). After 792 iterations, we get the positive definite solution

$$X \approx X_{792} = \begin{pmatrix} 1.49147 & 0.310408 & 0.378343 \\ 0.310408 & 1.44441 & 0.208234 \\ 0.378343 & 0.208234 & 1.8949 \end{pmatrix}$$

and $R_2(X_{792}) = 7.99361 \times 10^{-15}$, $\lambda_{\min}(I + \sum_{i=1}^{2} A_i^* A_i - X_{792}) = 0.183803$, $\lambda_{\min}(X_{792} - I) = 0.134276$.

The other results are listed in Table 6.

5 Conclusion

In this paper, we investigate the nonlinear matrix equations $X \pm \sum_{i=1}^{m} A_i^* X^{-n_i} A_i = I$, where n_i, $i = 1, 2, \ldots, m$, are positive integers. Necessary and sufficient conditions for the existence of positive definite solutions are derived. Iterative algorithms are proposed to compute the positive definite solutions of these equations. Moreover, some numerical examples are given to illustrate the effectiveness and rapidly convergence rate (small run time) of the proposed iterative algorithms (see values of $\epsilon(X_s)$, $R_1(X_s)$, and $R_2(X_s)$). Also, the values of $\lambda_{\min}$ show that the solutions of the matrix equations satisfy the necessary conditions.

Competing interests
The author declares to have no competing interests.

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