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Regularization and stability estimates for an inverse source problem of the radially symmetric parabolic equation

Wei Cheng*

*Correspondence:
chwei670815@163.com
College of Science, Henan University of Technology, Zhengzhou, 450001, P.R. China

Abstract
We consider an inverse problem of determining an unknown source term in the radially symmetric parabolic equation from a noisy final data and prove the uniqueness of solution for the problem. Using the Hölder inequality, we obtain a conditional stability for the space-dependent source term. A modified quasi-reversibility method is applied to deal with the ill-posedness of the problem. A Hölder-type error estimate between the approximate solution and the exact solution is provided by introducing some technical inequalities and choosing a suitable regularization parameter.

MSC: 35R25; 35R30; 65J20; 65M30

Keywords: ill-posed problem; inverse source problem; parabolic equation; quasi-reversibility method; error estimate

1 Introduction

Inverse source problems occur in many branches of engineering sciences, for example, heat conduction, reaction diffusion, pollutant detection, crack identification, geophysical prospecting and electromagnetic theory. These problems are typically ill-posed in the sense of Hadamard [1]. In other words, the solution (if it exists) does not depend continuously on measured data. So, the numerical simulation is very difficult and some special regularization methods and stability estimates are required.

The inverse source problems have been investigated in many papers; for example, the existence and uniqueness of the solution were investigated in [2, 3], the conditional stability and the data compatibility were studied in [4–10], and the numerical algorithms for the identification problem can be found in [11–17]. In [18], Yang and Fu solved an inverse problem for determining a heat source in a parabolic equation by a mollification regularization method, and they gave two kinds of explicit error estimates by using an *a priori* and an *a posteriori* regularization parameter choice rule, respectively. In [19], Wei and Wang used a modified quasi-boundary value method to deal with an inverse source problem of the time-fractional diffusion equation and provided two kinds of convergence rates. Cheng *et al.* in [20] solved the identification of an unknown source term in radial heat conduction by a spectral method and gave a logarithmic-type error estimate. Yang *et al.* in [21] applied a quasi-boundary value regularization method for identifying an unknown source in the Poisson equation. However, to our knowledge, the research into the inverse source identification problem is mainly devoted to numerical methods. The stability theory with explicit error estimate for the problem is still limited.

In this paper, we apply a modified quasi-reversibility method to deal with an inverse source identification problem of a radially symmetric parabolic equation and obtain a Hölder-type error estimate between the approximate solution and the exact solution. The physical model we considered is a sphere of radius r_0 with initial state, and it is considered radially symmetric and with the surface state distribution function remaining zero. The correspondingly mathematical model is the following radially symmetric parabolic equation:

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial r^2}-\frac{2}{r}\frac{\partial u}{\partial r}=f(r),\quad 0<r<r_0, 0<t<T, \tag{1.1}$$

with the boundary conditions

$$u(r_0,t)=0,\qquad \lim_{r\to 0} u(r,t)\text{ bounded},\quad 0\le t\le T, \tag{1.2}$$

and the initial condition and final observation at $t=T$,

$$u(r,0)=0,\qquad u(r,T)=g(r),\quad 0\le r\le r_0, \tag{1.3}$$

where r denotes the radial coordinate, u represents state function, and f is physical laws, which means source term here. The inverse source identification problem (1.1)-(1.3) is to determine the unknown source term $f(r)$ from the noisy final data $g(r)$.

The quasi-reversibility method was first proposed by Lattès and Lions in [22]. This method consists in replacing the former second-order ill-posed problem into a family of well-posed fourth-order problems that depend on a regularization parameter α. The solution of quasi-reversibility is close to the exact solution when α is small. This method has been used to solve various types of inverse problems such as inverse heat conduction problem [23, 24], backward heat conduction problem [25], the Cauchy problem of Laplace equation [26, 27] and Cauchy problem for the modified Helmholtz equation [28].

