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A new iterative algorithm for split solution problems of quasi-nonexpansive mappings

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Abstract

Some strong convergence algorithms are introduced to solve the split common fixed point problem for quasi-nonexpansive mappings. These results develop the related ones for fixed point iterative methods in the literature.

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1 Introduction and preliminaries

Throughout this paper, let H be a real Hilbert space with zero vector θ, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively. Let K be a nonempty closed convex subset of a Banach space E and T be a mapping from K into itself. In this paper, the set of fixed points of T is denoted by $F(T)$. The symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

Let $T: K \to K$ be a mapping and K a subset of a Banach space E. T is called a nonexpansive mapping if, for all $x, y \in K$, $\|Tx - Ty\| \le \|x - y\|$. T is called quasi-nonexpansive, if $F(T) \neq \emptyset$ and for all $x \in K$, $p \in F(T)$, $\|Tx - Tp\| \le \|x - p\|$. For examples of quasi-nonexpansive mappings, see [1].

Let H_1 and H_2 be two real Hilbert spaces. $T_1 : H_1 \to H_1$, $T_2 : H_2 \to H_2$ are two nonlinear operators with $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. $A : H_1 \to H_2$ is a bounded linear operator. The split fixed point problem for T_1 and T_2 is to

$$\text{find an element } x \in F(T_1) \text{ such that } Ax \in F(T_2). \tag{1.1}$$

Let $\Gamma = \{x \in F(T_1) : Ax \in F(T_2)\}$ denote the solution set of the problem (1.1). The problem was proposed by Censor and Segal [2] in a finite-dimensional space firstly. Next, Moudafi [3] studied the problem (1.1) in real Hilbert spaces; this generalized the problem (1.1) from a finite-dimensional space to infinite-dimensional Hilbert spaces. More precisely, the following result was obtained.

Theorem M (see [3]) *Let H_1 and H_2 be two real Hilbert spaces. Given a bounded linear operator $A : H_1 \to H_2$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two quasi-nonexpansive operators with $F(U) \neq \emptyset$ and $F(T) \neq \emptyset$. Assume that $U - I$ and $T - I$ are demiclosed at θ. Let*

$\{x_n\}$ be generated by

$$\begin{cases} x_1 \in H_1, \\ u_n = x_n + \gamma\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U(u_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\beta \in (0,1)$, $\{\alpha_n\} \subset (\delta, 1 - \delta)$ for a small enough $\delta > 0$, $\gamma \in (0, \frac{1}{\lambda\beta})$, and λ is the spectral radius of the operator A^*A. Then $\{x_n\}$ weakly converges to a split common fixed point $x^* \in \{x^* \in F(U) : Ax^* \in F(T)\}$.

It is well known that the split feasibility problem and the convex feasibility problem are useful to some areas of applied mathematics such as image recovery, convex optimization, and so on. According to [2], the split common fixed point problem (1.1) is a generalization of both these; also see [3]. This shows the split common fixed point problem (1.1) is important. Recently, some convergence theorems for the split common solution problems were given in [4–9]. We notice that Theorem M is a weak convergence theorem, and it is well known that a strong convergence theorem is always more convenient to use. Hence, the purpose of this paper is to give some algorithms for the problem (1.1), and establishes some strong convergence theorems. At the same time, we generalize the problem (1.1) to two countable families of quasi-nonexpansive mappings.

A mapping T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to y, and if the sequence $\{Tx_n\}$ strongly converges to z, we have $T(y) = z$; see [3].

Definition 1.1 (see [4, 5]) Let K be a nonempty closed convex subset of a real Hilbert space and T a mapping from K into K. The mapping T is called *zero-demiclosed* if $\{x_n\}$ in K satisfying $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup z \in K$ implies $Tz = z$.

Proposition 1.1 (see [4, 5]) *Let K be a nonempty closed convex subset of a real Hilbert space with zero vector θ and T a mapping from K into K. Then the following statements hold.*

(a) *T is zero-demiclosed if and only if $I - T$ is demiclosed at θ.*
(b) *If T is a nonexpansive mapping and there is a bounded sequence $\{x_n\} \subset H$ such that $\|x_n - Tx_n\| \to 0$ as $n \to 0$, then T is zero-demiclosed.*

Example 1.1 (see [4]) Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$ and $Tx = \frac{x^2+2}{1+x}$ for all $x \in C$. Then T is a continuous zero-demiclosed quasi-nonexpansive mapping but not nonexpansive.

Example 1.2 (see [4]) Let $H = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the standard norm $|\cdot|$. Let $C := [0, +\infty)$. Let T be a mapping from C into C defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1}, & x \in (1, +\infty), \\ 0, & x \in [0, 1]. \end{cases}$$

Then T is a discontinuous quasi-nonexpansive mapping but not zero-demiclosed.

The following results are important in this paper.

