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Relaxed extragradient methods for systems of variational inequalities

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Abstract
In this paper, we introduce a relaxed extragradient iterative algorithm for finding a common element of the set of solutions of a general mixed equilibrium problem, the set of solutions of general system of variational inequalities, the set of solutions of finitely many variational inclusions, and the set of common fixed points of finitely many nonexpansive mappings and a strict pseudocontraction in a real Hilbert space. The iterative algorithm is based on Korpelevich's extragradient method, the viscosity approximation method, Mann's iterative method, and the strongly positive bounded linear operator approach. We derive the strong convergence of the iterative algorithm to a common element of these sets, which also solves some hierarchical minimization.

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Keywords: hybrid viscosity iterative algorithm; general mixed equilibrium; general system of variational inequalities; nonexpansive mapping; inverse-strongly monotone mapping

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, C be a nonempty closed convex subset of H, and P_C be the metric projection of H onto C. Let $S: C\to C$ be a self-mapping on C. We denote by $\mathrm{Fix}(S)$ the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers. A mapping V is called strongly positive on H if there exists a constant $\bar{\gamma}>0$ such that

$$\langle Vx,x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x\in H.$$

A mapping $A: C\to H$ is called L-Lipschitz-continuous if there exists a constant $L\geq 0$ such that

$$\|Ax-Ay\|\leq L\|x-y\|, \quad \forall x,y\in C.$$

In particular, if $L=1$ then A is called a nonexpansive mapping; if $L\in[0,1)$ then A is called a contraction. A mapping $T: C\to C$ is called ξ-strictly pseudocontractive if there exists a constant $\xi\in[0,1)$ such that

$$\|Tx-Ty\|^2\leq\|x-y\|^2+\xi\left\|(I-T)x-(I-T)y\right\|^2, \quad \forall x,y\in C.$$

In particular, if $\xi=0$, then T is a nonexpansive mapping.

Let $A : C \to H$ be a nonlinear mapping on C. We consider the following variational inequality problem (VIP): find a point $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $\mathrm{VI}(C, A)$.

The VIP (1.1) was first discussed by Lions [1]. There are many applications of VIP (1.1) in various fields; see *e.g.*, [2, 3]. It is well known that, if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (1.1) has a unique solution. In 1976, Korpelevich [4] proposed an iterative algorithm for solving VIP (1.1) in Euclidean space $\mathbf{R}^n$:

$$\begin{cases} y_n = P_C(x_n - \tau A x_n), \\ x_{n+1} = P_C(x_n - \tau A y_n), \quad \forall n \geq 0, \end{cases}$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast and Korpelevich's extragradient method has received great attention given by many authors, who improved it in various ways; see *e.g.*, [5–20] and the references therein.

On the other hand, we consider the following general mixed equilibrium problem (GMEP) (see, also, [21, 22]) of finding $x \in C$ such that

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C, \tag{1.2}$$

where $\Theta, h : C \times C \to \mathbf{R}$ are two bifunctions. We denote the set of solutions of GMEP (1.2) by $\mathrm{GMEP}(\Theta, h)$. The GMEP (1.2) is very general; for example, it includes the following equilibrium problems as special cases.

As an example, in [12, 23, 24] the authors considered and studied the generalized equilibrium problem (GEP), which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of GEP is denoted by $\mathrm{GEP}(\Theta, A)$.

In [21, 25, 26], the authors considered and studied the mixed equilibrium problem (MEP), which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solutions of MEP is denoted by $\mathrm{MEP}(\Theta, \varphi)$.

In [27–30], the authors considered and studied the equilibrium problem (EP), which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of EP is denoted by $\mathrm{EP}(\Theta)$. It is worth to mention that the EP is an unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, *etc.*

Throughout this paper, it is assumed as in [31] that $\Theta : C \times C \to \mathbf{R}$ is a bifunction satisfying conditions (θ1)-(θ3) and $h : C \times C \to \mathbf{R}$ is a bifunction with restrictions (h1)-(h3), where

(θ1) $\Theta(x,x) = 0$ for all $x \in C$;

(θ2) Θ is monotone (*i.e.*, $\Theta(x,y) + \Theta(y,x) \le 0$, $\forall x,y \in C$) and upper hemicontinuous in the first variable, *i.e.*, for each $x,y,z \in C$,

$$\limsup_{t\to 0^+} \Theta\bigl(tz + (1-t)x, y\bigr) \le \Theta(x,y);$$

(θ3) Θ is lower semicontinuous and convex in the second variable;

(h1) $h(x,x) = 0$ for all $x \in C$;

(h2) h is monotone and weakly upper semicontinuous in the first variable;

(h3) h is convex in the second variable.

For $r > 0$ and $x \in H$, let $T_r : H \to 2^C$ be a mapping defined by

$$T_r x = \left\{ z \in C : \Theta(z,y) + h(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\},$$

called the resolvent of Θ and h.

Let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers in $[0,1]$. For any $n \ge 1$, define a mapping W_n on C as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \dots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \dots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{1.4}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \dots, T_1$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$.

In 2013, Rattanaseeha [30] introduced an iterative algorithm:

$$\begin{cases} x_1 \in H \text{ arbitrarily given}, \\ \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = P_C[\alpha_n \gamma f(x_n) + (I - \alpha_n V)W_n u_n], \quad \forall n \ge 1, \end{cases} \tag{1.5}$$

and proved the following strong convergence theorem.

Theorem R (see [30], Theorem 3.1) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying assumptions* (A1)-(A4). *Let f be an α-contraction on H with $\alpha \in (0,1)$, and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive self-mappings on C such that $\Omega := \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \cap \operatorname{EP}(\Theta) \neq \emptyset$. Let $V : H \to H$ be* *a $\bar{\gamma}$-strongly positive bounded linear operator with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of real numbers such that $0 < \lambda_n \leq b < 1$, $n = 1, 2, \ldots$. Let W_n be the W-mapping of C into itself generated by* (1.4). *Let W be defined by $Wx = \lim_{n\to\infty} W_n x$, $\forall x \in C$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by* (1.5), *where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\}$ is a sequence in $(0,\infty)$ such that the following conditions hold*:

(C1) $\lim_{n\to\infty} \alpha_n = 0$,

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and*

(C3) $\lim_{n\to\infty} r_n = r > 0$.

Then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in \Omega$, where $x^* = P_\Omega(I - (V - \gamma f))x^*$ is a unique solution of the VIP

$$\langle (V - \gamma f)x^*, x^* - x \rangle \leq 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{1}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Let $F_1, F_2 : C \to H$ be two mappings. Consider the general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.6}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.6) is denoted by $\mathrm{GSVI}(C, F_1, F_2)$.

In particular, if $F_1 = F_2 = A$, then the GSVI (1.6) reduces to the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 A y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \nu_2 A x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

which is defined by Verma [32] and it is called a new system of variational inequalities (NSVI). Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical VIP (1.1). In 2008, Ceng *et al.* [8] transformed the GSVI (1.6) into the fixed point problem of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$, that is, $Gx^* = x^*$, where $y^* = P_C(I - \nu_2 F_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by Ξ.

In 2012, Marino *et al.* [33] introduced a multi-step iterative scheme

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, & i = 2, \ldots, N, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T y_{n,N}, \end{cases} \tag{1.7}$$

with $f : C \to C$ a ρ-contraction and $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0,1)$, $\{r_n\} \subset (0,\infty)$, which generalizes the two-step iterative scheme in [34] for two nonexpansive mappings to a finite family of nonexpansive mappings $T, S_i : C \to C$, $i = 1, \dots, N$, and proved that the proposed scheme (1.7) converges strongly to a common fixed point of the mappings that is also an equilibrium point of the GMEP (1.2).

More recently, Marino *et al.*'s multi-step iterative scheme (1.7) was extended to develop the following relaxed viscosity iterative algorithm by virtue of Korpelevich's extragradient method.

Algorithm CKW (see (3.1) in [13]) Let $f : C \to C$ be a ρ-contraction and $T : C \to C$ be a ξ-strict pseudocontraction. Let $S_i : C \to C$ be a nonexpansive mapping for each $i = 1, \dots, N$. Let $F_j : C \to H$ be ζ_j-inverse strongly monotone with $0 < \nu_j < \zeta_j$ for each $j = 1, 2$. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions $(\theta 1)$-$(\theta 3)$ and $h : C \times C \to \mathbf{R}$ be a bifunction with restrictions (h1)-(h3). Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \dots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n) G y_{n,N}, \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \quad \forall n \geq 0, \end{cases} \tag{1.8}$$

where $G = P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2)$, $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0, 1)$ with

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$$

$\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $[0, 1]$ with $\liminf_{n\to\infty} \delta_n > 0$ and $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 0$, $\{\beta_{n,i}\}$ is a sequence in $(0, 1)$ for each $i = 1, \dots, N$, $(\gamma_n + \delta_n)\xi \leq \gamma_n$, $\forall n \geq 0$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$.

The authors [13] proved that the proposed scheme (1.8) converges strongly to a common fixed point of the mappings $T, S_i : C \to C$, $i = 1, \dots, N$, which is also an equilibrium point of the GMEP (1.2) and a solution of the GSVI (1.6).

Furthermore, let B be a single-valued mapping of C into H and R be a multivalued mapping with $D(R) = C$. Consider the following variational inclusion: find a point $x \in C$ such that

$$0 \in Bx + Rx. \tag{1.9}$$

We denote by $\mathrm{I}(B, R)$ the solution set of the variational inclusion (1.9). In particular, if $B = R = 0$, then $\mathrm{I}(B, R) = C$. If $B = 0$, then problem (1.9) becomes the inclusion problem introduced by Rockafellar [35]. It is known that problem (1.9) provides a convenient framework for the unified study of optimal solutions in many optimization related areas including mathematical programming, complementarity problems, variational inequalities, optimal control, mathematical economics, equilibria, game theory, *etc.* Let a set-valued mapping $R : D(R) \subset H \to 2^H$ be maximal monotone. We define the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$ associated with R and λ as follows:

$$J_{R,\lambda} = (I + \lambda R)^{-1}, \quad \forall x \in H,$$

where λ is a positive number.

In 1998, Huang [36] studied problem (1.9) in the case where R is maximal monotone and B is strongly monotone and Lipschitz-continuous with $D(R) = C = H$. Subsequently, Zeng *et al.* [37] further studied problem (1.9) in the case which is more general than Huang's one [36]. Moreover, the authors [37] obtained the same strong convergence conclusion as in Huang's result [36]. In addition, the authors also gave the geometric convergence rate estimate for approximate solutions. Also, various types of iterative algorithms for solving variational inclusions have been further studied and developed; for more details, refer to [11, 15, 38–40] and the references therein.

