Ye and Xu *Journal of Inequalities and Applications* (2015) 2015:141
DOI 10.1186/s13660-015-0662-z

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

A posteriori error estimates for the fractional optimal control problems

Xingyang Ye[1] and Chuanju Xu[2*]

*Correspondence: cjxu@xmu.edu.cn
[2]School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the spectral approximation for a constrained optimal control problem governed by the time fractional diffusion equation. *A posteriori* error estimates are obtained for both the state and the control approximations. Some numerical experiments are carried out to show that the obtained *a posteriori* error estimates are reliable.

Keywords: fractional optimal control problem; space-time spectral method; *a posteriori* error

1 Introduction

Optimal control problems have been subject of many research works in scientific and engineering computing. The literature on this field is huge, and it is impossible to give even a very brief review. It has been found that the fractional order model can provide a more realistic description for some kind of complex systems in the fields covering control theory [1], viscoelastic materials [2, 3], anomalous diffusion [4–6], advection and dispersion of solutes in porous or fractured media [7], *etc.* [8–10]. Consequently, an optimal control problem for fractional differential equations initiates a new research direction, and we see a growing interest in this topic from both scientific and engineering communities.

A general formulation and a solution scheme for the fractional optimal control problem (FOCP) were first proposed in [11], where the fractional variational principle and the Lagrange multiplier technique were used. Following this idea, Frederico and Torres [12] formulated a Noether-type theorem in the general context and studied fractional conservation laws. Mophou [13] applied the classical control theory to a fractional diffusion equation, involving a Riemann-Liouville fractional time derivative. Dorville *et al.* [14] later extended the results of [13] to a boundary fractional optimal control.

Recently, some efforts have been put into developing spectral methods for solving FOCPs. For instance, a numerical direct method based on the Legendre orthonormal basis and operational matrix of Riemann-Liouville fractional integration were introduced in [15] to solve a general class of FOCP, and the convergence of the proposed method was also extensively discussed. In [16], the Legendre spectral-collocation method was applied to obtain approximate solutions for some types of FOCPs. Ye and Xu [17] proposed a Galerkin spectral method to solve a linear quadratic FOCP associated with the time fractional diffusion equation with Caputo fractional derivative, and a detailed error analysis was carried out. However, to the best of our knowledge, much less research is available for the *a posteriori* error estimation for problems involving fractional derivative, especially the one for FOCP.

The purpose of this paper is to derive *a posteriori* error estimates for the FOCP governed by the time fractional diffusion equation (TFDE) with Riemann-Liouville fractional derivative. Let $\Lambda = (-1,1), I = (0,T), \Omega = \Lambda \times I$. We consider the following linear-quadratic optimal control problem for the control variable q under constraints:

$$\min_{q\in K}\left\{\frac{1}{2}\int_\Omega \big(u(x,t)-\bar{u}(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\lambda}{2}\int_\Omega q^2(x,t)\,\mathrm{d}x\,\mathrm{d}t\right\}, \tag{1.1}$$

where λ and $\bar{u}$ are given, u is governed by the TFDE as follows:

$$\begin{aligned}
&{}_0^R\partial_t^\alpha u(x,t)-\partial_x^2 u(x,t)=f(x,t)+q(x,t), \quad \forall (x,t)\in\Omega,\\
&I_t^{1-\alpha}u(x,0)=0, \quad \forall x\in\Lambda,\\
&u(-1,t)=u(1,t)=0, \quad \forall t\in I,
\end{aligned} \tag{1.2}$$

with ${}_0^R\partial_t^\alpha$ $(0<\alpha<1)$ denoting the left Riemann-Liouville fractional derivative, $I_t^{1-\alpha}$ denoting the Riemann-Liouville fractional integral, and

$$K=\left\{q\in L^2(\Omega):\int_\Omega q(x,t)\,\mathrm{d}x\,\mathrm{d}t\ge 0\right\}.$$

The main physical purpose for adopting and investigating diffusion equations of fractional order is to describe phenomena of anomalous diffusion usually met in transport processes through complex and/or disordered systems including fractal media [18]. In [19], Nigmatullin used the fractional diffusion equation to describe diffusion in media with fractal geometry. Mainardi [3] pointed out that the propagation of mechanical diffusive wave in viscoelastic media can be modeled by TFDE. An interesting review on the anomalous diffusion by Metzler and Klafter [20] has appeared to which (and references therein) we refer the interested reader. Applying a time fractional integration of order α to both sides of the first equation in (1.2) allows us to eliminate the time fractional derivative on the L.H.S. leading to the integral form

$$u(x,t)-\frac{1}{\Gamma(\alpha)}\int_0^t\frac{\partial_x^2 u(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{f(x,\tau)+q(x,\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau. \tag{1.3}$$

The outline of the paper is as follows. In the next section we discuss the optimality conditions and spectral discretization of the optimal problem. *A posteriori* error estimate is derived in Section 3. Finally, in Section 4, we carry out some numerical tests to verify the theoretical results.