In this study, we propose a modified version of quasi-reversibility method to solve the inverse source problem (1.1)-(1.3), *i.e.*, adding a perturbation term in the parabolic equation (1.1) to form an approximate problem:

$$\begin{cases} \frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial r^2}-\frac{2}{r}\frac{\partial u}{\partial r}=f(r)-\alpha\frac{r_0^2}{\pi^2 r}(rf(r))'', & 0<r<r_0, 0<t<T,\\ u(r,0)=0, \quad u(r,T)=g^\delta(r), & 0\le r\le r_0,\\ u(r_0,t)=0, \quad \lim_{r\to 0} u(r,t)\text{ bounded}, & 0\le t\le T, \end{cases} \tag{1.4}$$

where α plays a role of regularization parameter, data g^δ represents the measured data of function g. For this modification, we can obtain a Hölder-type error estimate with an *a priori* choice of regularization parameter.

The paper is organized as follows. In Section 2, the ill-posedness of problem (1.1)-(1.3) is described. In Section 3, we prove the uniqueness of solution and obtain a conditional stability for the inverse source problem (1.1)-(1.3) by using the Hölder inequality. Introducing some technical inequalities and choosing a suitable regularization parameter, in

Section 4 we present a Hölder-type error estimate between the approximate solution and the exact solution for the modified quasi-reversibility method.

2 Ill-posedness of problem (1.1)-(1.3)

In this section, we derive an analytical solution for the inverse source problem by the eigenfunction expansion and analyze the ill-posedness of inverse source problem (1.1)-(1.3). Throughout this paper, we denote by $L^2[0,r_0;r^2]$ the Hilbert space of Lebesgue measurable functions h with weight r^2 on $[0,r_0]$, and $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and the norm on $L^2[0,r_0;r^2]$, respectively, with the norm

$$\|h\| = \left(\int_0^{r_0} r^2 \left|h(r)\right|^2 dr\right)^{\frac{1}{2}}.$$

If the solution of problem (1.1)-(1.3) exists, then it must be unique, which will be given in Section 3.

Applying separation of variables, we can obtain the eigenvalues and corresponding eigenfunctions of problem (1.1)-(1.3):

$$\lambda_n = \left(\frac{n\pi}{r_0}\right)^2 \quad \text{and} \quad R_n(r) = j_0\left(\frac{n\pi r}{r_0}\right), \quad n = 1,2,\ldots,$$

where $j_0(x) = \frac{\sin x}{x}$ denotes a spherical Bessel function of the first kind [29]. Then, using the eigenfunction method, we suppose that the solution $u(r,t)$ and nonhomogeneous term $f(r)$ of problem (1.1)-(1.3) can be represented as follows:

$$u(r,t) = \sum_{n=1}^{\infty} u_n(t) j_0\left(\frac{n\pi r}{r_0}\right), \tag{2.1}$$

$$f(r) = \sum_{n=1}^{\infty} f_n j_0\left(\frac{n\pi r}{r_0}\right). \tag{2.2}$$

Applying the properties of $j_0(x)$, the eigenfunctions system $j_0(\frac{n\pi r}{r_0})$ is complete and orthogonal with weight r^2 on $[0,r_0]$. Substituting (2.1) and (2.2) into equation (1.1) and the initial condition in (1.2), we have that $u_n(t)$ satisfies

$$\begin{cases} u_n'(t) + (n\pi/r_0)^2 u_n(t) = f_n, \\ u_n(0) = 0. \end{cases} \tag{2.3}$$

Solving this initial problem yields

$$u_n(t) = \int_0^t f_n e^{-(\frac{n\pi}{r_0})^2(t-\tau)}\, d\tau, \quad n = 1,2,\ldots. \tag{2.4}$$

Combining (2.1) with (2.4), we have

$$u(r,t) = \sum_{n=1}^{\infty} f_n \left(\int_0^t e^{-(\frac{n\pi}{r_0})^2(t-\tau)}\, d\tau\right) j_0\left(\frac{n\pi r}{r_0}\right).$$

From (1.2) at $t = T$,

$$g(r) = \sum_{n=1}^{\infty} f_n \left(\int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} \, d\tau \right) j_0 \left(\frac{n\pi r}{r_0} \right). \tag{2.5}$$