Very recently, Ceng *et al.* [41] introduced and analyzed one multi-step hybrid steepest-descent extragradient algorithm and another multi-step composite Mann-type viscosity iterative algorithm for finding a solution of triple hierarchical variational inequalities defined over the common set of solutions of finitely many generalized mixed equilibrium problems, finitely many variational inclusions, a general system of variational inequalities, and a fixed point problem of a strict pseudocontraction in a real Hilbert space H. Here, the generalized mixed equilibrium problem is defined as follows: Let $\varphi : C \to \mathbf{R}$ be a real-valued function, $A : C \to H$ be a nonlinear mapping and $\Theta : C \times C \to \mathbf{R}$ be a bifunction. Then the objective is to find $x \in C$ such that

$$\Theta(x,y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall y \in C.$$

The solution set of such generalized mixed equilibrium problem is denoted by $\mathrm{GMEP}(\Theta, \varphi, A)$. Under appropriate assumptions, the authors proved that the proposed algorithms converge strongly to an element of the common set, which is a unique solution of a triple hierarchical variational inequality problem; see [41], Theorems 3.1 and 4.1.

In this paper, we introduce a relaxed extragradient iterative algorithm for finding a common element of the solution set $\mathrm{GMEP}(\Theta, h)$ of GMEP (1.2), the solution set $\mathrm{GSVI}(C, F_1, F_2)$ (*i.e.*, Ξ) of GSVI (1.6), the solution set $\bigcap_{k=1}^{M} \mathrm{I}(B_k, R_k)$ of finitely many variational inclusions for maximal monotone mappings $\{R_k\}_{k=1}^{M}$ and inverse-strongly monotone mappings $\{B_k\}_{k=1}^{M}$, and the common fixed point set $\bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{Fix}(T)$ of finitely many nonexpansive mappings $S_i : C \to C$, $i = 1, \ldots, N$, and a strictly pseudocontractive mapping $T : H \to H$, in the setting of the infinite-dimensional Hilbert space. The iterative algorithm is based on Korpelevich's extragradient method, viscosity approximation method [42] (see also [43]), Mann's iterative method, and strongly positive bounded linear operator approach. Our aim is to prove that the iterative algorithm converges strongly to a common element of these sets, which also solves some hierarchical minimization. We observe that related results have been derived say in [13, 26, 28, 29, 33, 34, 44–51]. In addition, we also point out what are different between the present article and the previous one [41] as follows:

(i) The problem of finding an element of $\mathrm{Fix}(T) \cap \bigcap_{i=1}^{N} \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{k=1}^{M} \mathrm{I}(B_k, R_k) \cap \Xi$ in Theorems 3.1 and 3.2 of this paper is very different from the one of finding an element of $\bigcap_{k=1}^{M} \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^{N} \mathrm{I}(B_i, R_i) \cap \Xi \cap \mathrm{Fix}(T)$ in [41], Theorems 3.1 and 4.1, where S_i is a nonexpansive mapping for each $i \in \{1, \ldots, N\}$ and T is a strict pseudocontraction. It is clear that the general mixed equilibrium problem (1.2) is very different from the above generalized mixed equilibrium problem.

(ii) The iterative scheme (3.1) in this paper is very different from the iterative schemes (3.1) and (4.1) in the authors [41] because the scheme (3.1) involves finding a common fixed point of finitely many nonexpansive mappings $\{S_i\}_{i=1}^{N}$ and a strict pseudocontraction T. In the meantime, Theorems 3.1 and 3.2 of this paper show that the proposed algorithm converges strongly to a unique solution of a VIP defined over the set $\mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \cap \Xi$. However, Theorems 3.1 and 4.1 in [41] show that the proposed algorithms converge strongly to a unique solution of a triple hierarchical variational inequality problem defined over the set $\bigcap_{k=1}^M \mathrm{GMEP}(\Theta_k, \varphi_k, A_k) \cap \bigcap_{i=1}^N \mathrm{I}(B_i, R_i) \cap \Xi \cap \mathrm{Fix}(T)$.

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$ and $\omega_s(x_n)$ to denote the strong ω-limit set of the sequence $\{x_n\}$, *i.e.*,

$$\omega_w(x_n) := \bigl\{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\bigr\},$$

and

$$\omega_s(x_n) := \bigl\{x \in H : x_{n_i} \to x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\bigr\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 *Given any $x \in H$ and $z \in C$. One has*

- (i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \le 0$, $\forall y \in C$;
- (ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \le \|x - y\|^2 - \|y - z\|^2$, $\forall y \in C$;
- (iii) $\langle P_C x - P_C y, x - y \rangle \ge \|P_C x - P_C y\|^2$, $\forall y \in H$, *which hence implies that P_C is nonexpansive and monotone.*

Definition 2.1 A mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H;$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse-strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \ge \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2 A mapping $A : C \to H$ is said to be

(i) monotone if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C;$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iii) ζ-inverse-strongly monotone if there exists a constant $\zeta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \zeta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

It can easily be seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse-strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if $A : C \to H$ is ζ-inverse-strongly monotone, then A is monotone and $\frac{1}{\zeta}$-Lipschitz-continuous. Moreover, we also note that, for all $u, v \in C$ and $\lambda > 0$,

$$\left\|(I - \lambda A)u - (I - \lambda A)v\right\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\zeta)\|Au - Av\|^2. \tag{2.1}$$

So, if $\lambda \leq 2\zeta$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

In 2008, Ceng *et al.* [8] transformed problem (1.6) into a fixed point problem in the following way.

Proposition 2.2 (see [8]) *For given* $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ *is a solution of the GSVI* (1.6) *if and only if* $\bar{x}$ *is a fixed point of the mapping* $G : C \to C$ *defined by*

$$Gx = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)x, \quad \forall x \in C,$$

where $\bar{y} = P_C(I - \nu_2 F_2)\bar{x}$.

In particular, if the mapping $F_j : C \to H$ is ζ_j-inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $\nu_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by Ξ denote the fixed point set of the mapping G.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then we have the following inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold*:

(a) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H$;

(b) $\|\lambda x + \mu y\|^2 = \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu\|x - y\|^2$ *for all* $x, y \in H$ *and* $\lambda, \mu \in [0, 1]$ *with* $\lambda + \mu = 1$;

(c) *if* $\{x_n\}$ *is a sequence in* H *such that* $x_n \rightharpoonup x$, *it follows that*

$$\limsup_{n\to\infty} \|x_n - y\|^2 = \limsup_{n\to\infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

It is clear that, in a real Hilbert space H, $T: C \to C$ is ξ-strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\xi}{2}\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a ξ-strictly pseudocontractive mapping, then $I - T$ is $\frac{1-\xi}{2}$-inverse strongly monotone; for further details, we refer to [52] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the class of nonexpansive mappings and that the class of pseudocontractions strictly includes the class of strict pseudocontractions.

Lemma 2.3 (see [52], Proposition 2.1) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T: C \to C$ be a mapping.*

(i) *If T is a ξ-strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leq \frac{1+\xi}{1-\xi}\|x - y\|, \quad \forall x, y \in C.$$

(ii) *If T is a ξ-strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at* 0, *that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \to 0$, then* $(I - T)\tilde{x} = 0$.

(iii) *If T is ξ-(quasi-)strict pseudocontraction, then the fixed point set* $\mathrm{Fix}(T)$ *of T is closed and convex so that the projection $P_{\mathrm{Fix}(T)}$ is well defined.*

Lemma 2.4 (see [14]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be a ξ-strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)\xi \leq \gamma$. Then*

$$\big\|\gamma(x - y) + \delta(Tx - Ty)\big\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.5 (see [53], demiclosedness principle) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive self-mapping on C. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y, it follows that $(I - S)x = y$. Here I is the identity operator of H.*

Lemma 2.6 *Let $A: C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection* (*see Proposition* 2.1(i)) *implies*

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(u - \lambda Au), \quad \lambda > 0.$$

Lemma 2.7 (see [54]) *Let V be a $\bar{\gamma}$-strongly positive bounded linear operator on H and assume $0 < \rho \leq \|V\|^{-1}$. Then $\|I - \rho V\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.8 (see [55]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - s_n)a_n + s_n b_n + t_n, \quad \forall n \geq 1,$$

where $\{s_n\}$, $\{t_n\}$, and $\{b_n\}$ satisfy the following conditions:
(i) $\{s_n\} \subset [0,1]$ *and* $\sum_{n=1}^{\infty} s_n = \infty$;
(ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^{\infty} |s_n b_n| < \infty$;
(iii) $t_n \geq 0$ *for all* $n \geq 1$, *and* $\sum_{n=1}^{\infty} t_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

In the sequel, we will indicate with $\mathrm{GMEP}(\Theta, h)$ the solution set of GMEP (1.2).

Lemma 2.9 (see [31]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\Theta : C \times C \to \mathbf{R}$ be a bifunction satisfying conditions* (θ1)-(θ3) *and $h : C \times C \to \mathbf{R}$ is a bifunction with restrictions* (h1)-(h3). *Moreover, let us suppose that*
(H) *for fixed $r > 0$ and $x \in C$, there exist a bounded $K \subset C$ and $\hat{x} \in K$ such that for all $z \in C \setminus K$, $-\Theta(\hat{x}, z) + h(z, \hat{x}) + \frac{1}{r}\langle \hat{x} - z, z - x\rangle < 0$.*
For $r > 0$ and $x \in H$, the mapping $T_r : H \to 2^C$ (i.e., the resolvent of Θ and h) has the following properties:
(i) $T_r x \neq \emptyset$;
(ii) *$T_r x$ is a singleton*;
(iii) *T_r is firmly nonexpansive*;
(iv) $\mathrm{GMEP}(\Theta, h) = \mathrm{Fix}(T_r)$ *and it is closed and convex.*

Lemma 2.10 (see [31]) *Let us suppose that* (θ1)-(θ3), (h1)-(h3), *and* (H) *hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then*

$$\|T_{r_2} y - T_{r_1} x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2} y - y\|.$$

Lemma 2.11 (see [33]) *Suppose that the hypotheses of Lemma* 2.9 *are satisfied. Let $\{r_n\}$ be a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$. Suppose that $\{x_n\}$ is a bounded sequence. Then the following statements are equivalent and true*:
(a) *if $\|x_n - T_{r_n} x_n\| \to 0$ as $n \to \infty$, each weak cluster point of $\{x_n\}$ satisfies the problem*

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C,$$

i.e., $\omega_w(x_n) \subseteq \mathrm{GMEP}(\Theta, h)$;
(b) *the demiclosedness principle holds in the sense that, if $x_n \rightharpoonup x^*$ and $\|x_n - T_{r_n} x_n\| \to 0$ as $n \to \infty$, then $(I - T_{r_k})x^* = 0$ for all $k \geq 1$.*

Recall that a set-valued mapping $T : D(T) \subset H \to 2^H$ is called monotone if for all $x, y \in D(T)$, $f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y\rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H. We denote by $G(T)$ the graph of T.