For a domain $\mathcal{O}$, which may be Λ, I or Ω, we use $L^2(\mathcal{O})$, $H^s(\mathcal{O})$, and $H_0^s(\mathcal{O})$ to denote the usual Sobolev spaces, equipped with the norms $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$ respectively. For the Sobolev space X with the norm $\|\cdot\|_X$, we define the space $H^s(I;X):=\{v\,|\,\|v(\cdot,t)\|_X\in H^s(I)\}$ endowed with the norm $\|v\|_{H^s(I;X)}:=\|\|v(\cdot,t)\|_X\|_{s,I}$. Particularly, when X stands for $H^\mu(\Lambda)$ or $H_0^\mu(\Lambda)$, the norm of the space $H^s(I;X)$ will be denoted by $\|\cdot\|_{\mu,s,\Omega}$. Hereafter, in cases where no confusion would arise, the domain symbols I, Λ, Ω may be dropped from the notations.

2 Optimization and spectral approximation of the problem

For a weak formula of the state equation (1.2), we introduce the control space $L^2(\Omega)$ and the state space [21]

$$B^s(\Omega) = H^s\big(I, L^2(\Lambda)\big) \cap L^2\big(I, H_0^1(\Lambda)\big), \quad \forall s > 0,$$

equipped with the norm

$$\|v\|_{B^s(\Omega)} = \big(\|v\|^2_{H^s(I,L^2(\Lambda))} + \|v\|^2_{L^2(I,H_0^1(\Lambda))}\big)^{\frac{1}{2}}.$$

Then a weak formulation for the state equation (1.2) reads as follows: given $q, f \in L^2(\Omega)$, find $u \in B^{\frac{\alpha}{2}}(\Omega)$ such that

$$\mathcal{A}(u, v) = (f + q, v)_\Omega, \quad \forall v \in B^{\frac{\alpha}{2}}(\Omega), \tag{2.1}$$

where the bilinear form $\mathcal{A}(\cdot, \cdot)$ is defined by

$$\mathcal{A}(u, v) := \big({}_0^R\partial_t^{\frac{\alpha}{2}} u, {}_t^R\partial_T^{\frac{\alpha}{2}} v\big)_\Omega + (\partial_x u, \partial_x v)_\Omega.$$

Here, ${}_0^R\partial_t^{\frac{\alpha}{2}}$ and ${}_t^R\partial_T^{\frac{\alpha}{2}}$ respectively denote the left and right Riemann-Liouville fractional derivatives of order $\frac{\alpha}{2}$.

By defining the cost functional

$$\mathcal{J}(q, u) := \frac{1}{2}\|u - \bar{u}\|^2_{0,\Omega} + \frac{\lambda}{2}\|q\|^2_{0,\Omega}, \quad (q, u) \in K \times B^{\frac{\alpha}{2}}(\Omega), \tag{2.2}$$

with the given desired state $\bar{u} \in L^2(\Omega)$, the optimal control problem reads as follows: find $(q^*, u(q^*)) \in K \times B^{\frac{\alpha}{2}}(\Omega)$ such that

$$\mathcal{J}\big(q^*, u(q^*)\big) = \min_{(q,u) \in K \times B^{\frac{\alpha}{2}}(\Omega)} \mathcal{J}(q, u) \quad \text{subject to (2.1)}. \tag{2.3}$$

The well-posedness of the state problem ensures the existence of a control-to-state mapping $q \mapsto u = u(q)$ defined through (2.1). By means of this mapping we introduce the reduced cost functional $J : L^2(\Omega) \to \mathbb{R}$ as follows:

$$J(q) := \mathcal{J}\big(q, u(q)\big), \quad q \in L^2(\Omega).$$

Then the optimal control problem (2.3) is equivalent to finding $q^* \in K$ such that

$$J\big(q^*\big) = \min_{q \in K} J(q). \tag{2.4}$$

The first order necessary optimality condition for (2.4) reads

$$J'\big(q^*\big)\big(\delta q - q^*\big) \geq 0, \quad \forall \delta q \in K, \tag{2.5}$$

where $J'(q^*)(\cdot)$ is called the gradient of $J(q)$, defined through the Gâteaux derivative.

It has been proved [17] that

$$J'(q)(\delta q) = \big(\lambda q + z(q), \delta q\big)_{\Omega}, \quad \forall \delta q \in L^2(\Omega), \tag{2.6}$$

where $z(q) = z \in B^{\frac{\alpha}{2}}(\Omega)$ is the solution of the following adjoint state equation:

$$\mathcal{A}(\varphi, z) = (u - \bar{u}, \varphi)_{\Omega}, \quad \forall \varphi \in B^{\frac{\alpha}{2}}(\Omega). \tag{2.7}$$

Now, we consider the spectral approximation of the optimal control problem. We define the polynomial space

$$P_M^0(\Lambda) = P_M(\Lambda) \cap H_0^1(\Lambda), \qquad S_L = P_M^0(\Lambda) \otimes P_N(I) \subset B^{\frac{\alpha}{2}}(\Omega),$$

where P_M denotes the space of all polynomials of degree less than or equal to M, L stands for the parameter pair (M, N).