Let C be a closed convex subset of a real Hilbert space H. P_C denotes a metric projection of H onto C, it is well known that $P_C(x)$ has the properties: for $x \in H$, and $z \in C$,

$$z = P_C(x) \quad \Leftrightarrow \quad \langle x - z, z - y \rangle \geq 0, \quad \forall y \in C \tag{1.3}$$

and

$$\left\| y - P_C(x) \right\|^2 + \left\| x - P_C(x) \right\|^2 \leq \|x - y\|^2, \quad \forall y \in C, \forall x \in H. \tag{1.4}$$

In a real Hilbert space H, it is also well known that

$$\left\| \lambda x + (1-\lambda)y \right\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \quad \forall x, y \in H, \forall \lambda \in \mathbb{R} \tag{1.5}$$

and

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2, \quad \forall x, y \in H. \tag{1.6}$$

2 Strong convergence theorems

In this section, we construct some algorithms to solve the split common fixed point problem (1.1) for quasi-nonexpansive mappings.

Theorem 2.1 *Let H_1 and H_2 be two real Hilbert spaces. C is a nonempty closed convex subset of H_1 and K a nonempty closed convex subset of H_2. $T_1 : C \to H_1$ and $T_2 : H_2 \to H_2$ are two quasi-nonexpansive mappings with $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. $A : H_1 \to H_2$ is a bounded linear operator. Assume that $T_1 - I$ and $T_2 - I$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)T_1 z_n, \\ z_n = P_C(x_n + \lambda A^*(T_2 - I)Ax_n), \\ C_{n+1} = \{x \in C_n : \|y_n - x\| \leq \|z_n - x\| \leq \|x_n - x\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{2.1}$$

where P is a projection operator and A^* denotes the adjoint of A. $\{\alpha_n\} \subset (0, \eta] \subset (0,1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. Assume that $\Gamma = \{p \in F(T_1) : Ap \in F(T_2)\} \neq \emptyset$, then $x_n \to x^* \in \Gamma$ and $Ax_n \to Ax^* \in F(T_2)$.

Proof It is easy to verify that C_n is closed for $n \in \mathbb{N} \cup \{0\}$. We verify C_n is convex for $n \in \mathbb{N} \cup \{0\}$. In fact, let $v_1, v_2 \in C_{n+1}$, for each $\lambda \in (0,1)$, we have

$$\begin{aligned} \left\| y_n - \left(\lambda v_1 + (1-\lambda)v_2\right) \right\|^2 &= \left\| \lambda(y_n - v_1) - (1-\lambda)(y_n - v_2) \right\|^2 \\ &= \lambda \|y_n - v_1\|^2 + (1-\lambda)\|y_n - v_2\|^2 - \lambda(1-\lambda)\|v_1 - v_2\|^2 \\ &\leq \lambda \|z_n - v_1\|^2 + (1-\lambda)\|z_n - v_2\|^2 - \lambda(1-\lambda)\|v_1 - v_2\|^2 \\ &= \left\| z_n - \left(\lambda v_1 + (1-\lambda)v_2\right) \right\|^2, \end{aligned}$$

namely, $\|y_n - (\lambda v_1 + (1-\lambda)v_2)\| \le \|z_n - (\lambda v_1 + (1-\lambda)v_2)\|$. Similarly, we have $\|z_n - (\lambda v_1 + (1-\lambda)v_2)\| \le \|x_n - (\lambda v_1 + (1-\lambda)v_2)\|$; this shows $\lambda v_1 + (1-\lambda)v_2 \in C_{n+1}$ and C_{n+1} is a convex set for $n \in \mathbb{N} \cup \{0\}$. Now we prove $\Gamma \subset C_n$ for $n \in \mathbb{N} \cup \{0\}$. Let $p \in \Gamma$, then

$$\begin{aligned}
&2\lambda\langle x_n - p, A^*(T_2Ax_n - Ax_n)\rangle \\
&\quad = 2\lambda\langle A(x_n - p) + (T_2Ax_n - Ax_n) - (T_2Ax_n - Ax_n), T_2Ax_n - Ax_n\rangle \\
&\quad = 2\lambda\bigl(\langle T_2Ax_n - Ap, (T_2Ax_n - Ax_n)\rangle - \|T_2Ax_n - Ax_n\|^2\bigr) \\
&\quad = 2\lambda\biggl(\frac{1}{2}\|T_2Ax_n - Ap\|^2 + \frac{1}{2}\|T_2Ax_n - Ax_n\|^2 \\
&\qquad - \frac{1}{2}\|Ax_n - Ap\|^2 - \|T_2Ax_n - Ax_n\|^2\biggr) \quad \text{by (1.6)} \\
&\quad \le 2\lambda\biggl(\frac{1}{2}\|T_2Ax_n - Ax_n\|^2 - \|T_2Ax_n - Ax_n\|^2\biggr) \\
&\quad = -\lambda\|T_2Ax_n - Ax_n\|^2.
\end{aligned} \tag{2.2}$$