It is well known that a monotone mapping T is maximal if and only if, for $(x,f) \in H \times H$, $\langle f - g, x - y\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Next we provide an example to illustrate the concept of a maximal monotone mapping.

Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $N_C v$ be the normal cone to C at $v \in C$, *i.e.*,

$$N_C v = \{u \in H : \langle v - p, u\rangle \geq 0, \forall p \in C\}.$$

Define

$$\widetilde{T}v = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then it is well known [4] that $\widetilde{T}$ is maximal monotone and $0 \in \widetilde{T}v$ if and only if $v \in \mathrm{VI}(C, A)$.

Let $R : D(R) \subset H \to 2^H$ be a maximal monotone mapping. Let $\lambda, \mu > 0$ be two positive numbers.

Lemma 2.12 (see [56]) *We have the resolvent identity*

$$J_{R,\lambda}x = J_{R,\mu}\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_{R,\lambda}x\right), \quad \forall x \in H.$$

Remark 2.1 For $\lambda, \mu > 0$, we have the following relation:

$$\|J_{R,\lambda}x - J_{R,\mu}y\| \leq \|x - y\| + |\lambda - \mu|\left(\frac{1}{\lambda}\|J_{R,\lambda}x - y\| + \frac{1}{\mu}\|x - J_{R,\mu}y\|\right), \quad \forall x, y \in H. \tag{2.2}$$

In terms of Huang [36] (see also [37]), we have the following property for the resolvent operator $J_{R,\lambda} : H \to \overline{D(R)}$.

Lemma 2.13 *$J_{R,\lambda}$ is single-valued and firmly nonexpansive, i.e.,*

$$\langle J_{R,\lambda}x - J_{R,\lambda}y, x - y\rangle \geq \|J_{R,\lambda}x - J_{R,\lambda}y\|^2, \quad \forall x, y \in H.$$

Consequently, $J_{R,\lambda}$ is nonexpansive and monotone.

Lemma 2.14 (see [15]) *Let R be a maximal monotone mapping with $D(R) = C$. Then for any given $\lambda > 0$, $u \in C$ is a solution of problem* (1.9) *if and only if $u \in C$ satisfies*

$$u = J_{R,\lambda}(u - \lambda Bu).$$

Lemma 2.15 (see [37]) *Let R be a maximal monotone mapping with $D(R) = C$ and let $B : C \to H$ be a strongly monotone, continuous, and single-valued mapping. Then for each $z \in H$, the equation $z \in (B + \lambda R)x$ has a unique solution x_λ for $\lambda > 0$.*

Lemma 2.16 (see [15]) *Let R be a maximal monotone mapping with $D(R) = C$ and $B : C \to H$ be a monotone, continuous and single-valued mapping. Then $(I + \lambda(R + B))C = H$ for each $\lambda > 0$. In this case, $R + B$ is maximal monotone.*

3 Main results

We now propose the following relaxed extragradient iterative scheme:

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ v_n = J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}B_M)J_{R_{M-1},\lambda_{M-1,n}}(I - \lambda_{M-1,n}B_{M-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ y_{n,1} = \beta_{n,1}S_1v_n + (1 - \beta_{n,1})v_n, \\ y_{n,i} = \beta_{n,i}S_iv_n + (1 - \beta_{n,i})y_{n,i-1}, \quad i = 2, \ldots, N, \\ y_n = \alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu V)Gy_{n,N}; \\ x_{n+1} = \beta_nx_n + \gamma_ny_n + \delta_nTy_n, \end{cases} \tag{3.1}$$

for all $n \geq 0$, where

V is a $\bar{\gamma}$-strongly positive bounded linear operator on H and $f : C \to C$ is an l-Lipschitz-continuous mapping with $0 \leq \gamma l < \mu\bar{\gamma}$;
$T : H \to H$ is a ξ-strict pseudocontraction and $S_i : C \to C$ is a nonexpansive mapping for each $i = 1, \ldots, N$;
$R_k : C \to 2^H$ is a maximal monotone mapping and $B_k : C \to H$ is η_k-inverse-strongly monotone with $\{\lambda_{k,n}\} \subset [a_k, b_k] \subset (0, 2\eta_k)$ for each $k = 1, 2, \ldots, M$;
$F_j : C \to H$ is ζ_j-inverse-strongly monotone and $G := P_C(I - \nu_1F_1)P_C(I - \nu_2F_2)$ with $\nu_j \in (0, 2\zeta_j)$ for $j = 1, 2$;
$\Theta, h : C \times C \to \mathbf{R}$ are two bifunctions satisfying the hypotheses of Lemma 2.9;
$\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $(0,1)$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
$\{\gamma_n\}$, $\{\delta_n\}$ are sequences in $[0,1]$ with $\beta_n + \gamma_n + \delta_n = 1$, $\forall n \geq 0$;
$\{\beta_{n,i}\}_{i=1}^N$ are sequences in $(0,1)$ and $(\gamma_n + \delta_n)\xi \leq \gamma_n$, $\forall n \geq 0$;
$\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} \delta_n > 0$.

We start our main result from the following series of propositions.

Proposition 3.1 *Let us suppose that* $\Omega = \mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \cap \mathrm{GMEP}(\Theta, h) \cap \Xi \neq \emptyset$. *Then the sequences* $\{x_n\}$, $\{y_n\}$, $\{y_{n,i}\}$ *for all* i, $\{u_n\}$, $\{v_n\}$ *are bounded.*

Proof Since $\lim_{n\to\infty} \alpha_n = 0$ and $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\beta_n\} \subset [c, d] \subset (0,1)$ and $0 < \alpha_n\mu \leq \|V\|^{-1}$ for all $n \geq 0$. Since V is a $\bar{\gamma}$-strongly positive bounded linear operator on H, by Lemma 2.7 we know that

$$\|I - \alpha_n\mu V\| \leq 1 - \alpha_n\mu\bar{\gamma}, \quad \forall n \geq 0.$$

Put

$$\Lambda_n^k = J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)J_{R_{k-1},\lambda_{k-1,n}}(I - \lambda_{k-1,n}B_{k-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 0$, and $\Lambda_n^0 = I$, where I is the identity mapping on H. Then we have that $v_n = \Lambda_n^M u_n$.

First of all, take a fixed $p \in \Omega$ arbitrarily. Utilizing (2.1) and Lemma 2.13 we have

$$\begin{aligned} \|v_n - p\| &= \left\|J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}u_n - J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}p\right\| \\ &\leq \left\|(I - \lambda_{M,n}A_M)\Lambda_n^{M-1}u_n - (I - \lambda_{M,n}A_M)\Lambda_n^{M-1}p\right\| \end{aligned}$$

$$\begin{aligned}
&\leq \left\| \Lambda_n^{M-1} u_n - \Lambda_n^{M-1} p \right\| \\
&\cdots \\
&\leq \left\| \Lambda_n^{0} u_n - \Lambda_n^{0} p \right\| \\
&= \|u_n - p\| \leq \|x_n - p\|.
\end{aligned} \tag{3.2}$$

Let us observe that, if $p \in \Omega$, then

$$\|y_{n,1} - p\| \leq \|v_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

For all from $i = 2$ to $i = N$, by induction, one proves that

$$\|y_{n,i} - p\| \leq \beta_{n,i}\|v_n - p\| + (1 - \beta_{n,i})\|y_{n,i-1} - p\| \leq \|v_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

Thus we obtain, for every $i = 1, \ldots, N$,

$$\|y_{n,i} - p\| \leq \|v_n - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.3}$$

For simplicity, we write $\tilde{p} = P_C(p - \nu_2 F_2 p)$, $\tilde{y}_{n,N} = P_C(y_{n,N} - \nu_2 F_2 y_{n,N})$, and $z_n = P_C(\tilde{y}_{n,N} - \nu_1 F_1 \tilde{y}_{n,N})$ for each $n \geq 1$. Then $z_n = G y_{n,N}$ and

$$p = P_C(I - \nu_1 F_1)\tilde{p} = P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) p = Gp.$$

Since $F_j : C \to H$ is ζ_j-inverse-strongly monotone and $0 < \nu_j < 2\zeta_j$ for each $j = 1, 2$, we know that, for all $n \geq 0$,

$$\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \|G y_{n,N} - p\|^2 \\
&\quad = \left\| P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_1 F_1) P_C(I - \nu_2 F_2) p \right\|^2 \\
&\quad \leq \left\| (I - \nu_1 F_1) P_C(I - \nu_2 F_2) y_{n,N} - (I - \nu_1 F_1) P_C(I - \nu_2 F_2) p \right\|^2 \\
&\quad = \left\| \left[P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right] \right. \\
&\qquad \left. - \nu_1 \left[F_1 P_C(I - \nu_2 F_2) y_{n,N} - F_1 P_C(I - \nu_2 F_2) p \right] \right\|^2 \\
&\quad \leq \left\| P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right\|^2 \\
&\qquad + \nu_1(\nu_1 - 2\zeta_1) \left\| F_1 P_C(I - \nu_2 F_2) y_{n,N} - F_1 P_C(I - \nu_2 F_2) p \right\|^2 \\
&\quad \leq \left\| P_C(I - \nu_2 F_2) y_{n,N} - P_C(I - \nu_2 F_2) p \right\|^2 \\
&\quad \leq \left\| (I - \nu_2 F_2) y_{n,N} - (I - \nu_2 F_2) p \right\|^2 \\
&\quad = \left\| (y_{n,N} - p) - \nu_2 (F_2 y_{n,N} - F_2 p) \right\|^2 \\
&\quad \leq \|y_{n,N} - p\|^2 + \nu_2(\nu_2 - 2\zeta_2) \|F_2 y_{n,N} - F_2 p\|^2 \\
&\quad \leq \|y_{n,N} - p\|^2 \leq \|v_n - p\|^2 \leq \|u_n - p\|^2 \leq \|x_n - p\|^2.
\end{aligned} \tag{3.4}$$