Then we consider the spectral approximation to the state equation (2.1) as follows: find $u_L(q) \in S_L$ such that

$$\mathcal{A}\big(u_L(q), v_L\big) = (f + q, v_L)_{\Omega}, \quad \forall v_L \in S_L. \tag{2.8}$$

Similar to the continuous case, we introduce the semidiscrete reduced cost functional $J_L : L^2(\Omega) \to \mathbb{R}$ as follows:

$$J_L(q) := \mathcal{J}\big(q, u_L(q)\big), \quad q \in L^2(\Omega), \tag{2.9}$$

where $u_L(q)$ is given by (2.8). Then we consider the following auxiliary optimal problem: find $q^* \in K$ such that

$$J_L\big(q^*\big) = \min_{q \in K} J_L(q). \tag{2.10}$$

The solution q^* of the above problem fulfills the first order optimality condition

$$J_L'\big(q^*\big)\big(\delta q - q^*\big) \geq 0, \quad \forall \delta q \in K, \tag{2.11}$$

where

$$J_L'(q)(\phi) = \big(\lambda q + z_L(q), \phi\big)_{\Omega}, \quad \forall q, \phi \in K, \tag{2.12}$$

with $z_L(q) \in S_L$ being the solution of the semidiscrete adjoint problem

$$\mathcal{A}\big(\varphi_L, z_L(q)\big) = \big(u_L(q) - \bar{u}, \varphi_L\big)_{\Omega}, \quad \forall \varphi_L \in S_L. \tag{2.13}$$

Now we consider the approximation of the control space to obtain the full discrete optimal control problem. To this end, we introduce the finite dimensional subspace for the control variable as follows:

$$K_L = K \cap \big(P_M(\Lambda) \otimes P_N(I)\big).$$

Then the full discrete optimal control problem reads as follows: find $q_L^* \in K_L$ such that

$$J_L(q_L^*) = \min_{q_L \in K_L} J_L(q_L), \tag{2.14}$$

where $J_L(\cdot)$ is defined in (2.9). The unique solution of (2.14), q_L^*, satisfies the following optimality condition:

$$J_L'(q_L^*)(\delta q - q_L^*) \geq 0, \quad \forall \delta q \in K_L. \tag{2.15}$$

3 *A posteriori* error estimates

We aim in this section at deriving estimates of the error between a continuous solution and its spectral approximation in terms of known and computable quantities, *i.e.*, *a posteriori* error estimates. We will confine ourselves to the so-called residual-based estimates [22]. To simplify the notations, we let c be a generic positive constant independent of any functions and of any discretization parameters. We use the expression $A \lesssim B$ to mean that $A \leq cB$.

The error analysis will make use of some projection operators. The orthogonal projector $\Pi_M^{1,0} : H_0^1(\Lambda) \to P_M^0(\Lambda)$ is defined by $\forall v \in H_0^1(\Lambda)$, $\Pi_M^{1,0} v \in P_M^0(\Lambda)$ such that

$$\big((\Pi_M^{1,0} v - v)', \phi_M'\big)_\Lambda = 0, \quad \forall \phi_M \in P_M^0(\Lambda).$$

The following estimates hold [22]: $\forall v \in H^m(\Lambda) \cap H_0^1(\Lambda)$, $m \geq 1$,

$$\begin{aligned} &\big|\Pi_M^{1,0} v - v\big|_{1,\Lambda} \lesssim M^{1-m} \|v\|_{m,\Lambda}, \\ &\big\|\Pi_M^{1,0} v - v\big\|_{0,\Lambda} \lesssim M^{-m} \|v\|_{m,\Lambda}. \end{aligned}$$

For the L^2-orthogonal projector Π_N, defined by $\Pi_N v \in P_N(I)$, such that $(\Pi_N v - v, w_N)_I = 0$, $\forall w_N \in P_N(I)$, we have

$$\|\Pi_N v - v\|_{0,I} \lesssim N^{-m} \|v\|_{m,I}, \quad \forall v \in H^m(I), m \geq 0.$$

The L^2-orthogonal projector Π_M in Λ is defined similarly.

The first step is to derive *a posteriori* error estimates for the approximation to the control variable.

Lemma 3.1 *Suppose q^* and q_L^* are the solutions of* (2.4) *and* (2.14) *respectively, then the following estimate holds*:

$$\big\|q^* - q_L^*\big\|_{0,\Omega} \lesssim \big\|z_L(q_L^*) - z(q_L^*)\big\|_{0,\Omega}, \tag{3.1}$$

where $z_L(q_L^*)$ and $z(q_L^*)$ are respectively the solutions of (2.13) *and* (2.7) *associated to q_L^*.*

Proof Similar to Lemma 4.3 in [23], it follows from (2.6), (2.1) and (2.7) that for all $p, q \in L^2(\Omega)$,

$$J'(p)(p - q) - J'(q)(p - q) \geq \lambda \|p - q\|_{0,\Omega}^2.$$

Then in virtue of (2.5) and (2.15) we get, for arbitrary $p_L \in K_L$,

$$\begin{aligned}
&\lambda\|q^* - q_L^*\|_{0,\Omega}^2 \\
&\quad \le J'(q^*)(q^* - q_L^*) - J'(q_L^*)(q^* - q_L^*) \\
&\quad \le -J'(q_L^*)(q^* - q_L^*) \\
&\quad = J_L'(q_L^*)(q_L^* - q^*) - J'(q_L^*)(q^* - q_L^*) + J_L'(q_L^*)(q^* - q_L^*) \\
&\quad = J_L'(q_L^*)(q_L^* - p_L) + J_L'(q_L^*)(p_L - q^*) \\
&\qquad - (\lambda q_L^* + z(q_L^*), q^* - q_L^*)_\Omega + (\lambda q_L^* + z_L(q_L^*), q^* - q_L^*)_\Omega \\
&\quad \le J_L'(q_L^*)(p_L - q^*) + (z_L(q_L^*) - z(q_L^*), q^* - q_L^*)_\Omega \\
&\quad = (z_L(q_L^*) + \lambda q_L^*, p_L - q^*)_\Omega + (z_L(q_L^*) - z(q_L^*), q^* - q_L^*)_\Omega.
\end{aligned} \tag{3.2}$$