From (2.1) and (2.2) we have

$$\begin{aligned}
\|z_n - p\|^2 &= \bigl\|P_C\bigl(x_n + \lambda A^*(T_2Ax_n - Ax_n)\bigr) - P_C(p)\bigr\|^2 \\
&\le \bigl\|x_n + \lambda A^*(T_2Ax_n - Ax_n) - p\bigr\|^2 \\
&= \|x_n - p\|^2 + \bigl\|\lambda A^*(T_2Ax_n - Ax_n)\bigr\|^2 + 2\lambda\langle x_n - p, A^*(T_2Ax_n - Ax_n)\rangle \\
&\le \|x_n - p\|^2 + \lambda^2\bigl\|A^*\bigr\|^2\|T_2Ax_n - Ax_n\|^2 - \lambda\|T_2Ax_n - Ax_n\|^2 \\
&= \|x_n - p\|^2 - \lambda\bigl(1 - \lambda\bigl\|A^*\bigr\|^2\bigr)\|T_2Ax_n - Ax_n\|^2.
\end{aligned} \tag{2.3}$$

Again from $p \in \Gamma$, (2.1), and (2.3), it follows that

$$\|y_n - p\| \le \|z_n - p\| \le \|x_n - p\|. \tag{2.4}$$

Hence, $p \in C_n$ and $\Gamma \subset C_n$ for $n \in \mathbb{N} \cup \{0\}$.

Notice that $\Gamma \subset C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_0) \subset C_n$, then

$$\|x_{n+1} - x_0\| \le \|p - x_0\| \quad \text{for } n \in \mathbb{N} \text{ and } p \in \Gamma. \tag{2.5}$$

By (2.5), $\{x_n\}$ is bounded. For $n \in \mathbb{N}$, by (1.4), we have

$$\|x_{n+1} - x_n\|^2 + \|x_0 - x_n\|^2 = \bigl\|x_{n+1} - P_{C_n}(x_0)\bigr\|^2 + \bigl\|x_0 - P_{C_n}(x_0)\bigr\|^2 \le \|x_{n+1} - x_0\|^2,$$

which implies that $0 \le \|x_n - x_{n+1}\|^2 \le \|x_{n+1} - x_0\|^2 - \|x_0 - x_n\|^2$. Thus $\{\|x_n - x_0\|\}$ is nondecreasing. Therefore, by the boundedness of $\{x_n\}$, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. For $m, n \in \mathbb{N}$ with $m > n$, from $x_m = P_{C_m}(x_0) \subset C_n$ and (1.4), we have

$$\|x_m - x_n\|^2 + \|x_0 - x_n\|^2 = \bigl\|x_m - P_{C_n}(x_0)\bigr\|^2 + \bigl\|x_0 - P_{C_n}(x_0)\bigr\|^2 \le \|x_m - x_0\|^2. \tag{2.6}$$

By (2.5) and (2.6), $\lim_{n\to\infty}\|x_n - x_m\| = 0$. So, $\{x_n\}$ is a Cauchy sequence.

Let $x_n \to x^*$. Since $x_{n+1} = P_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, we have

$$\begin{aligned} \|z_n - x_n\| &\leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0, \\ \|y_n - x_n\| &\leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \leq 2\|x_{n+1} - x_n\| \to 0, \\ \|y_n - z_n\| &\leq \|y_n - x_n\| + \|x_n - z_n\| \to 0. \end{aligned} \tag{2.7}$$

Notice that $\lambda(1 - \lambda\|A^*\|^2) > 0$, from (2.3) and (2.7),

$$\begin{aligned} \|T_2Ax_n - Ax_n\|^2 &\leq \frac{\|x_n - p\|^2 - \|z_n - p\|^2}{\lambda(1 - \lambda\|A^*\|^2)} \\ &\leq \frac{1}{\lambda(1 - \lambda\|A^*\|^2)} \|x_n - z_n\| \bigl\{ \|x_n - p\| + \|z_n - p\| \bigr\} \to 0. \end{aligned} \tag{2.8}$$

Again from (2.1) and (2.7), we have

$$\|T_1z_n - z_n\| = \bigl\|(T_1 - I)z_n\bigr\| \to 0. \tag{2.9}$$

Since $x_n \to x^*$, from (2.7) we have $z_n \to x^*$, which implies that $z_n \rightharpoonup x^*$. By Proposition 1.1, we obtain $x^* \in F(T_1)$.

Next, we want to show $Ax^* \in F(T_2)$. Since A is a bounded linear operator, we know that $\|Ax_n - Ax^*\| \to 0$ by $x_n \to x^*$. Together with $\|T_2Ax_n - Ax_n\| \to 0$ and $T_2 - I$ being demiclosed at θ, we have $Ax^* \in F(T_2)$. Thus, $x^* \in \Gamma$ and $\{x_n\}$ converges strongly to $x^* \in \Gamma$. The proof is completed. □

Remark 2.1 If the quasi-nonexpansive mappings T_1 and T_2 are continuous, then the demiclosed property can be removed for the quasi-nonexpansive mappings T_1 and T_2 in Theorem 2.1.