Utilizing $Gp = p$ and the nonexpansivity of G, from (3.1) and (3.4) we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \big\|\alpha_n\gamma\big(f(y_{n,N}) - f(p)\big) + (I - \alpha_n\mu V)(Gy_{n,N} - p) + \alpha_n(\gamma f - \mu V)p\big\| \\
&\quad \le \alpha_n\gamma\big\|f(y_{n,N}) - f(p)\big\| + \|I - \alpha_n\mu V\|\|Gy_{n,N} - p\| + \alpha_n\big\|(\gamma f - \mu V)p\big\| \\
&\quad \le \alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\mu\bar{\gamma})\|y_{n,N} - p\| + \alpha_n\big\|(\gamma f - \mu V)p\big\| \\
&\quad = \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\|y_{n,N} - p\| + \alpha_n\big\|(\gamma f - \mu V)p\big\| \\
&\quad = \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\|y_{n,N} - p\| + \alpha_n(\mu\bar{\gamma} - \gamma l)\frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l} \\
&\quad \le \max\left\{\|y_{n,N} - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l}\right\} \\
&\quad \le \max\left\{\|x_n - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l}\right\}.
\end{aligned} \tag{3.5}$$

Since $(\gamma_n + \delta_n)\xi \le \gamma_n$ for all $n \ge 0$, utilizing Lemma 2.4 we obtain from (3.1) and (3.5)

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\beta_n(x_n - p) + \gamma_n(y_n - p) + \delta_n(Ty_n - p)\big\| \\
&\le \beta_n\|x_n - p\| + \big\|\gamma_n(y_n - p) + \delta_n(Ty_n - p)\big\| \\
&\le \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\|y_n - p\| \\
&\le \beta_n\|x_n - p\| + (\gamma_n + \delta_n)\max\left\{\|x_n - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l}\right\} \\
&\le \max\left\{\|x_n - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l}\right\}.
\end{aligned} \tag{3.6}$$

By induction, we get

$$\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|(\gamma f - \mu V)p\|}{\mu\bar{\gamma} - \gamma l}\right\}, \quad \forall n \ge 0.$$

This implies that $\{x_n\}$ is bounded and so are $\{F_2 y_{n,N}\}$, $\{F_1\tilde{y}_{n,N}\}$, $\{\tilde{y}_{n,N}\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$, $\{y_{n,i}\}$ for each $i = 1, \dots, N$. Since $\|Ty_n - p\| \le \frac{1+\xi}{1-\xi}\|y_n - p\|$, $\{Ty_n\}$ is also bounded. □

Proposition 3.2 *Let us suppose that $\Omega \ne \emptyset$. Moreover, let us suppose that the following hold*:

(H0) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(H1) $\sum_{n=1}^{\infty}|\lambda_{k,n} - \lambda_{k,n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\lambda_{k,n}-\lambda_{k,n-1}|}{\alpha_n} = 0$ *for each* $k = 1, \dots, M$;
(H2) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} = 0$;
(H3) $\sum_{n=1}^{\infty}|\beta_{n,i} - \beta_{n-1,i}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\beta_{n,i}-\beta_{n-1,i}|}{\alpha_n} = 0$ *for each* $i = 1, \dots, N$;
(H4) $\sum_{n=1}^{\infty}|r_n - r_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|r_n-r_{n-1}|}{\alpha_n} = 0$;
(H5) $\sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}| < \infty$ *or* $\lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n} = 0$;
(H6) $\sum_{n=1}^{\infty}|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| < \infty$ *or* $\lim_{n\to\infty}\frac{1}{\alpha_n}|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}| = 0$.

Then $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, *i.e.*, $\{x_n\}$ *is asymptotically regular.*

Proof First, it is well known that $\{\beta_n\} \subset [c, d] \subset (0, 1)$ as in the proof of Proposition 3.1. Taking into account $\liminf_{n\to\infty} r_n > 0$, we may assume, without loss of generality, that

$\{r_n\} \subset [\epsilon, \infty)$ for some $\epsilon > 0$. First, we write $x_n = \beta_{n-1}x_{n-1} + (1-\beta_{n-1})w_{n-1}$, $\forall n \geq 1$, where $w_{n-1} = \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}}$. It follows that for all $n \geq 1$

$$\begin{aligned} w_n - w_{n-1} &= \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} - \frac{x_n - \beta_{n-1}x_{n-1}}{1-\beta_{n-1}} \\ &= \frac{\gamma_n y_n + \delta_n T y_n}{1-\beta_n} - \frac{\gamma_{n-1}y_{n-1} + \delta_{n-1}Ty_{n-1}}{1-\beta_{n-1}} \\ &= \frac{\gamma_n(y_n - y_{n-1}) + \delta_n(Ty_n - Ty_{n-1})}{1-\beta_n} + \left(\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}\right)y_{n-1} \\ &\quad + \left(\frac{\delta_n}{1-\beta_n} - \frac{\delta_{n-1}}{1-\beta_{n-1}}\right)Ty_{n-1}. \end{aligned} \tag{3.7}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.4 we have

$$\left\| \gamma_n(y_n - y_{n-1}) + \delta_n(Ty_n - Ty_{n-1}) \right\| \leq (\gamma_n + \delta_n)\|y_n - y_{n-1}\|. \tag{3.8}$$

Next, we estimate $\|y_n - y_{n-1}\|$. Observe that

$$\begin{aligned} &\|z_n - z_{n-1}\|^2 \\ &\quad = \left\| P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)y_{n,N} - P_C(I-\nu_1F_1)P_C(I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\quad \leq \left\| (I-\nu_1F_1)P_C(I-\nu_2F_2)y_{n,N} - (I-\nu_1F_1)P_C(I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\quad = \big\| \big[P_C(I-\nu_2F_2)y_{n,N} - P_C(I-\nu_2F_2)y_{n-1,N}\big] \\ &\qquad - \nu_1\big[F_1P_C(I-\nu_2F_2)y_{n,N} - F_1P_C(I-\nu_2F_2)y_{n-1,N}\big]\big\|^2 \\ &\quad \leq \left\| P_C(I-\nu_2F_2)y_{n,N} - P_C(I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\qquad - \nu_1(2\zeta_1 - \nu_1)\left\| F_1P_C(I-\nu_2F_2)y_{n,N} - F_1P_C(I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\quad \leq \left\| P_C(I-\nu_2F_2)y_{n,N} - P_C(I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\quad \leq \left\| (I-\nu_2F_2)y_{n,N} - (I-\nu_2F_2)y_{n-1,N} \right\|^2 \\ &\quad = \left\| (y_{n,N} - y_{n-1,N}) - \nu_2(F_2y_{n,N} - F_2y_{n-1,N}) \right\|^2 \\ &\quad \leq \|y_{n,N} - y_{n-1,N}\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2y_{n-1,N}\|^2 \\ &\quad \leq \|y_{n,N} - y_{n-1,N}\|^2. \end{aligned} \tag{3.9}$$

Also, from (3.1) we have

$$\begin{cases} y_n = \alpha_n\gamma f(y_{n,N}) + (I - \alpha_n\mu V)z_n, \\ y_{n-1} = \alpha_{n-1}\gamma f(y_{n-1,N}) + (I - \alpha_{n-1}\mu V)z_{n-1}, \quad \forall n \geq 1. \end{cases}$$

Simple calculations show that

$$\begin{aligned} y_n - y_{n-1} &= (I - \alpha_n\mu V)(z_n - z_{n-1}) + (\alpha_n - \alpha_{n-1})\big(\gamma f(y_{n-1,N}) - \mu V z_{n-1}\big) \\ &\quad + \alpha_n\gamma\big(f(y_{n,N}) - f(y_{n-1,N})\big). \end{aligned}$$

Then passing to the norm we get from (3.9)

$$\begin{aligned}\|y_n - y_{n-1}\| &\le \|I - \alpha_n \mu V\|\|z_n - z_{n-1}\| + |\alpha_n - \alpha_{n-1}|\big\|\gamma f(y_{n-1,N}) - \mu V z_{n-1}\big\| \\ &\quad + \alpha_n\gamma\big\|f(y_{n,N}) - f(y_{n-1,N})\big\| \\ &\le (1 - \alpha_n\mu\bar{\gamma})\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}| + \alpha_n\gamma l\|y_{n,N} - y_{n-1,N}\| \\ &= \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}|, \end{aligned} \tag{3.10}$$

where $\sup_{n\ge 0}\|\gamma f(y_{n,N}) - \mu V z_n\| \le \widetilde{M}$ for some $\widetilde{M} > 0$. In the meantime, by the definition of $y_{n,i}$ one obtains, for all $i = N, \dots, 2$,

$$\begin{aligned}\|y_{n,i} - y_{n-1,i}\| &\le \beta_{n,i}\|v_n - v_{n-1}\| + \|S_i v_{n-1} - y_{n-1,i-1}\||\beta_{n,i} - \beta_{n-1,i}| \\ &\quad + (1 - \beta_{n,i})\|y_{n,i-1} - y_{n-1,i-1}\|. \end{aligned} \tag{3.11}$$

In the case $i = 1$, we have

$$\begin{aligned}\|y_{n,1} - y_{n-1,1}\| &\le \beta_{n,1}\|v_n - v_{n-1}\| + \|S_1 v_{n-1} - v_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| + (1 - \beta_{n,1})\|v_n - v_{n-1}\| \\ &= \|v_n - v_{n-1}\| + \|S_1 v_{n-1} - v_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \end{aligned} \tag{3.12}$$

Substituting (3.12) in all (3.11)-type expressions one obtains for $i = 2, \dots, N$

$$\begin{aligned}\|y_{n,i} - y_{n-1,i}\| &\le \|v_n - v_{n-1}\| + \sum_{k=2}^{i}\|S_k v_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\quad + \|S_1 v_{n-1} - v_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|. \end{aligned}$$

This together with (3.10) implies that

$$\begin{aligned}&\|y_n - y_{n-1}\| \\ &\quad \le \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\|y_{n,N} - y_{n-1,N}\| + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\ &\quad \le \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\Bigg[\|v_n - v_{n-1}\| + \sum_{k=2}^{N}\|S_k v_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\qquad + \|S_1 v_{n-1} - v_{n-1}\||\beta_{n,1} - \beta_{n-1,1}|\Bigg] + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\ &\quad \le \big(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\big)\|v_n - v_{n-1}\| + \sum_{k=2}^{N}\|S_k v_{n-1} - y_{n-1,k-1}\||\beta_{n,k} - \beta_{n-1,k}| \\ &\qquad + \|S_1 v_{n-1} - v_{n-1}\||\beta_{n,1} - \beta_{n-1,1}| + \widetilde{M}|\alpha_n - \alpha_{n-1}|. \end{aligned} \tag{3.13}$$