Thus, there holds

$$f_n = \frac{2((n\pi)^2 / r_0^3)}{\int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} \, d\tau} \int_0^{r_0} r^2 g(r) j_0 \left(\frac{n\pi r}{r_0} \right) dr, \quad n = 1, 2, \ldots, \tag{2.6}$$

since

$$\int_0^T e^{-(\frac{n\pi}{r_0})^2 (T-\tau)} \, d\tau = \left(\frac{r_0}{n\pi} \right)^2 \left(1 - e^{-(\frac{n\pi}{r_0})^2 T}\right),$$

then formula (2.5) becomes

$$g(r) = \sum_{n=1}^{\infty} f_n \left(\frac{r_0}{n\pi} \right)^2 \left(1 - e^{-(\frac{n\pi}{r_0})^2 T}\right) j_0 \left(\frac{n\pi r}{r_0} \right). \tag{2.7}$$

Define

$$\varphi_n(r) = \frac{\sqrt{2} n\pi}{\sqrt{r_0^3}} j_0 \left(\frac{n\pi r}{r_0} \right), \tag{2.8}$$

this eigenfunctions system is orthonormal with weight r^2 on $[0, r_0]$ and a complete system in $L^2[0, r_0; r^2]$. Combining (2.6) with (2.8), formula (2.7) can be rewritten as

$$g(r) = \sum_{n=1}^{\infty} \left(\frac{r_0}{n\pi} \right)^2 \left(1 - e^{-(\frac{n\pi}{r_0})^2 T}\right) (f, \varphi_n) \varphi_n(r). \tag{2.9}$$

In practical applications, the input data g can only be measured, so we actually have the measured data function $g^\delta(\cdot)$ which belongs to $L^2[0, r_0; r^2]$ and satisfies

$$\| g - g^\delta \| \le \delta, \tag{2.10}$$

where $\delta > 0$ denotes the noise level.

We introduce an operator $K : f(\cdot) \to g(\cdot)$, then we have

$$g(r) = Kf(r) = \sum_{n=1}^{\infty} \left(\frac{r_0}{n\pi} \right)^2 \left(1 - e^{-(\frac{n\pi}{r_0})^2 T}\right) (f, \varphi_n) \varphi_n. \tag{2.11}$$

It is easy to see that K is linear self-adjoint compact operator with eigenvalues

$$k_n = \left(\frac{r_0}{n\pi} \right)^2 \left(1 - e^{-(\frac{n\pi}{r_0})^2 T}\right) \tag{2.12}$$

and eigenelements φ_n. From formula (2.11), we have

$$(g, \varphi_n) = (f, \varphi_n) k_n,$$

then

$$f(r) = \sum_{n=1}^{\infty}(f,\varphi_n)\varphi_n = \sum_{n=1}^{\infty} k_n^{-1}(g,\varphi_n)\varphi_n. \tag{2.13}$$

Since the eigenvalues k_n of the operator K decay, we realize that problem (1.1)-(1.3) is an ill-posed problem.

It is well known that for any ill-posed problems an *a priori* bound assumption for the exact solution is needed and necessary. Otherwise, the convergence of the regularized approximation solution will not be obtained or the convergence rate can be arbitrarily slow [30]. Assume also that there exists an *a priori* condition for problem (1.1)-(1.3):

$$\|f\|_p \leq E, \quad p > 0, \tag{2.14}$$

where $\|f\|_p$ is defined by

$$\|f\|_p = \left(\sum_{n=1}^{\infty}\left(1+n^2\right)^p\left|(f,\varphi_n)\right|^2\right)^{\frac{1}{2}}.$$

3 Uniqueness and conditional stability for problem (1.1)-(1.3)

In this section, we provide the uniqueness and conditional stability in Theorems 3.1 and 3.2, respectively.

Let g be a known function in $L^2[0,r_0;r^2]$. We consider the problem of finding a pair of functions $(u(r,t),f(r))$.