Now, we consider the split fixed point problem for a finite family of quasi-nonexpansive mappings.

Lemma 2.1 (see [3]) *Let $T : H \to H$ be a quasi-nonexpansive mapping, and set $T_\alpha := (1 - \alpha)I + \alpha T$ for $\alpha \in (0,1]$. Then $\|T_\alpha x - p\| \leq \|x - p\| - \alpha(1 - \alpha)\|Tx - x\|$, $p \in F(T)$ and $x \in H$. Moreover, $F(T_\alpha) = F(T)$.*

Lemma 2.2 *Let $T_1, T_2 : H \to H$ be two quasi-nonexpansive mappings and set $S_{\xi_1} := (1 - \xi_1)I + \xi_1T_1$ and $S_{\xi_2} := (1 - \xi_2)I + \xi_2T_2$ for $\xi_1, \xi_2 \in (0,1)$. Again let $S = \tau S_{\xi_1} + (1 - \tau)S_{\xi_2}$ for $\tau \in (0,1)$. Then S is a quasi-nonexpansive mapping, and $F(S) = \bigcap_{i=1}^2 F(S_{\xi_i}) = \bigcap_{i=1}^2 F(T_i)$.*

Proof (1) It is easy to verify that $\bigcap_{i=1}^2 F(S_{\xi_i}) = \bigcap_{i=1}^2 F(T_i)$. We only need to prove $F(S) = \bigcap_{i=1}^2 F(S_{\xi_i})$. Clearly, $\bigcap_{i=1}^2 F(S_{\xi_i}) \subset F(S)$. On the other hand, for $p \in F(S)$ and $p_1 \in \bigcap_{i=1}^2 F(S_{\xi_i})$, we have

$$\begin{aligned} \|p - p_1\|^2 &= \bigl\|\tau S_{\xi_1}p + (1 - \tau)S_{\xi_2}p - p_1\bigr\|^2 = \bigl\|\tau(S_{\xi_1}p - p_1) + (1 - \tau)(S_{\xi_2}p - p_1)\bigr\|^2 \\ &= \tau\|S_{\xi_1}p - p_1\|^2 + (1 - \tau)\|S_{\xi_2}p - p_1\|^2 - \tau(1 - \tau)\|S_{\xi_1}p - S_{\xi_2}p\|^2 \\ &\leq \tau\|p - p_1\|^2 - \tau\xi_1(1 - \xi_1)\|T_1p - p\|^2 + (1 - \tau)\|p - p_1\|^2 \end{aligned}$$

$$\begin{aligned} &- (1-\tau)\xi_2(1-\xi_2)\|T_2p-p\|^2 \quad \text{(by Lemma 2.1)} \\ &= \|p-p_1\|^2 - \tau\xi_1(1-\xi_1)\|T_1p-p\|^2 - (1-\tau)\xi_2(1-\xi_2)\|T_2p-p\|^2, \end{aligned}$$

which yields $\|T_1p-p\| = \|T_2p-p\| = 0$, namely, $p \in \bigcap_{i=1}^2 F(T_i) = \bigcap_{i=1}^2 F(S_{\xi_i})$. So, $F(S) = \bigcap_{i=1}^2 F(S_{\xi_i})$.

(2) Let $x \in H$ and $p \in F(S)$. Then

$$\begin{aligned} \|Sx-p\| &= \|\tau S_{\xi_1}x + (1-\tau)S_{\xi_2}x - p\| = \|\tau(S_{\xi_1}x-p) + (1-\tau)(S_{\xi_2}x-p)\| \\ &\le \tau\|x-p\| + (1-\tau)\|x-p\| = \|x-p\| \quad \text{(by Lemma 2.1)}. \end{aligned}$$

So, S is a quasi-nonexpansive mapping. The proof is completed. □

Lemma 2.3 *Let $T_1, T_2, \dots, T_k : H \to H$ be k quasi-nonexpansive mappings and set $S = \sum_{i=1}^k \tau_i S_{\xi_i}$, where $\tau_i \in (0,1)$ satisfies $\sum_{i=1}^k \tau_i = 1$, $S_{\xi_i} := (1-\xi_i)I + \xi_i T_i$ for $\xi_i \in (0,1)$, $i = 1, 2, \dots, k$. Then S is a quasi-nonexpansive mapping, and $F(S) = \bigcap_{i=1}^k F(S_{\xi_i}) = \bigcap_{i=1}^k F(T_i)$.*