Furthermore, as shown in Lemma 5 in [17], it holds

$$(z_L(q_L^*) + \lambda q_L^*, \Pi_M \Pi_N q^* - q^*)_\Omega = 0. \tag{3.3}$$

Therefore, by taking $p_L = \Pi_N \Pi_M q^*$ in (3.2), then using (3.3) and the Cauchy-Schwarz inequality, we obtain (3.1). □

Theorem 3.1 *Let q^* be the solution of* (2.4), *$u(q^*)$ and $z(q^*)$ be the corresponding state and the adjoint state respectively. Let q_L^* be the solution of* (2.14) *with the corresponding discrete state $u_L(q_L^*)$ and the adjoint state $z_L(q_L^*)$. Then the following estimate holds*:

$$\|q^* - q_L^*\|_{0,\Omega} + \|u(q^*) - u_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} + \|z(q^*) - z_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \eta_1 + \eta_2,$$

where

$$\eta_1 = (N^{-\frac{\alpha}{2}} + M^{-1})\xi_1, \qquad \eta_2 = (N^{-\frac{\alpha}{2}} + M^{-1})\xi_2,$$

with

$$\begin{aligned}
\xi_1 &= \|{}_t^R\partial_T^\alpha z_L(q_L^*) - \partial_x^2 z_L(q_L^*) - u_L(q_L^*) + \bar{u}\|_{0,\Omega}, \\
\xi_2 &= \|{}_0^R\partial_t^\alpha u_L(q_L^*) - \partial_x^2 u_L(q_L^*) - f - q_L^*\|_{0,\Omega}.
\end{aligned}$$

Proof We first estimate $\|q^* - q_L^*\|_{0,\Omega}$. According to (3.1), it suffices to estimate $\|z_L(q_L^*) - z(q_L^*)\|_{0,\Omega}$. Let $e_z = z_L(q_L^*) - z(q_L^*)$, $e_z^L = \Pi_N \Pi_M^{1,0} e_z \in S_L$. It follows from (2.7) and (2.13) that

$$\mathcal{A}(e_z^L, e_z) = (u_L(q_L^*) - u(q_L^*), e_z^L)_\Omega.$$

It has been proved [21] that for all $u, v \in B^{\frac{\alpha}{2}}(\Omega)$, the following continuity and coercivity hold:

$$\mathcal{A}(u,v) \lesssim \|u\|_{B^{\frac{\alpha}{2}}(\Omega)} \|v\|_{B^{\frac{\alpha}{2}}(\Omega)}, \qquad \mathcal{A}(v,v) \gtrsim \|v\|_{B^{\frac{\alpha}{2}}(\Omega)}^2.$$

Thus, using (2.1), (2.7), (2.8), and (2.13), we have

$$\begin{aligned}
&\|z_L(q_L^*) - z(q_L^*)\|^2_{B^{\frac{\alpha}{2}}(\Omega)} \\
&\quad\lesssim \mathcal{A}(e_z, e_z) = \mathcal{A}(e_z - e_z^L, e_z) + \mathcal{A}(e_z^L, e_z) \\
&\quad= \mathcal{A}(e_z - e_z^L, z_L(q_L^*)) - \mathcal{A}(e_z - e_z^L, z(q_L^*)) + \mathcal{A}(e_z^L, e_z) \\
&\quad= \left({}_0^R\partial_t^{\frac{\alpha}{2}}(e_z - e_z^L), {}_t^R\partial_T^{\frac{\alpha}{2}} z_L(q_L^*)\right)_\Omega + \left(\partial_x(e_z - e_z^L), \partial_x z_L(q_L^*)\right)_\Omega \\
&\qquad - \left(u(q_L^*) - \bar{u}, e_z - e_z^L\right)_\Omega + \left(u_L(q_L^*) - u(q_L^*), e_z^L\right)_\Omega \\
&\quad= \left(e_z - e_z^L, {}_t^R\partial_T^\alpha z_L(q_L^*)\right)_\Omega - \left(e_z - e_z^L, \partial_x^2 z_L(q_L^*)\right)_\Omega - \left(u(q_L^*) - \bar{u}, e_z - e_z^L\right)_\Omega \\
&\qquad + \left(u_L(q_L^*) - u(q_L^*), e_z^L - e_z\right)_\Omega + \left(u_L(q_L^*) - u(q_L^*), e_z\right)_\Omega \\
&\quad= \left(e_z - e_z^L, {}_t^R\partial_T^\alpha z_L(q_L^*) - \partial_x^2 z_L(q_L^*)\right)_\Omega + \left(u_L(q_L^*) - \bar{u}, e_z^L - e_z\right)_\Omega \\
&\qquad + \left(u_L(q_L^*) - u(q_L^*), e_z\right)_\Omega \\
&\quad= \left(e_z - e_z^L, {}_t^R\partial_T^\alpha z_L(q_L^*) - \partial_x^2 z_L(q_L^*) - u_L(q_L^*) + \bar{u}\right)_\Omega \\
&\qquad + \left(u_L(q_L^*) - u(q_L^*), e_z\right)_\Omega \\
&\quad\lesssim \|e_z - e_z^L\|_{0,\Omega}\xi_1 + \|u_L(q_L^*) - u(q_L^*)\|_{0,\Omega}\|e_z\|_{0,\Omega}.
\end{aligned} \tag{3.4}$$