Theorem 3.1 *If* $u_i \in C^{2,1}((0,r_0)\times(0,T]), f_i \in L^2[0,r_0;r^2]$ $(i=1,2)$. *Let* (u_i,f_i) $(i=1,2)$ *satisfy problem* (1.1)-(1.3), *then* $(u_1,f_1) = (u_2,f_2)$.

Proof Put $\tilde{u} = u_1 - u_2, \tilde{f} = f_1 - f_2$, it is easy to know that $\tilde{u}$ satisfies

$$\begin{cases} \tilde{u}_t - \tilde{u}_{rr} - \frac{2}{r}\tilde{u}_r = \tilde{f}(r), & 0<r<r_0, 0<t<T, \\ \tilde{u}(r_0,t) = 0, & 0 \leq t \leq T, \\ \tilde{u}(r,0) = 0, \qquad \tilde{u}(r,T) = 0, & 0 \leq r \leq r_0. \end{cases} \tag{3.1}$$

We now introduce a function

$$v(r,t) = \tilde{u}(r,t) + F(r), \tag{3.2}$$

where

$$F(r) = \int_0^r \xi^{-2}\int_0^{\xi}\tau^2\tilde{f}(\tau)\,d\tau\,d\xi - \int_0^{r_0}\xi^{-2}\int_0^{\xi}\tau^2\tilde{f}(\tau)\,d\tau\,d\xi. \tag{3.3}$$

According to formulas (3.1)-(3.3), we can obtain that the function $v(r,t)$ satisfies

$$v_t - v_{rr} - \frac{2}{r}v_r = 0, \quad 0<r<r_0, 0<t<T, \tag{3.4}$$

$$v(r_0,t) = 0, \quad 0 \leq t \leq T, \tag{3.5}$$

$$\nu(r,0) = \nu(r,T) = F(r), \quad 0 \le r \le r_0. \tag{3.6}$$

From (3.4)-(3.6), we have

$$\begin{aligned}
&\int_0^T \int_0^{r_0} r^2 \nu \left(\nu_t - \nu_{rr} - \frac{2}{r}\nu_r \right) dr\, dt \\
&\quad = \frac{1}{2}\int_0^{r_0} r^2 \nu^2 |_{t=0}^{t=T}\, dr - \int_0^T \int_0^{r_0} \nu\, d\big(r^2 \nu_r\big)\, dt \\
&\quad = \frac{1}{2}\int_0^{r_0} r^2 \big(\nu^2(r,T) - \nu^2(r,0)\big)\, dr \\
&\qquad - \int_0^T \left(r^2 \nu \nu_r |_{r=0}^{r=r_0} - \int_0^{r_0} r^2 \nu_r^2\, dr \right) dt \\
&\quad = \int_0^T \int_0^{r_0} r^2 \nu_r^2\, dr\, dt = 0.
\end{aligned}$$

Thus,

$$\nu_r^2 = 0,$$

i.e.,

$$\nu(r,t) = C(t), \quad 0 \le r \le r_0.$$

So there holds $\nu(r_0,t) = C(t)$. Combining this with (3.5) yields

$$\nu(r,t) \equiv 0, \quad (r,t) \in [0,r_0] \times [0,T]. \tag{3.7}$$

From (3.7) and (3.6), we get

$$F(r) \equiv 0, \quad (r,t) \in [0,r_0] \times [0,T] \tag{3.8}$$

and

$$\int_0^r \xi^{-2} \int_0^{\xi} \tau^2 \tilde{f}(\tau)\, d\tau\, d\xi - \int_0^{r_0} \xi^{-2} \int_0^{\xi} \tau^2 \tilde{f}(\tau)\, d\tau\, d\xi = 0. \tag{3.9}$$

Differentiating (3.9) with respect to r, we obtain $\tilde{f} \equiv 0$. Substituting (3.7) and (3.8) into (3.2) yields $\tilde{u} \equiv 0$. The uniqueness of inverse source problem (1.1)-(1.3) is proved. □

We give a conditional stability for the inverse source problem (1.1)-(1.3) in the following theorem.