Proof Using mathematical induction, Lemma 2.3 is obtained by Lemma 2.2. □

Theorem 2.2 *Let H_1 and H_2 be two real Hilbert spaces. C is a nonempty closed convex subset of H_1 and K a nonempty closed convex subset of H_2. $T_1, \dots, T_k : C \to H_1$ are k quasi-nonexpansive mappings with $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. $G_1, \dots, G_l : H_2 \to H_2$ are l quasi-nonexpansive mappings with $\bigcap_{j=1}^l F(G_j) \neq \emptyset$. $A : H_1 \to H_2$ is a bounded linear operator. Assume that $T_i - I$ $(i = 1, 2, \dots, k)$ and $G_j - I$ $(j = 1, 2, \dots, l)$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)\sum_{i=1}^k \tau_i T_{\xi_i} z_n, \\ z_n = P_C(x_n + \lambda A^*(\sum_{i=1}^l \varepsilon_j G_{\theta_j} - I)Ax_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{2.10}$$

where P is a projection operator and A^* denotes the adjoint of A, $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. $\tau_i \in (0,1)$ and $\varepsilon_j \in (0,1)$ satisfy $\sum_{i=1}^k \tau_i = 1$ and $\sum_{j=1}^l \varepsilon_j = 1$, $T_{\xi_i} := (1-\xi_i)I + \xi_i T_i$ for $\xi_i \in (0,1)$, $i = 1, 2, \dots, k$, $G_{\theta_j} := (1-\theta_j)I + \theta_j G_j$ for $\theta_j \in (0,1)$, $j = 1, 2, \dots, l$. Assume that $\Gamma = \{p \in \bigcap_{i=1}^k F(T_i) : Ap \in \bigcap_{j=1}^l F(G_j)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $q \in \Gamma$.

Proof Let $T = \sum_{i=1}^k \tau_i T_{\xi_i}$, $S = \sum_{i=1}^l \varepsilon_j G_{\theta_j}$, by Lemma 2.3, $F(T) = \bigcap_{i=1}^k F(T_i) \neq \emptyset$, and $F(S) = \bigcap_{j=1}^l F(G_j) \neq \emptyset$. Moreover, T and S are quasi-nonexpansive mappings.

Next, we want to prove $T - I$ and $S - I$ are demiclosed at θ. By the hypothesis, $T_i - I$ $(i = 1, 2, \dots, k)$ and $G_j - I$ $(j = 1, 2, \dots, l)$ are demiclosed at θ. So, $T_{\xi_i} - I = \xi_i(T_i - I)$ and $G_{\theta_j} - I = \theta_j(G_j - I)$ are demiclosed at θ, and that $T - I = \sum_{i=1}^k \tau_i(T_{\xi_i} - I)$ and $S - I = \sum_{i=1}^l \varepsilon_j(G_{\theta_j} - I)$ are demiclosed at θ.

Thus, by Theorem 2.1, we obtain the desired result. The proof is completed. □

If $C = H_1$ in Theorem 2.1 and Theorem 2.2, then we have the following corollaries.

Corollary 2.1 *Let H_1 and H_2 be two real Hilbert spaces. $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two quasi-nonexpansive mappings with $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$. $A : H_1 \to H_2$ is a bounded linear operator. Assume that $T_1 - I$ and $T_2 - I$ are demiclosed at θ. Let $x_0 \in H_1$, $C_0 = H_1$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)T_1 z_n, \\ z_n = x_n + \lambda A^*(T_2 A x_n - A x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator and A^* denotes the adjoint of A, $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. Assume that $\Gamma = \{p \in F(T_1) : Ap \in F(T_2)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $x^* \in \Gamma$.

Corollary 2.2 *Let H_1 and H_2 be two real Hilbert spaces. $T_1, \dots, T_k : H_1 \to H_1$ are k quasi-nonexpansive mappings with $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. $G_1, \dots, G_l : H_2 \to H_2$ are l quasi-nonexpansive mappings with $\bigcap_{j=1}^l F(G_j) \neq \emptyset$. $A : H_1 \to H_2$ is a bounded linear operator. Assume that $T_i - I$ $(i = 1,2,\dots,k)$ and $G_j - I$ $(j = 1,2,\dots,l)$ are demiclosed at θ. Let $x_0 \in H_1$, $C_0 = H_1$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)\sum_{i=1}^k \tau_i T_{\xi_i} z_n, \\ z_n = x_n + \lambda A^*(\sum_{i=1}^k \tau_i G_{\theta_j} A x_n - A x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator and A^* denotes the adjoint of A, $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. Here $\tau_i \in (0,1)$ and $\varepsilon_j \in (0,1)$ satisfy $\sum_{i=1}^k \tau_i = 1$ and $\sum_{j=1}^l \varepsilon_j = 1$, $T_{\xi_i} := (1-\xi_i)I + \xi_i T_i$ for $\xi_i \in (0,1)$, $i = 1,2,\dots,k$, $G_{\theta_j} := (1-\theta_j)I + \theta_j G_j$ for $\theta_j \in (0,1)$, $j = 1,2,\dots,l$. Assume that $\Gamma = \{p \in \bigcap_{i=1}^k F(T_i) : Ap \in \bigcap_{j=1}^l F(G_j)\} \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $q \in \Gamma$.