Furthermore,

$$\begin{aligned}
&\|e_z - e_z^L\|_{0,\Omega} \\
&\quad\le \|e_z - \Pi_N e_z\|_{0,\Omega} + \|\Pi_N e_z - \Pi_N \Pi_M^{1,0} e_z\|_{0,\Omega} \\
&\quad\le \|e_z - \Pi_N e_z\|_{0,\Omega} + \|\Pi_N(e_z - \Pi_M^{1,0} e_z) - (e_z - \Pi_M^{1,0} e_z)\|_{0,\Omega} \\
&\qquad + \|e_z - \Pi_M^{1,0} e_z\|_{0,\Omega} \\
&\quad\lesssim \|e_z - \Pi_N e_z\|_{0,\Omega} + \|e_z - \Pi_M^{1,0} e_z\|_{0,\Omega} \\
&\quad\lesssim N^{-\frac{\alpha}{2}}\|e_z\|_{0,\frac{\alpha}{2},\Omega} + M^{-1}\|e_z\|_{1,0,\Omega} \\
&\quad\lesssim N^{-\frac{\alpha}{2}}\|e_z\|_{B^{\frac{\alpha}{2}}(\Omega)} + M^{-1}\|e_z\|_{B^{\frac{\alpha}{2}}(\Omega)}.
\end{aligned} \tag{3.5}$$

Plugging (3.5) into (3.4) yields

$$\|z_L(q_L^*) - z(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \eta_1 + \|u_L(q_L^*) - u(q_L^*)\|_{0,\Omega}. \tag{3.6}$$

Similarly, set $e_u = u_L(q_L^*) - u(q_L^*)$, and let $e_u^L = \Pi_N \Pi_M^{1,0} e_u \in S_L$. Then it follows from (2.1) and (2.8) that $\mathcal{A}(e_u, e_u^L) = 0$, and thus

$$\begin{aligned}
&\|u_L(q_L^*) - u(q_L^*)\|^2_{B^{\frac{\alpha}{2}}(\Omega)} \\
&\quad\lesssim \mathcal{A}(e_u, e_u) = \mathcal{A}(e_u, e_u - e_u^L) \\
&\quad= \mathcal{A}(u_L(q_L^*), e_u - e_u^L) - \mathcal{A}(u(q_L^*), e_u - e_u^L) \\
&\quad= \left({}_0^R\partial_t^\alpha u_L(q_L^*) - \partial_x^2 u_L(q_L^*), e_u - e_u^L\right)_\Omega - \left(f + q_L^*, e_u - e_u^L\right)_\Omega
\end{aligned}$$

$$\begin{aligned}&\lesssim \xi_2\|e_u - e_u^L\|_{0,\Omega}\\&\lesssim \xi_2\big(N^{-\frac{\alpha}{2}}\|e_u\|_{0,\frac{\alpha}{2},\Omega} + M^{-1}\|e_u\|_{1,0,\Omega}\big)\\&\lesssim \xi_2\big(N^{-\frac{\alpha}{2}} + M^{-1}\big)\|e_u\|_{B^{\frac{\alpha}{2}}(\Omega)}.\end{aligned}$$

This leads to

$$\big\|u_L(q_L^*) - u(q_L^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \eta_2. \tag{3.7}$$

Then combining (3.6) and (3.7) gives

$$\big\|z_L(q_L^*) - z(q_L^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \eta_1 + \big\|u_L(q_L^*) - u(q_L^*)\big\|_{0,\Omega} \lesssim \eta_1 + \eta_2.$$

Using the above estimate, the inequality $\|\cdot\|_{0,\Omega} \lesssim \|\cdot\|_{B^{\frac{\alpha}{2}}(\Omega)}$, and Lemma 3.1, we get

$$\big\|q^* - q_L^*\big\|_{0,\Omega} \lesssim \eta_1 + \eta_2. \tag{3.8}$$

Furthermore, using the triangle inequalities

$$\begin{aligned}&\big\|z_L(q_L^*) - z(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \le \big\|z_L(q_L^*) - z(q_L^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} + \big\|z(q_L^*) - z(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)},\\&\big\|u_L(q_L^*) - u(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \le \big\|u_L(q_L^*) - u(q_L^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} + \big\|u(q_L^*) - u(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)},\end{aligned}$$

and the following obvious estimates

$$\big\|z(q_L^*) - z(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \big\|u(q_L^*) - u(q^*)\big\|_{0,\Omega} \lesssim \big\|q_L^* - q^*\big\|_{0,\Omega},$$

we obtain

$$\big\|u_L(q_L^*) - u(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} + \big\|z_L(q_L^*) - z(q^*)\big\|_{B^{\frac{\alpha}{2}}(\Omega)} \lesssim \eta_1 + \eta_2.$$

This completes the proof. □

4 Optimization algorithm and numerical results

4.1 Projection gradient optimization algorithm

In what follows, we propose a projection gradient optimization algorithm to solve the resulting minimization problems. The key of the algorithm is to determine a suitable projector to guarantee that the imposed constraint on the control variable is satisfied. To this end, to any $q_L \in P_M(\Lambda) \otimes P_N(I)$ we associate the function $q_K = -\min\{0, \overline{q_L}\} + q_L$ such that $q_K \in K$.