Theorem 3.2 *Let $f(r)$ be the solution of the inverse source problem* (1.1)-(1.3), *and condition* (2.14) *be satisfied, then the following estimate holds*:

$$\|f\| \le \left(\frac{e^{(\pi/r_0)^2 T} \pi^2}{(e^{(\pi/r_0)^2 T} - 1) r_0^2} \right)^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \|g\|^{\frac{p}{p+2}}. \tag{3.10}$$

Proof From (2.13) and the Hölder inequality, the following holds:

$$\begin{aligned}
\|f\|^2 &= \left\| \sum_{n=1}^{\infty} k_n^{-1}(g,\varphi_n)\varphi_n \right\|^2 = \sum_{n=1}^{\infty} |k_n|^{-2}|g_n|^2 = \sum_{n=1}^{\infty} |k_n|^{-2}|g_n|^{\frac{4}{p+2}}|g_n|^{\frac{2p}{p+2}} \\
&\le \left[\sum_{n=1}^{\infty} \left(|k_n|^{-2}|g_n|^{\frac{4}{p+2}} \right)^{\frac{p+2}{2}} \right]^{\frac{2}{p+2}} \left[\sum_{n=1}^{\infty} \left(|g_n|^{\frac{2p}{p+2}} \right)^{\frac{p+2}{p}} \right]^{\frac{p}{p+2}} \\
&= \left[\sum_{n=1}^{\infty} \left| k_n^{-1} \right|^{p+2} |g_n|^2 \right]^{\frac{2}{p+2}} \left[\sum_{n=1}^{\infty} |g_n|^2 \right]^{\frac{p}{p+2}} \\
&= \left[\sum_{n=1}^{\infty} \left| k_n^{-1} \right|^{p} \left| k_n^{-1} g_n \right|^2 \right]^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}} \\
&= \left[\sum_{n=1}^{\infty} \left| \frac{(n\pi/r_0)^2}{1-e^{-(n\pi/r_0)^2T}} \right|^{p} \left(1+n^2\right)^{-p} \left(1+n^2\right)^{p} |f_n|^2 \right]^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}} \\
&\le \max_{n\in\mathbb{N}} \left| \left(\frac{\pi}{r_0} \right)^2 \frac{e^{(n\pi/r_0)^2T}}{e^{(n\pi/r_0)^2T}-1} \right|^{\frac{2p}{p+2}} \left[\sum_{n=1}^{\infty} \left(1+n^2\right)^{p} |f_n|^2 \right]^{\frac{2}{p+2}} \|g\|^{\frac{2p}{p+2}} \\
&\le \left(\frac{e^{(\pi/r_0)^2T}\pi^2}{(e^{(\pi/r_0)^2T}-1)r_0^2} \right)^{\frac{2p}{p+2}} E^{\frac{4}{p+2}} \|g\|^{\frac{2p}{p+2}}.
\end{aligned}$$

The proof is completed. □

4 Modified quasi-reversibility method and error estimate

In this section, we propose a modified quasi-reversibility method to solve problem (1.1)-(1.3) and give a Hölder-type error estimate with some technical inequalities and an *a priori* regularization parameter choice rule.

Let $(u_\alpha^\delta(r,t), f_\alpha^\delta(r))$ be the solution of the following regularized problem:

$$\begin{cases} \frac{\partial u_\alpha^\delta}{\partial t} - \frac{\partial^2 u_\alpha^\delta}{\partial r^2} - \frac{2}{r}\frac{\partial u_\alpha^\delta}{\partial r} = f_\alpha^\delta(r) - \alpha \frac{r_0^2}{\pi^2 r}(r f_\alpha^\delta(r))'', & 0<r<r_0, 0<t<T, \\ u_\alpha^\delta(r,0)=0, \quad u_\alpha^\delta(r,T)=g^\delta(r), & 0\le r\le r_0, \\ u_\alpha^\delta(r_0,t)=0, \quad \lim_{r\to 0} u_\alpha^\delta(r,t) \text{ bounded}, & 0\le t\le T, \end{cases} \tag{4.1}$$

where $\alpha > 0$ is a regularization parameter.