If $H_1 = H_2 := H$ and A is an identity operator, then we have the following results by Theorems 2.1 and 2.2, respectively.

Corollary 2.3 *Let H be a real Hilbert space. C is a nonempty closed convex subset of H. $T_1 : C \to H$ and $T_2 : H \to H$ are two quasi-nonexpansive mappings with $\Gamma := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $T_1 - I$ and $T_2 - I$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)T_1 z_n, \\ z_n = P_C((1-\lambda)x_n + \lambda T_2 x_n), \\ C_{n+1} = \{x \in C_n : \|y_n - x\| \le \|z_n - x\| \le \|x_n - x\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator. $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0,1)$. Then $x_n \to x^* \in \Gamma$.

Corollary 2.4 *Let H be a real Hilbert space. C is a nonempty closed convex subset of H. $T_1, \dots, T_k : C \to H$ are k quasi-nonexpansive mappings with $\bigcap_{i=1}^k F(T_i) \neq \emptyset$. $G_1, \dots, G_l$:*

$H \to H$ are l quasi-nonexpansive mappings with $\bigcap_{j=1}^{l} F(G_j) \neq \emptyset$. Assume that $T_i - I$ ($i = 1, 2, \dots, k$) and $G_j - I$ ($j = 1, 2, \dots, l$) are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)\sum_{i=1}^{k} \tau_i T_{\xi_i} z_n, \\ z_n = P_C((1-\lambda)x_n + \lambda \sum_{i=1}^{l} \varepsilon_j G_{\theta_j} x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator. $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$, $\lambda \in (0, 1)$. $\tau_i \in (0, 1)$ and $\varepsilon_j \in (0, 1)$ satisfy $\sum_{i=1}^{k} \tau_i = 1$ and $\sum_{j=1}^{l} \varepsilon_j = 1$, $T_{\xi_i} := (1-\xi_i)I + \xi_i T_i$ for $\xi_i \in (0, 1)$, $i = 1, 2, \dots, k$, $G_{\theta_j} := (1-\theta_j)I + \theta_j G_j$ for $\theta_j \in (0, 1)$, $j = 1, 2, \dots, l$. Assume that $\Gamma := (\bigcap_{i=1}^{k} F(T_i)) \cap (\bigcap_{j=1}^{l} F(G_j)) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $q \in \Gamma$.

If $C = H := H_1 = H_2$ and A is an identity operator, then we have the following results by Corollaries 2.3 and 2.4, respectively.

Corollary 2.5 *Let H be a real Hilbert space. $T_1, T_2 : H \to H$ are two quasi-nonexpansive mappings with $\Gamma := F(T_1) \cap F(T_2) \neq \emptyset$. Assume that $T_1 - I$ and $T_2 - I$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n) T_1 z_n, \\ z_n = (1-\lambda)x_n + \lambda T_2 x_n, \\ C_{n+1} = \{x \in C_n : \|y_n - x\| \leq \|z_n - x\| \leq \|x_n - x\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator. $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$, $\lambda \in (0, 1)$. Then $x_n \to x^* \in \Gamma$.

Corollary 2.6 *Let H be a real Hilbert space. $T_1, \dots, T_k : H \to H$ are k quasi-nonexpansive mappings with $\bigcap_{i=1}^{k} F(T_i) \neq \emptyset$. $G_1, \dots, G_l : H \to H$ are l quasi-nonexpansive mappings with $\bigcap_{j=1}^{l} F(G_j) \neq \emptyset$. Assume that $T_i - I$ ($i = 1, 2, \dots, k$) and $G_j - I$ ($j = 1, 2, \dots, l$) are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n)\sum_{i=1}^{k} \tau_i T_{\xi_i} z_n, \\ z_n = (1-\lambda)x_n + \lambda \sum_{i=1}^{l} \varepsilon_j G_{\theta_j} x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where P is a projection operator. $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$, $\lambda \in (0, 1)$. $\tau_i \in (0, 1)$ and $\varepsilon_j \in (0, 1)$ satisfy $\sum_{i=1}^{k} \tau_i = 1$ and $\sum_{j=1}^{l} \varepsilon_j = 1$, $T_{\xi_i} := (1-\xi_i)I + \xi_i T_i$ for $\xi_i \in (0, 1)$, $i = 1, 2, \dots, k$, $G_{\theta_j} := (1-\theta_j)I + \theta_j G_j$ for $\theta_j \in (0, 1)$, $j = 1, 2, \dots, l$. Assume that $\Gamma := (\bigcap_{i=1}^{k} F(T_i)) \cap (\bigcap_{j=1}^{l} F(G_j)) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element $q \in \Gamma$.

Remark 2.2 The coefficient condition that $\{\alpha_n\} \subset (\delta, 1-\delta)$ for a small enough $\delta > 0$ in Theorem M is replaced with $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$. This shows we can let $\alpha_n = \frac{1}{n+1}$ in this paper, which is a natural choice.