Then we propose the following projection gradient algorithm for the optimal control problem (2.14):

- Start with an initial control $q_L^{(0)}$.
- Repeat for $k = 0, 1, \ldots$.
  - Determine a descent direction: $J_L'(q_L^{(k)})$.
  - Choose a step size ρ_k.

- Update: $q_L^{(k+\frac{1}{2})} = q_L^{(k)} - \rho_k J_L'(q_L^{(k)})$.
- Projection: $q_L^{(k+\frac{1}{2})} \to q_L^{(k+1)} := q_K^{(k+\frac{1}{2})}$.

• Until stopping criterion is satisfied.

The proposed stopping criterion is

$$\|J_L'(q_L^{(k)})\| \le \varepsilon, \tag{4.1}$$

where ε is a pre-defined tolerance. Whenever (4.1) is satisfied for some k, the optimal control variable q_L^* is supposed to be obtained, *i.e.*, $q_L^* = q_L^{(k)}$.

The key components of the above algorithm include:

(i) Determination of the descent direction.

(ii) Choice of the step size.

The details are described below. Given an initial control $q_L^{(0)}$, the corresponding state $u_L(q_L^{(0)})$ is given by the solution of the state equation in (2.8). To apply the stopping criterion $\|J_L'(q_L^{(0)})\| \le \varepsilon$, we need information on the adjoint state $z_L(q_L^{(0)})$, which is obtained from the adjoint state equation (2.13) for given $u_L(q_L^{(0)})$ and $q_L^{(0)}$. Then the descent direction, that is, the gradient of the objective functional at $q_L^{(0)}$, is calculated through

$$d_L^{(0)} := J_L'(q_L^{(0)}) = z_L(q_L^{(0)}) + \lambda q_L^{(0)}.$$

Then, assuming known $q_L^{(k)}$ and $d_L^{(k)}$ at the current (kth) iteration, we update $q_L^{(k)}$ via

$$q_L^{(k+\frac{1}{2})} = q_L^{(k)} - \rho_k d_L^{(k)}, \qquad q_L^{(k+1)} = -\min\{0, \overline{q_L^{(k+\frac{1}{2})}}\} + q_L^{(k+\frac{1}{2})},$$

where ρ_k is the iteration step size determined in a way such that

$$J_L(q_L^{(k)} - \rho_k d_L^{(k)}) = \min_{\rho>0} J_L(q_L^{(k)} - \rho d_L^{(k)}).$$

Such a ρ_k is characterized by

$$\left(z_L^{(k+\frac{1}{2})} + \lambda(q_L^{(k)} - \rho_k d_L^{(k)}), d_L^{(k)}\right)_\Omega = 0, \tag{4.2}$$

where $z_L^{(k+\frac{1}{2})} \in S_L$ is the solution of

$$\mathcal{A}(\varphi_L, z_L^{(k+\frac{1}{2})}) = (u_L^{(k+\frac{1}{2})} - \bar{u}, \varphi_L)_\Omega, \quad \forall \varphi_L \in S_L \tag{4.3}$$

with $u_L^{(k+\frac{1}{2})} \in S_L$ given by

$$\mathcal{A}(u_L^{(k+\frac{1}{2})}, v_L) = (f + q_L^{(k)} - \rho_k d_L^{(k)}, v_L)_\Omega, \quad \forall v_L \in S_L. \tag{4.4}$$

The optimal iteration step size ρ_k can be efficiently calculated through solving (4.2). Indeed we first notice that there exists an explicit expression of $z_L^{(k+\frac{1}{2})}$ on ρ_k. Let $\tilde{u}_L^{(k)}$ and $\tilde{z}_L^{(k)}$ denote respectively the solutions of

$$\mathcal{A}(\tilde{u}_L^{(k)}, v_L) = (d_L^{(k)}, v_L)_\Omega, \quad \forall v_L \in S_L, \tag{4.5}$$

$$\mathcal{A}(\varphi_L, \tilde{z}_L^{(k)}) = (\tilde{u}_L^{(k)}, \varphi_L)_\Omega, \quad \forall \varphi_L \in S_L. \tag{4.6}$$

$u_L(q_L^{(k)})$ and $z_L(q_L^{(k)})$ are respectively the solutions of

$$\mathcal{A}(u_L(q_L^{(k)}), v_L) = (f + q_L^{(k)}, v_L)_\Omega, \quad \forall v_L \in S_L, \tag{4.7}$$

$$\mathcal{A}(\varphi_L, z_L(q_L^{(k)})) = (u_L(q_L^{(k)}) - \bar{u}, \varphi_L)_\Omega, \quad \forall \varphi_L \in S_L. \tag{4.8}$$

Then it can be checked that $z_L(q_L^{(k)}) - \rho_k \tilde{z}_L^{(k)}$ solves (4.3) and (4.4), that is,

$$z_L^{(k+\frac{1}{2})} = z_L(q_L^{(k)}) - \rho_k \tilde{z}_L^{(k)}.$$

Bringing this expression into (4.2) gives

$$(z_L(q_L^{(k)}) - \rho_k \tilde{z}_L^{(k)} + \lambda(q_L^{(k)} - \rho_k d_L^{(k)}), d_L^{(k)})_\Omega = 0.$$

Let $\tilde{d}_L^{(k)} = \tilde{z}_L^{(k)} + \lambda d_L^{(k)}$, then we obtain

$$\rho_k = \frac{(d_L^{(k)}, d_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, d_L^{(k)})_\Omega}. \tag{4.9}$$

The overall process is summarized below.