Similar to the derivation process of formula (2.13), we can obtain the solution of problem (4.1) as follows:

$$f_\alpha^\delta(r) = \sum_{n=1}^{\infty} \frac{k_n^{-1}}{1+\alpha n^2}(g^\delta,\varphi_n)\varphi_n. \tag{4.2}$$

We call $f_\alpha^\delta(r)$ above the quasi-reversibility approximations of the solution $f(r)$ of problem (1.1)-(1.3).

Before giving an error estimate, we present firstly the following lemma which is crucial for error estimate.

Lemma 4.1 *Let $x \geq 1$, then we have the inequality*

$$\frac{1}{1-e^{-x}} \leq \frac{e}{e-1}. \tag{4.3}$$

Lemma 4.2 *If constants $\alpha > 0$, $p > 0$, then we obtain the inequality*

$$\max_{n\in\mathbb{N}} \frac{\alpha n^2}{1+\alpha n^2}\left(1+n^2\right)^{-\frac{p}{2}} \leq \begin{cases} \alpha^{\frac{p}{2}}, & 0 < p < 2, \\ \alpha, & p \geq 2. \end{cases} \tag{4.4}$$

Proof Let

$$\psi(n) = \frac{\alpha n^2}{1+\alpha n^2}\left(1+n^2\right)^{-\frac{p}{2}}. \tag{4.5}$$

The proof of (4.4) is divided into two cases.

Case I. For large values of n, *i.e.*, for $n \geq n_0 := \alpha^{-\frac{1}{2}}$, we have

$$\psi(n) \leq \left(1+n^2\right)^{-\frac{p}{2}} \leq n_0^{-p} = \alpha^{\frac{p}{2}}. \tag{4.6}$$

Case II. For $n < n_0$, there holds

$$\psi(n) \leq \alpha n^2\left(1+n^2\right)^{-\frac{p}{2}}. \tag{4.7}$$

If $0 < p < 2$, inequality (4.7) becomes

$$\psi(n) \leq \alpha n^2 \cdot n^{-p} \leq \alpha^{\frac{p}{2}}. \tag{4.8}$$

If $p \geq 2$, we obtain

$$\psi(n) \leq \frac{\alpha n^2}{1+n^2} \leq \alpha. \tag{4.9}$$

According to (4.5)-(4.9), the proof of (4.4) is completed. □

Theorem 4.3 *Let $f(r)$ given by* (2.13) *be the exact source history for $r \in [0, r_0]$ and $f_\alpha^\delta(r)$ given by* (4.2) *be the regularized approximation source to $f(r)$. Suppose that the a priori condition* (2.14) *and the noise assumption* (2.10) *hold, then*:

(1) *If $0 < p < 2$ and select the regularization parameter $\alpha = (\frac{\delta}{E})^{\frac{2}{p+2}}$, there holds the stability estimate*

$$\left\|f(\cdot) - f_\alpha^\delta(\cdot)\right\| \leq \left(1 + \frac{\pi^2 e^{(\pi/r_0)^2 T}}{r_0^2(e^{(\pi/r_0)^2 T} - 1)}\right) E^{\frac{2}{p+2}} \delta^{\frac{p}{p+2}}. \tag{4.10}$$

(2) *If $p \geq 2$ and choose the regularization parameter $\alpha = (\frac{\delta}{E})^{\frac{1}{2}}$, there holds the stability estimate*

$$\left\|f(\cdot) - f_\alpha^\delta(\cdot)\right\| \leq \left(1 + \frac{\pi^2 e^{(\pi/r_0)^2 T}}{r_0^2(e^{(\pi/r_0)^2 T} - 1)}\right) E^{\frac{1}{2}} \delta^{\frac{1}{2}}. \tag{4.11}$$