3 Further generalization of the problem (1.1)

In Section 2, we gave a strong convergence algorithm for the problem (1.1). By the algorithm, we also considered the split solution problem for two finite families of quasi-nonexpansive mappings; see the algorithm (2.10). However, the algorithm (2.10) has an obvious drawback, in that the algorithm (2.10) will be invalid for two countable families of quasi-nonexpansive mappings. So, in this section, we introduce an algorithm for the split solution problem of two countable families of quasi-nonexpansive mappings. The following lemma can be found in [10].

Lemma *The unique solutions to the positive integer equation*

$$n = i + \frac{(m-1)m}{2}, \quad m \geq i, n = 1, 2, 3, \ldots \tag{3.1}$$

are

$$i = n - \frac{(m-1)m}{2}, \quad m = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{2}}\right] \geq i, n = 1, 2, 3, \ldots, \tag{3.2}$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

Theorem 3.1 *Let* H_1 *and* H_2 *be two real Hilbert spaces.* C *is a nonempty closed convex subset of* H_1. $A : H_1 \to H_2$ *is a bounded linear operator.* $\{T_i\}_{i=1}^{\infty} : C \to H_1$ *and* $\{G_i\}_{i=1}^{\infty} : H_2 \to H_2$ *are two countable families of quasi-nonexpansive mappings with* $\Gamma = \{p \in \bigcap_{i=1}^{\infty} F(T_i) : Ap \in \bigcap_{j=1}^{\infty} F(G_j)\} \neq \emptyset$. *Assume that* $T_i - I$ $(i = 1, 2, \ldots)$ *and* $G_j - I$ $(j = 1, 2, \ldots)$ *are demiclosed at* θ. *Let* $x_0 \in C$, $C_0 = C$, *and* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1 - \alpha_n) T_{i_n} z_n, \\ z_n = P_C(x_n + \lambda A^*(G_{i_n} - I)Ax_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.3}$$

where P *is a projection operator and* A^* *denotes the adjoint of* A, $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. i_n *satisfies* (3.1), *i.e.* $i_n = n - \frac{(m-1)m}{2}$ *and* $m \geq i_n$ *for* $n = 1, 2, \ldots$. *Then the sequence* $\{x_n\}$ *converges strongly to an element* $q \in \Gamma$.

Proof Just like the proof in Theorem 2.1, we can obtain the following facts (I)-(IV):

(I) For $p \in \Gamma$,

$$2\lambda\langle x_n - p, A^*(G_{i_n} - I)Ax_n\rangle \leq -\lambda\|(G_{i_n} - I)Ax_n\|^2, \tag{3.4}$$

$$\|z_n - p\|^2 \leq \|x_n - p\|^2 - \lambda(1 - \lambda\|A^*\|^2)\|(G_{i_n} - I)Ax_n\|^2 \tag{3.5}$$

and

$$\|y_n - p\| \leq \|z_n - p\| \leq \|x_n - p\|. \tag{3.6}$$

(II) We have $\Gamma \subset C_n$ for $n \in \mathbb{N} \cup \{0\}$. C_n is also closed and convex for $n \in \mathbb{N} \cup \{0\}$.

(III) $\{x_n\}$ is a Cauchy sequence and

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{3.7}$$

(IV)

$$\lim_{n\to\infty}\left\|(T_{i_n} - I)z_n\right\| = 0, \qquad \lim_{n\to\infty}\left\|(G_{i_n} - I)Ax_n\right\| = 0. \tag{3.8}$$

Now, for each $i \in \mathbb{N}$, set $K_i = \{k \geq 1 : k = i + \frac{(m-1)m}{2}, m \geq i, m \in \mathbb{N}\}$. Since $n = i_n + \frac{(m-1)m}{2}$, $m \geq i_n$, and $m \in \mathbb{N}$ for $n = 1, 2, \ldots$, and the definition of K_i, we have $i_k \equiv i$ for $k \in K_i$. Obviously, $\{k\}$ is a subsequence of $\{n\}$. Thus, for $k \in K_i$ and $i \in \mathbb{N}$, it follows from (3.8) that

$$\begin{aligned}
&\lim_{k\to\infty}\left\|(T_i - I)z_k\right\| = \lim_{k\to\infty}\left\|(T_{i_k} - I)z_k\right\| = 0, \\
&\lim_{k\to\infty}\left\|(G_i - I)Ax_k\right\| = \lim_{k\to\infty}\left\|(G_{i_k} - I)Ax_k\right\| = 0.
\end{aligned} \tag{3.9}$$

Let $x_n \to x^*$. From (3.7) we have $z_n \to x^*$. By (3.9), we obtain $x^* \in F(T_i)$.