Projection gradient optimization algorithm Choose an initial control $q_L^{(0)}$, set $k = 0$.

(a) Solve problems (4.7) and (4.8), let $d_L^{(k)} = z_L(q_L^{(k)}) + \lambda q_L^{(k)}$.

(b) Solve problems (4.5) and (4.6), and set $\tilde{d}_L^{(k)} = \tilde{z}_L^{(k)} + \lambda d_L^{(k)}$, $\rho_k = \frac{(d_L^{(k)}, d_L^{(k)})_\Omega}{(\tilde{d}_L^{(k)}, d_L^{(k)})_\Omega}$.

(c) Update: $q_L^{(k+\frac{1}{2})} = q_L^{(k)} - \rho_k d_L^{(k)}$, $q_L^{(k+1)} = -\min\{0, \overline{q_L^{(k+\frac{1}{2})}}\} + q_L^{(k+\frac{1}{2})}$.

(d) If $\|d_L^{(k)}\| \leq$ tolerance, then take $q_L^* = q_L^{(k+1)}$ and solve problems (2.8) and (2.13) to get $u_L(q_L^*)$ and $z_L(q_L^*)$.
Else, set $k = k + 1$, repeat (a)-(d).

4.2 Numerical results

In this subsection we carry out some numerical experiments to validate the *a posteriori* error estimates for the numerical solutions. In our calculation, we take $T = 1$, $\lambda = 1$.

Example 4.1 We consider problem (2.3) with exact analytical solutions as

$$u(q^*) = \sin \pi x \cos \pi t, \qquad z(q^*) = \sin \pi x \sin \pi(1-t), \qquad q^* = \max\{0, \overline{z(q^*)}\} - z(q^*).$$

The right-hand side f and the desired state $\bar{u}$ here are respectively numerically calculated through (2.1) and (2.7) using $u(q^*)$, $z(q^*)$ and q^*.

In order to validate the *a posteriori* error estimate, we compare the error indicator $\eta = \eta_1 + \eta_2$, which is defined in Theorem 3.1 and the real error of the numerical solution measured by

$$e = \|q^* - q_L^*\|_{0,\Omega} + \|u(q^*) - u_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)} + \|z(q^*) - z_L(q_L^*)\|_{B^{\frac{\alpha}{2}}(\Omega)}.$$

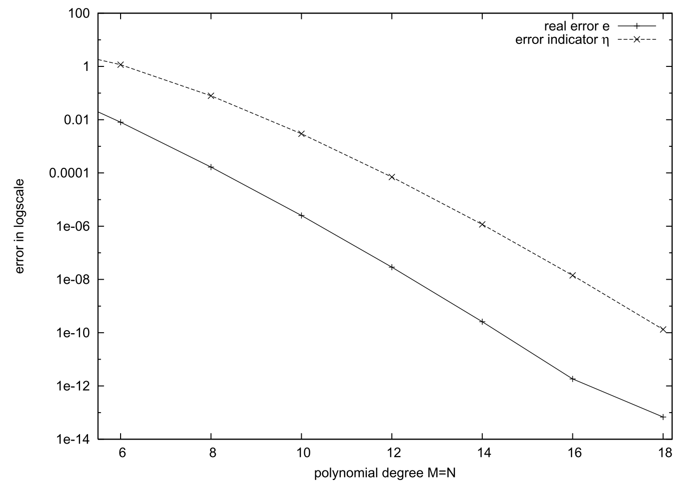

Figure 1 Performance of error indicator η with $M = N$, $\alpha = 0.6$.

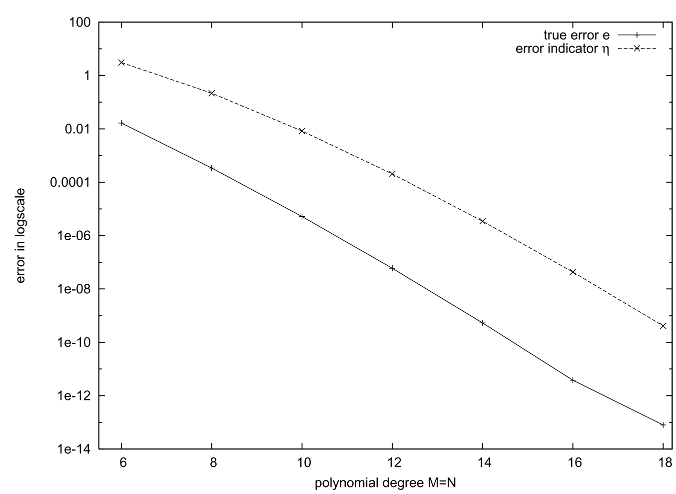

Figure 2 Performance of error indicator η with $M = N$, $\alpha = 0.3$.

These two errors are compared in Figure 1 as functions of M (= N). We observe that the indicator η has almost the same exponential decay as the error e, whereas it overestimates the error, which is consistent with our theoretical results.