Furthermore, utilizing (2.1) and (2.2), we obtain

$$\begin{aligned}&\|v_n - v_{n-1}\| \\ &\quad = \big\|\Lambda_n^M u_n - \Lambda_{n-1}^M u_{n-1}\big\| \end{aligned}$$

$$
\begin{aligned}
&= \big\|J_{R_M,\lambda_{M,n}}(I-\lambda_{M,n}B_M)\Lambda_n^{M-1}u_n - J_{R_M,\lambda_{M,n-1}}(I-\lambda_{M,n-1}B_M)\Lambda_{n-1}^{M-1}u_{n-1}\big\| \\
&\le \big\|J_{R_M,\lambda_{M,n}}(I-\lambda_{M,n}B_M)\Lambda_n^{M-1}u_n - J_{R_M,\lambda_{M,n}}(I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n\big\| \\
&\quad + \big\|J_{R_M,\lambda_{M,n}}(I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n - J_{R_M,\lambda_{M,n-1}}(I-\lambda_{M,n-1}B_M)\Lambda_{n-1}^{M-1}u_{n-1}\big\| \\
&\le \big\|(I-\lambda_{M,n}B_M)\Lambda_n^{M-1}u_n - (I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n\big\| \\
&\quad + \big\|(I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n - (I-\lambda_{M,n-1}B_M)\Lambda_{n-1}^{M-1}u_{n-1}\big\| + |\lambda_{M,n}-\lambda_{M,n-1}| \\
&\quad \times \bigg(\frac{1}{\lambda_{M,n}}\big\|J_{R_M,\lambda_{M,n}}(I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n - (I-\lambda_{M,n-1}B_M)\Lambda_{n-1}^{M-1}u_{n-1}\big\| \\
&\quad + \frac{1}{\lambda_{M,n-1}}\big\|(I-\lambda_{M,n-1}B_M)\Lambda_n^{M-1}u_n - J_{R_M,\lambda_{M,n-1}}(I-\lambda_{M,n-1}B_M)\Lambda_{n-1}^{M-1}u_{n-1}\big\|\bigg) \\
&\le |\lambda_{M,n}-\lambda_{M,n-1}|\big(\big\|B_M\Lambda_n^{M-1}u_n\big\| + \widehat{M}\big) + \big\|\Lambda_n^{M-1}u_n - \Lambda_{n-1}^{M-1}u_{n-1}\big\| \\
&\le |\lambda_{M,n}-\lambda_{M,n-1}|\big(\big\|B_M\Lambda_n^{M-1}u_n\big\| + \widehat{M}\big) \\
&\quad + |\lambda_{M-1,n}-\lambda_{M-1,n-1}|\big(\big\|B_{M-1}\Lambda_n^{M-2}u_n\big\| + \widehat{M}\big) + \big\|\Lambda_n^{M-2}u_n - \Lambda_{n-1}^{M-2}u_{n-1}\big\| \\
&\quad \cdots \\
&\le |\lambda_{M,n}-\lambda_{M,n-1}|\big(\big\|B_M\Lambda_n^{M-1}u_n\big\| + \widehat{M}\big) \\
&\quad + |\lambda_{M-1,n}-\lambda_{M-1,n-1}|\big(\big\|B_{M-1}\Lambda_n^{M-2}u_n\big\| + \widehat{M}\big) + \cdots \\
&\quad + |\lambda_{1,n}-\lambda_{1,n-1}|\big(\big\|B_1\Lambda_n^{0}u_n\big\| + \widehat{M}\big) + \big\|\Lambda_n^{0}u_n - \Lambda_{n-1}^{0}u_{n-1}\big\| \\
&\le \widetilde{M}_0\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}| + \|u_n-u_{n-1}\|,
\end{aligned}
\tag{3.14}
$$

where

$$
\begin{aligned}
\sup_{n\ge1,1\le k\le M}\bigg\{&\frac{1}{\lambda_{k,n}}\big\|J_{R_k,\lambda_{k,n}}(I-\lambda_{k,n-1}B_k)\Lambda_n^{k-1}u_n - (I-\lambda_{k,n-1}B_k)\Lambda_{n-1}^{k-1}u_{n-1}\big\| \\
&+ \frac{1}{\lambda_{k,n-1}}\big\|(I-\lambda_{k,n-1}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n-1}}(I-\lambda_{k,n-1}B_k)\Lambda_{n-1}^{k-1}u_{n-1}\big\|\bigg\} \le \widehat{M},
\end{aligned}
$$

for some $\widehat{M} > 0$ and $\sup_{n\ge0}\{\sum_{k=1}^{M}\|B_k\Lambda_n^{k-1}u_n\| + \widehat{M}\} \le \widetilde{M}_0$ for some $\widetilde{M}_0 > 0$.

By Lemma 2.10, we know that

$$
\|u_n-u_{n-1}\| \le \|x_n-x_{n-1}\| + L\left|1-\frac{r_{n-1}}{r_n}\right|,
\tag{3.15}
$$

where $L = \sup_{n\ge0}\|u_n-x_n\|$. So, combining (3.13)-(3.15), we obtain

$$
\begin{aligned}
&\|y_n-y_{n-1}\| \\
&\quad \le \big(1-\alpha_n(\mu\bar{\gamma}-\gamma l)\big)\|v_n-v_{n-1}\| + \sum_{k=2}^{N}\|S_kv_{n-1}-y_{n-1,k-1}\||\beta_{n,k}-\beta_{n-1,k}| \\
&\quad\quad + \|S_1v_{n-1}-v_{n-1}\||\beta_{n,1}-\beta_{n-1,1}| + \widetilde{M}|\alpha_n-\alpha_{n-1}| \\
&\quad \le \big(1-\alpha_n(\mu\bar{\gamma}-\gamma l)\big)\left[\widetilde{M}_0\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}| + \|u_n-u_{n-1}\|\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \sum_{k=2}^{N} \|S_k v_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \\
&\quad + \|S_1 v_{n-1} - v_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}| + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\
&\le \bigl(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\bigr)\left[\widetilde{M}_0 \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| + \|x_n - x_{n-1}\| + L\left|1 - \frac{r_{n-1}}{r_n}\right|\right] \\
&\quad + \sum_{k=2}^{N} \|S_k v_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \\
&\quad + \|S_1 v_{n-1} - v_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}| + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\
&\le \bigl(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\bigr)\|x_n - x_{n-1}\| + \widetilde{M}_0 \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| + L\left|1 - \frac{r_{n-1}}{r_n}\right| \\
&\quad + \sum_{k=2}^{N} \|S_k v_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \\
&\quad + \|S_1 v_{n-1} - v_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}| + \widetilde{M}|\alpha_n - \alpha_{n-1}| \\
&\le \bigl(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\bigr)\|x_n - x_{n-1}\| + \widetilde{M}_1\left[\sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}| + \frac{|r_n - r_{n-1}|}{r_n}\right. \\
&\quad \left. + \sum_{k=2}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\beta_{n,1} - \beta_{n-1,1}| + |\alpha_n - \alpha_{n-1}|\right] \\
&\le \bigl(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\bigr)\|x_n - x_{n-1}\| + \widetilde{M}_1\left[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}|\right. \\
&\quad \left. + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}|\right],
\end{aligned}
$$

where $\sup_{n\ge1}\{\widetilde{M}_0 + L + \sum_{k=2}^{N}\|S_k v_{n-1} - y_{n-1,k-1}\| + \|S_1 v_{n-1} - v_{n-1}\| + \widetilde{M}\} \le \widetilde{M}_1$ for some $\widetilde{M}_1 > 0$. This together with (3.7)-(3.8) implies that

$$
\begin{aligned}
&\|w_n - w_{n-1}\| \\
&\quad \le \frac{\|\gamma_n(y_n - y_{n-1}) + \delta_n(Ty_n - Ty_{n-1})\|}{1 - \beta_n} + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right| \|y_{n-1}\| \\
&\qquad + \left|\frac{\delta_n}{1 - \beta_n} - \frac{\delta_{n-1}}{1 - \beta_{n-1}}\right| \|Ty_{n-1}\| \\
&\quad \le \frac{(\gamma_n + \delta_n)\|y_n - y_{n-1}\|}{1 - \beta_n} + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right| \|y_{n-1}\| + \left|\frac{\delta_n}{1 - \beta_n} - \frac{\delta_{n-1}}{1 - \beta_{n-1}}\right| \|Ty_{n-1}\| \\
&\quad = \|y_n - y_{n-1}\| + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right| \bigl(\|y_{n-1}\| + \|Ty_{n-1}\|\bigr) \\
&\quad \le \bigl(1 - \alpha_n(\mu\bar{\gamma} - \gamma l)\bigr)\|x_n - x_{n-1}\| + \widetilde{M}_1\left[\frac{|r_n - r_{n-1}|}{\epsilon} + \sum_{k=1}^{M} |\lambda_{k,n} - \lambda_{k,n-1}|\right. \\
&\qquad \left. + \sum_{k=1}^{N} |\beta_{n,k} - \beta_{n-1,k}| + |\alpha_n - \alpha_{n-1}|\right] + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right| \bigl(\|y_{n-1}\| + \|Ty_{n-1}\|\bigr)
\end{aligned}
$$

$$\begin{aligned}&\leq \bigl(1-\alpha_n(\mu\bar{\gamma}-\gamma l)\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_2\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon}+\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|\\&\quad+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\alpha_n-\alpha_{n-1}|+\biggl|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\biggr|\Biggr],\end{aligned}\tag{3.16}$$

where $\sup_{n\geq 0}\{\widetilde{M}_1+\|y_n\|+\|Ty_n\|\}\leq\widetilde{M}_2$ for some $\widetilde{M}_2>0$.