Next, we want to prove $Ax^* \in F(G_i)$. Since A is a bounded linear operator, $\|Ax_n - Ax^*\| \to 0$ by $x_n \to x^*$. Together with $\|(G_i - I)Ax_k\| \to 0$, we have $Ax_n \to Ax^* \in F(G_i)$. Thus, $x^* \in \Gamma$ and $\{x_n\}$ converges strongly to $x^* \in \Gamma$. The proof is completed. □

If $C = H_1$, then we have the following result by Theorem 3.1.

Corollary 3.1 *Let H_1 and H_2 be two real Hilbert spaces. $A : H_1 \to H_2$ is a bounded linear operator. $\{T_i\}_{i=1}^{\infty} : H_1 \to H_1$ and $\{G_i\}_{i=1}^{\infty} : H_2 \to H_2$ are two countable families of quasi-nonexpansive mappings with $\Gamma = \{p \in \bigcap_{i=1}^{\infty} F(T_i) : Ap \in \bigcap_{j=1}^{\infty} F(G_j)\} \neq \emptyset$. Assume that $T_i - I$ $(i = 1, 2, \ldots)$ and $G_j - I$ $(j = 1, 2, \ldots)$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = H_1$, and $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases}
y_n = \alpha_n z_n + (1 - \alpha_n)T_{i_n}z_n, \\
z_n = x_n + \lambda A^*(G_{i_n} - I)Ax_n, \\
C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\
x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\},
\end{cases} \tag{3.10}$$

where P is a projection operator and A^* denotes the adjoint of A, $\{\alpha_n\} \subset (0, \eta] \subset (0, 1)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$. i_n satisfies (3.1), i.e. $i_n = n - \frac{(m-1)m}{2}$ and $m \geq i_n$ for $n = 1, 2, \ldots$. Then the sequence $\{x_n\}$ converges strongly to an element $q \in \Gamma$.

If $H_1 = H_2 := H$ and A is an identity operator, then we have the following results by Theorem 3.1 and Corollary 3.1, respectively.

Corollary 3.2 *Let H be a real Hilbert space. C is a nonempty closed convex subset of H. $\{T_i\}_{i=1}^{\infty} : C \to H$ and $\{G_i\}_{i=1}^{\infty} : H \to H$ are two countable families of quasi-nonexpansive mappings with $\Gamma := (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{\infty} F(G_j)) \neq \emptyset$. Assume that $T_i - I$ $(i = 1, 2, \ldots)$ and $G_j - I$ $(j = 1, 2, \ldots)$ are demiclosed at θ. Let $x_0 \in C$, $C_0 = C$, and $\{x_n\}$ be a sequence generated* *in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n) T_{i_n} z_n, \\ z_n = P_C((1-\lambda)x_n + \lambda G_{i_n} x_n), \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.11}$$

where P is a projection operator. $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0,1)$. i_n *satisfies* (3.1), *i.e.* $i_n = n - \frac{(m-1)m}{2}$ *and* $m \ge i_n$ *for* $n = 1, 2, \ldots$. *Then the sequence* $\{x_n\}$ *converges strongly to an element* $q \in \Gamma$.

Corollary 3.3 *Let H be a real Hilbert space.* $\{T_i\}_{i=1}^{\infty} : H \to H$ *and* $\{G_i\}_{i=1}^{\infty} : H \to H$ *are two countable families of quasi-nonexpansive mappings with* $\Gamma = \{p \in (\bigcap_{i=1}^{\infty} F(T_i)) \cap (\bigcap_{j=1}^{\infty} F(G_j))\} \ne \emptyset$. *Assume that* $T_i - I$ $(i = 1, 2, \ldots)$ *and* $G_j - I$ $(j = 1, 2, \ldots)$ *are demiclosed at* θ. *Let* $x_0 \in H$, $C_0 = H$, *and* $\{x_n\}$ *be a sequence generated in the following manner*:

$$\begin{cases} y_n = \alpha_n z_n + (1-\alpha_n) T_{i_n} z_n, \\ z_n = (1-\lambda)x_n + \lambda G_{i_n} x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le \|z_n - v\| \le \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_0), \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.12}$$

where P is a projection operator. $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$, $\lambda \in (0,1)$. i_n *satisfies* (3.1), *i.e.* $i_n = n - \frac{(m-1)m}{2}$ *and* $m \ge i_n$ *for* $n = 1, 2, \ldots$. *Then the sequence* $\{x_n\}$ *converges strongly to an element* $q \in \Gamma$.

4 Conclusion

(1) We give strong convergence algorithms for the split common fixed point problem of quasi-nonexpansive mappings. Our results improve and generalize some well-known results in [3, 11] and so on.

(2) Although Theorem 3.1 gives a strong convergence algorithm for two countable families of quasi-nonexpansive mappings, the condition that each mapping must be demiclosed at θ is very strong. In addition, we guess the speed of convergence is not too fast for the algorithm (3.3). Therefore, the algorithm (3.3) should be improved further in the future.

(3) The split common solution problem is a very interesting topic. It has received attention by many scholars. Many research articles have been published, for example, [12–21] and references therein.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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