Example 4.2 We choose other exact analytical solutions as

$$u(q^*) = \sin \pi x e^t, \qquad z(q^*) = \sin \pi x(1-t)e^{2t}, \qquad q^* = \max\{0, \overline{z(q^*)}\} - z(q^*).$$

The *a posteriori* error indicator η and the real error e are compared in Figure 2 as functions of M (= N) with $\alpha = 0.3$. It can be also observed from Figure 2 that the indicator η has almost the same exponential decay as the error e, and the reliability of the proposed estimator is confirmed again.

5 Concluding remarks

We have obtained *a posteriori* upper bound of the spectral method for the fractional control problem. This is an important step towards developing an adaptive spectral method for solving FOCPs. In the future, we will consider the efficiency of the *a posteriori* estimator to obtain an optimal estimate. As for the classical parabolic equation, we guess such an optimal estimate will have to make use of some Jacobi-weighted Sobolev spaces and polynomial inverse inequalities. Furthermore, many computational issues have to be addressed. For example, an adaptive refinement strategy should be investigated for efficiently implementing the adaptive spectral method for FOCPs, and the adaptive spectral method should be also used to solve some real examples from physical and engineering sciences.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have equal contributions to each part of this paper. All the authors read and approved the final manuscript.

Author details
[1]School of Science, Jimei University, Xiamen, 361021, China. [2]School of Mathematical Sciences, Xiamen University, Xiamen, 361005, China.

Acknowledgements
The work of XY Ye is partially supported by the Science Foundation of Jimei University, China (Grant Nos. ZQ2013005 and ZC2013021), the Foundation (Class B) of Fujian Educational Committee (Grant No. FB2013005), the Foundation of Fujian Educational Committee (No. JA14180). The work of CJ Xu was partially supported by the National NSF of China (Grant Nos. 11471274 and 11421110001).

Received: 11 November 2014 Accepted: 9 April 2015 Published online: 22 April 2015

References

1. Oustaloup, A: La Dérivation Non Entière: Théorie, Synthèse et Applications. Hermes, Paris (1995)
2. Koeller, RC: Application of fractional calculus to the theory of viscoelasticity. J. Appl. Mech. **51**, 299-307 (1984)
3. Mainardi, F: Fractional diffusive waves in viscoelastic solids. In: Nonlinear Waves in Solids, pp. 93-97 (1995)
4. Bouchaud, JP, Georges, A: Anomalous diffusion in disordered media: statistical mechanisms, models and physical applications. Phys. Rep. **195**(4-5), 127-293 (1990)
5. Dentz, M, Cortis, A, Scher, H, Berkowitz, B: Time behavior of solute transport in heterogeneous media: transition from anomalous to normal transport. Adv. Water Resour. **27**(2), 155-173 (2004)
6. Goychuk, I, Heinsalu, E, Patriarca, M, Schmid, G, Hänggi, P: Current and universal scaling in anomalous transport. Phys. Rev. E **73**(2), 020101 (2006)
7. Benson, DA, Wheatcraft, SW, Meerschaert, MM: The fractional-order governing equation of Lévy motion. Water Resour. Res. **36**(6), 1413-1423 (2000)
8. Diethelm, K: The Analysis of Fractional Differential Equations. Springer, Berlin (2010)
9. Miller, K, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
10. Podlubny, I: Fractional Differential Equations. Academic Press, New York (1999)
11. Agrawal, O: A general formulation and solution scheme for fractional optimal control problems. Nonlinear Dyn. **38**(1), 323-337 (2004)
12. Frederico, G, Torres, D: Fractional optimal control in the sense of Caputo and the fractional noethers theorem. Int. Math. Forum **3**(10), 479-493 (2008)
13. Mophou, GM: Optimal control of fractional diffusion equation. Comput. Math. Appl. **61**(1), 68-78 (2011)
14. Dorville, R, Mophou, GM, Valmorin, VS: Optimal control of a nonhomogeneous Dirichlet boundary fractional diffusion equation. Comput. Math. Appl. **62**(3), 1472-1481 (2011)
15. Lotfi, A, Yousefi, S, Dehghan, M: Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule. J. Comput. Appl. Math. **250**, 143-160 (2013)
16. Sweilam, N, Al-Ajami, T: Legendre spectral-collocation method for solving some types of fractional optimal control problems. J. Adv. Res. (2014). doi:10.1016/j.jare.2014.05.004
17. Ye, XY, Xu, C: A spectral method for optimal control problems governed by the time fractional diffusion equation with control constraints. In: Spectral and High Order Methods for Partial Differential Equations (ICOSAHOM 2012), pp. 403-414. Springer, Berlin (2014)
18. Gorenflo, R, Mainardi, F, Moretti, D, Paradisi, P: Time fractional diffusion: a discrete random walk approach. Nonlinear Dyn. **29**(1-4), 129-143 (2002). doi:10.1023/A:1016547232119
19. Nigmatullin, RR: The realization of the generalized transfer equation in a medium with fractal geometry. Phys. Status Solidi B **133**(1), 425-430 (1986)
20. Metzler, R, Klafter, J: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**(1), 1-77 (2000)

21. Li, XJ, Xu, CJ: A space-time spectral method for the time fractional diffusion equation. SIAM J. Numer. Anal. **47**(3), 2108-2131 (2009)
22. Canuto, C, Hussaini, MY, Quarteroni, A, Zang, TA: Spectral Methods - Fundamentals in Single Domains. Springer, Berlin (2006)
23. Ye, X, Xu, C: Spectral optimization methods for the time fractional diffusion inverse problem. Numer. Math., Theory Methods Appl. **6**(3), 499-519 (2013)