Further, we observe that

$$\begin{cases}x_{n+1}=\beta_n x_n+(1-\beta_n)w_n,\\x_n=\beta_{n-1}x_{n-1}+(1-\beta_{n-1})w_{n-1},\quad\forall n\geq 1.\end{cases}$$

Simple calculations show that

$$x_{n+1}-x_n=(1-\beta_n)(w_n-w_{n-1})+(\beta_n-\beta_{n-1})(x_{n-1}-w_{n-1})+\beta_n(x_n-x_{n-1}).$$

Then passing to the norm we get from (3.16)

$$\begin{aligned}&\|x_{n+1}-x_n\|\\&\quad\leq(1-\beta_n)\|w_n-w_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}-w_{n-1}\|+\beta_n\|x_n-x_{n-1}\|\\&\quad\leq(1-\beta_n)\Biggl\{\bigl(1-\alpha_n(\mu\bar{\gamma}-\gamma l)\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_2\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon}+\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|\\&\qquad+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\alpha_n-\alpha_{n-1}|+\biggl|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\biggr|\Biggr]\Biggr\}\\&\qquad+|\beta_n-\beta_{n-1}|\|x_{n-1}-w_{n-1}\|+\beta_n\|x_n-x_{n-1}\|\\&\quad\leq\bigl(1-(\mu\bar{\gamma}-\gamma l)(1-\beta_n)\alpha_n\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_2\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon}+\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|\\&\qquad+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\alpha_n-\alpha_{n-1}|+\biggl|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\biggr|\Biggr]+|\beta_n-\beta_{n-1}|\|x_{n-1}-w_{n-1}\|\\&\quad\leq\bigl(1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr)\|x_n-x_{n-1}\|+\widetilde{M}_3\Biggl[\frac{|r_n-r_{n-1}|}{\epsilon}+\sum_{k=1}^{M}|\lambda_{k,n}-\lambda_{k,n-1}|\\&\qquad+\sum_{k=1}^{N}|\beta_{n,k}-\beta_{n-1,k}|+|\alpha_n-\alpha_{n-1}|+\biggl|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\biggr|+|\beta_n-\beta_{n-1}|\Biggr],\end{aligned}\tag{3.17}$$

where $\sup_{n\geq 0}\{\widetilde{M}_2+\|x_n-w_n\|\}\leq\widetilde{M}_3$ for some $\widetilde{M}_3>0$. By hypotheses (H0)-(H6) and Lemma 2.8, we obtain the claim. □

Proposition 3.3 *Let us suppose that $\Omega\neq\emptyset$. Let us suppose that $\{x_n\}$ is asymptotically regular. Then $\|x_n-u_n\|=\|x_n-T_{r_n}x_n\|\to 0$ and $\|x_n-v_n\|\to 0$ as $n\to\infty$.*

Proof Take a fixed $p \in \Omega$ arbitrarily. We recall that, by the firm nonexpansivity of T_{r_n}, a standard calculation (see [44]) shows that for $p \in \mathrm{GMEP}(\Theta, h)$

$$\|u_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.18}$$

Observe that

$$\begin{aligned}
\|\Lambda_n^k u_n - p\|^2 &= \|J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)p\|^2 \\
&\le \|(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)p\|^2 \\
&\le \|\Lambda_n^{k-1}u_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_kp\|^2 \\
&\le \|u_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_kp\|^2 \\
&\le \|x_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_kp\|^2,
\end{aligned} \tag{3.19}$$

for each $k \in \{1, 2, \dots, M\}$.

Utilizing Lemmas 2.1 and 2.2(b), we obtain from $0 \le \gamma l < \mu\bar{\gamma}$, (3.1), (3.4), (3.18), and (3.19) that

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad= \|\alpha_n\gamma\big(f(y_{n,N}) - f(p)\big) + (I - \alpha_n\mu V)(z_n - p) + \alpha_n(\gamma f - \mu V)p\|^2 \\
&\quad\le \|\alpha_n\gamma\big(f(y_{n,N}) - f(p)\big) + (I - \alpha_n\mu V)(z_n - p)\|^2 + 2\alpha_n\big\langle(\gamma f - \mu V)p, y_n - p\big\rangle \\
&\quad\le \big[\alpha_n\gamma\|f(y_{n,N}) - f(p)\| + \|I - \alpha_n\mu V\|\|z_n - p\|\big]^2 + 2\alpha_n\big\langle(\gamma f - \mu V)p, y_n - p\big\rangle \\
&\quad\le \big[\alpha_n\gamma l\|y_{n,N} - p\| + (1 - \alpha_n\mu\bar{\gamma})\|z_n - p\|\big]^2 + 2\alpha_n\big\langle(\gamma f - \mu V)p, y_n - p\big\rangle \\
&\quad= \left[\alpha_n\mu\bar{\gamma}\frac{\gamma l}{\mu\bar{\gamma}}\|y_{n,N} - p\| + (1 - \alpha_n\mu\bar{\gamma})\|z_n - p\|\right]^2 + 2\alpha_n\big\langle(\gamma f - \mu V)p, y_n - p\big\rangle \\
&\quad\le \alpha_n\mu\bar{\gamma}\frac{(\gamma l)^2}{(\mu\bar{\gamma})^2}\|y_{n,N} - p\|^2 + (1 - \alpha_n\mu\bar{\gamma})\|z_n - p\|^2 + 2\alpha_n\big\langle(\gamma f - \mu V)p, y_n - p\big\rangle \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|y_{n,N} - p\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\
&\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|v_n - p\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\
&\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|\Lambda_n^k u_n - p\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 \\
&\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|u_n - p\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_kp\|^2 \\
&\qquad - \nu_2(2\zeta_2 - \nu_2)\|F_2y_{n,N} - F_2p\|^2 - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 \\
&\qquad + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\|B_k\Lambda_n^{k-1}u_n - B_kp\|^2
\end{aligned}$$

$$
\begin{aligned}
&- \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 p\|^2 - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 \\
&+ 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\|.
\end{aligned}
\tag{3.20}
$$

Since $(\gamma_n + \delta_n)\xi \le \gamma_n$ for all $n \ge 0$, utilizing Lemma 2.4 we have

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\|\beta_n(x_n - p) + \gamma_n(y_n - p) + \delta_n(Ty_n - p)\big\|^2 \\
&\quad = \left\|\beta_n(x_n - p) + (\gamma_n + \delta_n)\frac{1}{\gamma_n + \delta_n}\big[\gamma_n(y_n - p) + \delta_n(Ty_n - p)\big]\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\left\|\frac{1}{\gamma_n + \delta_n}\big[\gamma_n(y_n - p) + \delta_n(Ty_n - p)\big]\right\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (\gamma_n + \delta_n)\|y_n - p\|^2 \\
&\quad = \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|y_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\big[\alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|x_n - p\|^2 - \|x_n - u_n\|^2 \\
&\qquad + \lambda_{k,n}(\lambda_{k,n} - 2\eta_k)\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\|^2 - \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 p\|^2 \\
&\qquad - \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\|\big] \\
&\quad \le \|x_n - p\|^2 - (1 - \beta_n)\big[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\|^2 \\
&\qquad + \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\big] \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\|.
\end{aligned}
\tag{3.21}
$$

So, we deduce from $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_{k,n}\} \subset [a_k, b_k] \subset (0, 2\eta_k)$, $k = 1, \dots, M$, that

$$
\begin{aligned}
&(1 - d)\big[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\|^2 \\
&\qquad + \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\big] \\
&\quad \le (1 - \beta_n)\big[\|x_n - u_n\|^2 + \lambda_{k,n}(2\eta_k - \lambda_{k,n})\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\|^2 \\
&\qquad + \nu_2(2\zeta_2 - \nu_2)\|F_2 y_{n,N} - F_2 p\|^2 + \nu_1(2\zeta_1 - \nu_1)\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\|^2\big] \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| \\
&\quad \le \|x_n - x_{n+1}\|\big(\|x_n - p\| + \|x_{n+1} - p\|\big) + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 \\
&\qquad + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\|.
\end{aligned}
$$

By Propositions 3.1 and 3.2 we know that the sequences $\{x_n\}$, $\{y_n\}$, and $\{y_{n,N}\}$ are bounded, and that $\{x_n\}$ is asymptotically regular. Therefore, from $\alpha_n \to 0$ we obtain

$$
\begin{aligned}
\lim_{n\to\infty}\|x_n - u_n\| &= \lim_{n\to\infty}\|F_2 y_{n,N} - F_2 p\| = \lim_{n\to\infty}\|F_1\tilde{y}_{n,N} - F_1\tilde{p}\| \\
&= \lim_{n\to\infty}\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\| = 0,
\end{aligned}
\tag{3.22}
$$

for each $k \in \{1, \dots, M\}$.

Utilizing Lemmas 2.2(a) and 2.13, we obtain for each $k \in \{1, \dots, M\}$

$$\begin{aligned}
&\|\Lambda_n^k u_n - p\|^2 \\
&\quad = \|J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - J_{R_k,\lambda_{k,n}}(I - \lambda_{k,n}B_k)p\|^2 \\
&\quad \le \langle (I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)p, \Lambda_n^k u_n - p\rangle \\
&\quad = \frac{1}{2}\big(\|(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)p\|^2 + \|\Lambda_n^k u_n - p\|^2 \\
&\qquad - \|(I - \lambda_{k,n}B_k)\Lambda_n^{k-1}u_n - (I - \lambda_{k,n}B_k)p - (\Lambda_n^k u_n - p)\|^2\big) \\
&\quad \le \frac{1}{2}\big(\|\Lambda_n^{k-1}u_n - p\|^2 + \|\Lambda_n^k u_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_i p)\|^2\big) \\
&\quad \le \frac{1}{2}\big(\|u_n - p\|^2 + \|\Lambda_n^k u_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_i p)\|^2\big) \\
&\quad \le \frac{1}{2}\big(\|x_n - p\|^2 + \|\Lambda_n^k u_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k p)\|^2\big),
\end{aligned}$$

which immediately leads to

$$\begin{aligned}
&\|\Lambda_n^k u_n - p\|^2 \\
&\quad \le \|x_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n - \lambda_{k,n}(B_k\Lambda_n^{k-1}u_n - B_k p)\|^2 \\
&\quad = \|x_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\|^2 - \lambda_{k,n}^2\|B_k\Lambda_n^{k-1}u_n - B_k p\|^2 \\
&\qquad + 2\lambda_{k,n}\langle \Lambda_n^{k-1}u_n - \Lambda_n^k u_n, B_k\Lambda_n^{k-1}u_n - B_k p\rangle \\
&\quad \le \|x_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\|^2 \\
&\qquad + 2\lambda_{k,n}\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\|\|B_k\Lambda_n^{k-1}u_n - B_k p\|.
\end{aligned} \tag{3.23}$$