Proof Due to (2.13) and (4.2), there holds

$$\begin{aligned}
\|f(\cdot)-f_\alpha^\delta(\cdot)\| &= \left\|\sum_{n=1}^{\infty}k_n^{-1}(g,\varphi_n)\varphi_n-\sum_{n=1}^{\infty}\frac{k_n^{-1}}{1+\alpha n^2}\bigl(g^\delta,\varphi_n\bigr)\varphi_n\right\| \\
&\le \left\|\sum_{n=1}^{\infty}k_n^{-1}(g,\varphi_n)\varphi_n-\sum_{n=1}^{\infty}\frac{k_n^{-1}}{1+\alpha n^2}(g,\varphi_n)\varphi_n\right\| \\
&\quad + \left\|\sum_{n=1}^{\infty}\frac{k_n^{-1}}{1+\alpha n^2}\bigl(g-g^\delta,\varphi_n\bigr)\varphi_n\right\| \\
&\le \sup_{n\in\mathbb{N}}\frac{\alpha n^2}{1+\alpha n^2}\bigl(1+n^2\bigr)^{-p/2}\left\|\sum_{n=1}^{\infty}\bigl(1+n^2\bigr)^{p/2}k_n^{-1}(g,\varphi_n)\varphi_n\right\| \\
&\quad + \sup_{n\in\mathbb{N}}\frac{(\frac{n\pi}{r_0})^2}{(1+\alpha n^2)}\frac{1}{(1-e^{-(n\pi/r_0)^2T})}\left\|\sum_{n=1}^{\infty}\bigl(g-g^\delta,\varphi_n\bigr)\varphi_n\right\| \\
&\le \sup_{n\in\mathbb{N}}\left(\frac{\pi}{r_0}\right)^2\frac{1}{\alpha}\frac{1}{(1-e^{-(n\pi/r_0)^2T})}\|g-g_\delta\| \\
&\quad + \sup_{n\in\mathbb{N}}\frac{\alpha n^2}{1+\alpha n^2}\bigl(1+n^2\bigr)^{-p/2}\|f\|_p.
\end{aligned}$$

Combining with conditions (2.10), (2.14) and inequalities (4.3), (4.4), we obtain

$$\|f(\cdot)-f_\alpha^\delta(\cdot)\| \le \frac{\pi^2e^{(\pi/r_0)^2T}}{r_0^2(e^{(\pi/r_0)^2T}-1)}\frac{\delta}{\alpha}+\begin{cases}E\alpha^{\frac{p}{2}}, & 0<p<2,\\ E\alpha, & p\ge 2.\end{cases}$$

Choose the regularization parameter α by

$$\alpha=\begin{cases}(\frac{\delta}{E})^{\frac{2}{p+2}}, & 0<p<2,\\ (\frac{\delta}{E})^{\frac{1}{2}}, & p\ge 2.\end{cases}$$

Thus, we have

$$\|f(\cdot)-f_\alpha^\delta(\cdot)\| \le \begin{cases}(1+\frac{\pi^2e^{(\pi/r_0)^2T}}{r_0^2(e^{(\pi/r_0)^2T}-1)})E^{\frac{2}{p+2}}\delta^{\frac{p}{p+2}}, & 0<p<2,\\ (1+\frac{\pi^2e^{(\pi/r_0)^2T}}{r_0^2(e^{(\pi/r_0)^2T}-1)})E^{\frac{1}{2}}\delta^{\frac{1}{2}}, & p\ge 2.\end{cases}$$

The proof is completed. □

Remark 4.4 In general, the *a priori* bound E is unknown in practice. In this case, for Theorem 4.3, with

$$\alpha=\begin{cases}\delta^{\frac{2}{p+2}}, & 0<p<2,\\ \delta^{\frac{1}{2}}, & p\ge 2,\end{cases}$$

the following stability estimates hold:

$$\|f(\cdot)-f_\alpha^\delta(\cdot)\| \le \begin{cases}(E+\frac{\pi^2e^{(\pi/r_0)^2T}}{r_0^2(e^{(\pi/r_0)^2T}-1)})\delta^{\frac{p}{p+2}}, & 0<p<2,\\ (E+\frac{\pi^2e^{(\pi/r_0)^2T}}{r_0^2(e^{(\pi/r_0)^2T}-1)})\delta^{\frac{1}{2}}, & p\ge 2,\end{cases}$$

where E is only a bounded positive constant and it is not necessary to know it exactly.

Competing interests
The author declares that she has no competing interests.

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