From (3.4), (3.20), (3.21), and (3.23) we conclude that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|y_n - p\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\big[\alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|z_n - p\|^2 \\
&\qquad + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\|\big] \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|y_{n,N} - p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 \\
&\qquad + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|v_n - p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\|\Lambda_n^k u_n - p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 \\
&\qquad + 2\alpha_n\|(\gamma f - \mu V)p\|\|y_n - p\| \\
&\quad \le \beta_n\|x_n - p\|^2 + (1 - \beta_n)\big[\|x_n - p\|^2 - \|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\|^2 \\
&\qquad + 2\lambda_{k,n}\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\|\|B_k\Lambda_n^{k-1}u_n - B_k p\|\big]
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\le \|x_n-p\|^2 - (1-\beta_n)\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|^2 \\
&\quad + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\| \\
&\quad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|,
\end{aligned}
\tag{3.24}
$$

which, together with $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\{\lambda_{k,n}\} \subset [a_k,b_k] \subset (0,2\eta_k)$, $k = 1,\dots,M$, yields

$$
\begin{aligned}
&(1-d)\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|^2 \\
&\quad\le (1-\beta_n)\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|^2 \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\lambda_{k,n}\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\| \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big) + 2b_k\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\|\big\|B_k\Lambda_n^{k-1}u_n - B_k p\big\| \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$, and $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, and $\{u_n\}$ are bounded, we obtain from (3.22) and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2),

$$
\lim_{n\to\infty}\big\|\Lambda_n^{k-1}u_n - \Lambda_n^k u_n\big\| = 0, \quad \forall k \in \{1,\dots,M\}. \tag{3.25}
$$

Therefore,

$$
\begin{aligned}
\|u_n-v_n\| &= \big\|\Lambda_n^0 u_n - \Lambda_n^M u_n\big\| \\
&\le \big\|\Lambda_n^0 u_n - \Lambda_n^1 u_n\big\| + \big\|\Lambda_n^1 u_n - \Lambda_n^2 u_n\big\| + \cdots + \big\|\Lambda_n^{M-1}u_n - \Lambda_n^M u_n\big\| \\
&\to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|x_n-v_n\| &\le \|x_n-u_n\| + \|u_n-v_n\| \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}
\tag{3.26}
$$

□

Remark 3.1 By the last proposition we have $\omega_w(x_n) = \omega_w(v_n)$ and $\omega_s(x_n) = \omega_s(v_n)$, *i.e.*, the sets of strong/weak cluster points of $\{x_n\}$ and $\{v_n\}$ coincide.

Of course, if $\beta_{n,i} \to \beta_i \ne 0$ as $n \to \infty$, for all indices i, the assumptions of Proposition 3.2 are enough to assure that

$$
\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_{n,i}} = 0, \quad \forall i \in \{1,\dots,N\}.
$$

In the next proposition, we estimate the case in which at least one sequence $\{\beta_{n,k_0}\}$ is a null sequence.

Proposition 3.4 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that* (H0) *holds. Moreover, for an index $k_0 \in \{1,\dots,N\}$, $\lim_{n\to\infty} \beta_{n,k_0} = 0$, and the following hold*:

(H7) *for each $i \in \{1,\dots,N\}$ and $k \in \{1,\dots,M\}$*,

$$\begin{aligned}\lim_{n\to\infty}\frac{|\beta_{n,i}-\beta_{n-1,i}|}{\alpha_n\beta_{n,k_0}} &= \lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n\beta_{n,k_0}} = \lim_{n\to\infty}\frac{|\beta_n-\beta_{n-1}|}{\alpha_n\beta_{n,k_0}} = \lim_{n\to\infty}\frac{|r_n-r_{n-1}|}{\alpha_n\beta_{n,k_0}}\\ &= \lim_{n\to\infty}\frac{1}{\alpha_n\beta_{n,k_0}}\left|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}\right| = \lim_{n\to\infty}\frac{|\lambda_{k,n}-\lambda_{k,n-1}|}{\alpha_n\beta_{n,k_0}} = 0;\end{aligned}$$

(H8) *there exists a constant $\tau > 0$ such that $\frac{1}{\alpha_n}|\frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}}| < \tau$ for all $n \geq 1$.*

Then

$$\lim_{n\to\infty}\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}}.$$

Proof We start by (3.17). Dividing both terms by β_{n,k_0} we have

$$\begin{aligned}&\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}}\\ &\quad\leq \bigl[1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n,k_0}} + \widetilde{M}_3\biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^M|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}}\\ &\qquad + \frac{\sum_{k=1}^N|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\biggr]. \qquad (3.27)\end{aligned}$$

So, by (H8) we have

$$\begin{aligned}&\frac{\|x_{n+1}-x_n\|}{\beta_{n,k_0}}\\ &\quad\leq \bigl[1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}}\\ &\qquad + \bigl[1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr]\|x_n-x_{n-1}\|\left|\frac{1}{\beta_{n,k_0}}-\frac{1}{\beta_{n-1,k_0}}\right|\\ &\qquad + \widetilde{M}_3\biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^M|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^N|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}}\\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\biggr]\\ &\quad\leq \bigl[1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} + \|x_n-x_{n-1}\|\left|\frac{1}{\beta_{n,k_0}}-\frac{1}{\beta_{n-1,k_0}}\right|\\ &\qquad + \widetilde{M}_3\biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^M|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^N|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}}\\ &\qquad + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}} + \frac{|\alpha_n-\alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n-\beta_{n-1}|}{\beta_{n,k_0}}\biggr]\\ &\quad\leq \bigl[1-(\mu\bar{\gamma}-\gamma l)(1-d)\alpha_n\bigr]\frac{\|x_n-x_{n-1}\|}{\beta_{n-1,k_0}} + \alpha_n\tau\|x_n-x_{n-1}\|\\ &\qquad + \widetilde{M}_3\biggl[\frac{|r_n-r_{n-1}|}{\epsilon\beta_{n,k_0}} + \frac{\sum_{k=1}^M|\lambda_{k,n}-\lambda_{k,n-1}|}{\beta_{n,k_0}} + \frac{\sum_{k=1}^N|\beta_{n,k}-\beta_{n-1,k}|}{\beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n}-\frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\beta_{n,k_0}}\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\beta_{n,k_0}}\Bigg] \\
&= \big[1 - \alpha_n(\mu\bar{\gamma} - \gamma l)(1-d)\big]\frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \alpha_n(\mu\bar{\gamma} - \gamma l)(1-d) \\
&\quad \cdot \frac{1}{(\mu\bar{\gamma} - \gamma l)(1-d)}\Bigg\{\tau\|x_n - x_{n-1}\| \\
&\quad + \widetilde{M}_3\Bigg[\frac{|r_n - r_{n-1}|}{\epsilon\alpha_n\beta_{n,k_0}} + \frac{\sum_{k=1}^M|\lambda_{k,n} - \lambda_{k,n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{\sum_{k=1}^N|\beta_{n,k} - \beta_{n-1,k}|}{\alpha_n\beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\alpha_n\beta_{n,k_0}} \\
&\quad + \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n\beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\alpha_n\beta_{n,k_0}}\Bigg]\Bigg\}.
\end{aligned}
$$

Therefore, utilizing Lemma 2.8, from (H0), (H7), and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2), we deduce that

$$
\lim_{n\to\infty}\frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0. \qquad \square
$$

Proposition 3.5 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that* (H0)-(H6) *hold. Then $\|z_n - y_{n,N}\| \to 0$ as $n \to \infty$.*

Proof Let $p \in \Omega$. In terms of the firm nonexpansivity of P_C and the ζ_j-inverse-strong monotonicity of F_j for $j = 1,2$, we obtain from $\nu_j \in (0, 2\zeta_j)$, $j = 1,2$, and (3.4)

$$
\begin{aligned}
&\|\tilde{y}_{n,N} - \tilde{p}\|^2 \\
&\quad = \big\|P_C(I - \nu_2F_2)y_{n,N} - P_C(I - \nu_2F_2)p\big\|^2 \\
&\quad \le \big\langle (I - \nu_2F_2)y_{n,N} - (I - \nu_2F_2)p, \tilde{y}_{n,N} - \tilde{p}\big\rangle \\
&\quad = \frac{1}{2}\Big[\big\|(I - \nu_2F_2)y_{n,N} - (I - \nu_2F_2)p\big\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 \\
&\qquad - \big\|(I - \nu_2F_2)y_{n,N} - (I - \nu_2F_2)p - (\tilde{y}_{n,N} - \tilde{p})\big\|^2\Big] \\
&\quad \le \frac{1}{2}\Big[\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 - \big\|(y_{n,N} - \tilde{y}_{n,N}) - \nu_2(F_2y_{n,N} - F_2p) - (p - \tilde{p})\big\|^2\Big] \\
&\quad = \frac{1}{2}\Big[\|y_{n,N} - p\|^2 + \|\tilde{y}_{n,N} - \tilde{p}\|^2 - \big\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\big\|^2 \\
&\qquad + 2\nu_2\big\langle (y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p}), F_2y_{n,N} - F_2p\big\rangle - \nu_2^2\|F_2y_{n,N} - F_2p\|^2\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\|z_n - p\|^2 \\
&\quad = \big\|P_C(I - \nu_1F_1)\tilde{y}_{n,N} - P_C(I - \nu_1F_1)\tilde{p}\big\|^2 \\
&\quad \le \big\langle (I - \nu_1F_1)\tilde{y}_{n,N} - (I - \nu_1F_1)\tilde{p}, z_n - p\big\rangle \\
&\quad = \frac{1}{2}\Big[\big\|(I - \nu_1F_1)\tilde{y}_{n,N} - (I - \nu_1F_1)\tilde{p}\big\|^2 + \|z_n - p\|^2 \\
&\qquad - \big\|(I - \nu_1F_1)\tilde{y}_{n,N} - (I - \nu_1F_1)\tilde{p} - (z_n - p)\big\|^2\Big]
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{1}{2}\bigl[\|\tilde{y}_{n,N}-\tilde{p}\|^2+\|z_n-p\|^2-\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2 \\
&\quad +2\nu_1\bigl\langle F_1\tilde{y}_{n,N}-F_1\tilde{p},(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\rangle-\nu_1^2\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|^2\bigr] \\
&\le \frac{1}{2}\bigl[\|y_{n,N}-p\|^2+\|z_n-p\|^2-\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2 \\
&\quad +2\nu_1\bigl\langle F_1\tilde{y}_{n,N}-F_1\tilde{p},(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\rangle\bigr].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\|\tilde{y}_{n,N}-\tilde{p}\|^2 &\le \|y_{n,N}-p\|^2-\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\quad +2\nu_2\bigl\langle(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p}),F_2y_{n,N}-F_2p\bigr\rangle-\nu_2^2\|F_2y_{n,N}-F_2p\|^2
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
\|z_n-p\|^2 &\le \|y_{n,N}-p\|^2-\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|^2 \\
&\quad +2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\bigl\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\bigr\|.
\end{aligned} \tag{3.29}$$

Consequently, from (3.4), (3.24), and (3.28), it follows that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2+\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 \\
&\qquad +2\alpha_n\bigl\|(\gamma f-\mu V)p\bigr\|\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|\tilde{y}_{n,N}-\tilde{p}\|^2+\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 \\
&\qquad +2\alpha_n\bigl\|(\gamma f-\mu V)p\bigr\|\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\bigl[\|y_{n,N}-p\|^2-\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\qquad +2\nu_2\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|\|F_2y_{n,N}-F_2p\|\bigr] \\
&\qquad +\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2+2\alpha_n\bigl\|(\gamma f-\mu V)p\bigr\|\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|^2+(1-\beta_n)\bigl[\|x_n-p\|^2-\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\qquad +2\nu_2\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|\|F_2y_{n,N}-F_2p\|\bigr] \\
&\qquad +\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2+2\alpha_n\bigl\|(\gamma f-\mu V)p\bigr\|\|y_n-p\| \\
&\quad\le \|x_n-p\|^2-(1-\beta_n)\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\qquad +2\nu_2\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|\|F_2y_{n,N}-F_2p\| \\
&\qquad +\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2+2\alpha_n\bigl\|(\gamma f-\mu V)p\bigr\|\|y_n-p\|,
\end{aligned}$$

which yields

$$\begin{aligned}
&(1-d)\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\quad\le (1-\beta_n)\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|^2 \\
&\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\nu_2\bigl\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\bigr\|\|F_2y_{n,N}-F_2p\|
\end{aligned}$$

$$
\begin{aligned}
&+ \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big) + 2\nu_2\big\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\big\|\|F_2y_{n,N}-F_2p\| \\
&\quad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, and $\{\tilde{y}_{n,N}\}$ are bounded, we deduce from (3.22) that

$$
\lim_{n\to\infty}\big\|(y_{n,N}-\tilde{y}_{n,N})-(p-\tilde{p})\big\| = 0. \tag{3.30}
$$

Furthermore, from (3.4), (3.24), and (3.29), it follows that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|z_n-p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 \\
&\qquad + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\big[\|y_{n,N}-p\|^2 - \big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|^2 \\
&\qquad + 2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|\big] \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\big[\|x_n-p\|^2 - \big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|^2 \\
&\qquad + 2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|\big] \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad\le \|x_n-p\|^2 - (1-\beta_n)\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|^2 \\
&\qquad + 2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\| \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&(1-d)\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|^2 \\
&\quad\le (1-\beta_n)\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\|^2 \\
&\quad\le \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + 2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\| \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad\le \|x_n-x_{n+1}\|\big(\|x_n-p\|+\|x_{n+1}-p\|\big) + 2\nu_1\|F_1\tilde{y}_{n,N}-F_1\tilde{p}\|\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\| \\
&\qquad + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$, and $\{x_n\}$, $\{z_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, and $\{\tilde{y}_{n,N}\}$ are bounded, we deduce from (3.22) that

$$
\lim_{n\to\infty}\big\|(\tilde{y}_{n,N}-z_n)+(p-\tilde{p})\big\| = 0. \tag{3.31}
$$

Note that

$$\|y_{n,N} - z_n\| \le \big\|(y_{n,N} - \tilde{y}_{n,N}) - (p - \tilde{p})\big\| + \big\|(\tilde{y}_{n,N} - z_n) + (p - \tilde{p})\big\|.$$

Hence from (3.30) and (3.31) we get

$$\lim_{n\to\infty} \|y_{n,N} - z_n\| = \lim_{n\to\infty} \|y_{n,N} - Gy_{n,N}\| = 0. \tag{3.32}$$

□

Proposition 3.6 *Let us suppose that* $\Omega \neq \emptyset$. *Let us suppose that* $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$ *for each* $i = 1,\dots,N$. *Moreover, suppose that* (H0)-(H6) *are satisfied. Then* $\lim_{n\to\infty} \|S_i v_n - v_n\| = 0$ *for each* $i = 1,\dots,N$ *provided* $\|Ty_n - y_n\| \to$ *as* $n \to \infty$.

Proof First of all, observe that

$$\begin{aligned} x_{n+1} - x_n &= \gamma_n(y_n - x_n) + \delta_n(Ty_n - x_n) \\ &= \gamma_n(y_n - x_n) + \delta_n(Ty_n - y_n) + \delta_n(y_n - x_n) \\ &= (\gamma_n + \delta_n)(y_n - x_n) + \delta_n(Ty_n - y_n) \\ &= (1 - \beta_n)(y_n - x_n) + \delta_n(Ty_n - y_n). \end{aligned}$$

By Proposition 3.2 we know that $\{x_n\}$ is asymptotically regular. Hence we have

$$(1 - \beta_n)\|y_n - x_n\| = \big\|x_{n+1} - x_n - \delta_n(Ty_n - y_n)\big\| \le \|x_{n+1} - x_n\| + \delta_n\|Ty_n - y_n\|,$$

which, together with $\|Ty_n - y_n\| \to 0$, implies that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.33}$$

Let us show that for each $i \in \{1,\dots,N\}$, one has $\|S_i v_n - y_{n,i-1}\| \to 0$ as $n \to \infty$. Let $p \in \Omega$. When $i = N$, by Lemma 2.2(b) we have from (3.3), (3.4), and (3.20)

$$\begin{aligned} \|y_n - p\|^2 &\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| \\ &\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| + \|y_{n,N} - p\|^2 \\ &= \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| + \beta_{n,N}\|S_N v_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|y_{n,N-1} - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N v_n - y_{n,N-1}\|^2 \\ &\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| + \beta_{n,N}\|v_n - p\|^2 \\ &\quad + (1 - \beta_{n,N})\|v_n - p\|^2 - \beta_{n,N}(1 - \beta_{n,N})\|S_N v_n - y_{n,N-1}\|^2 \\ &= \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| + \|v_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N v_n - y_{n,N-1}\|^2 \\ &\le \alpha_n\mu\bar{\gamma}\|y_{n,N} - p\|^2 + 2\alpha_n\big\|(\gamma f - \mu V)p\big\|\|y_n - p\| + \|x_n - p\|^2 \\ &\quad - \beta_{n,N}(1 - \beta_{n,N})\|S_N v_n - y_{n,N-1}\|^2. \end{aligned}$$

So, we have

$$\begin{aligned}
&\beta_{n,N}(1-\beta_{n,N})\|S_N v_n - y_{n,N-1}\|^2 \\
&\quad \le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\quad \le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \|x_n - y_n\|\big(\|x_n - p\| + \|y_n - p\|\big).
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty} \beta_{n,N} \le \limsup_{n\to\infty} \beta_{n,N} < 1$, and $\lim_{n\to\infty} \|x_n - y_n\| = 0$ (due to (3.33)), it is well known that $\{\|S_N v_n - y_{n,N-1}\|\}$ is a null sequence.

Let $i \in \{1, \dots, N-1\}$. Then one has

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \|y_{n,N} - p\|^2 \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \beta_{n,N}\|S_N v_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \beta_{n,N}\|x_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\|y_{n,N-1} - p\|^2 \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \beta_{n,N}\|x_n - p\|^2 \\
&\quad + (1-\beta_{n,N})\big[\beta_{n,N-1}\|S_{N-1} v_n - p\|^2 + (1-\beta_{n,N-1})\|y_{n,N-2} - p\|^2\big] \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| \\
&\quad + \big(\beta_{n,N} + (1-\beta_{n,N})\beta_{n,N-1}\big)\|x_n - p\|^2 + \prod_{k=N-1}^{N}(1-\beta_{n,k})\|y_{n,N-2} - p\|^2,
\end{aligned}$$

and so, after $(N-i+1)$ iterations,

$$\begin{aligned}
\|y_n - p\|^2 &\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| \\
&\quad + \left(\beta_{n,N} + \sum_{j=i+2}^{N}\left(\prod_{l=j}^{N}(1-\beta_{n,l})\right)\beta_{n,j-1}\right)\|x_n - p\|^2 + \prod_{k=i+1}^{N}(1-\beta_{n,k})\|y_{n,i} - p\|^2 \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| \\
&\quad + \left(\beta_{n,N} + \sum_{j=i+2}^{N}\left(\prod_{l=j}^{N}(1-\beta_{n,l})\right)\beta_{n,j-1}\right)\|x_n - p\|^2 \\
&\quad + \prod_{k=i+1}^{N}(1-\beta_{n,k})\big[\beta_{n,i}\|S_i u_n - p\|^2 \\
&\quad + (1-\beta_{n,i})\|y_{n,i-1} - p\|^2 - \beta_{n,i}(1-\beta_{n,i})\|S_i v_n - y_{n,i-1}\|^2\big] \\
&\le \alpha_n \mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n \big\|(\gamma f - \mu V)p\big\| \|y_n - p\| + \|x_n - p\|^2 \\
&\quad - \beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2.
\end{aligned} \tag{3.34}$$

Again we obtain

$$\begin{aligned}
&\beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2 \\
&\quad \le \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\|(\gamma f-\mu V)p\|\|y_n-p\| + \|x_n-p\|^2 - \|y_n-p\|^2 \\
&\quad \le \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\|(\gamma f-\mu V)p\|\|y_n-p\| + \|x_n-y_n\|\bigl(\|x_n-p\|+\|y_n-p\|\bigr).
\end{aligned}$$

Since $\alpha_n \to 0$, $0 < \liminf_{n\to\infty}\beta_{n,i} \le \limsup_{n\to\infty}\beta_{n,i} < 1$ for each $i = 1,\dots,N-1$, and $\lim_{n\to\infty}\|x_n - y_n\| = 0$ (due to (3.33)), it is well known that

$$\lim_{n\to\infty}\|S_i v_n - y_{n,i-1}\| = 0.$$

Obviously for $i = 1$, we have $\|S_1u_n - u_n\| \to 0$.

To conclude, we have

$$\|S_2v_n - v_n\| \le \|S_2v_n - y_{n,1}\| + \|y_{n,1} - v_n\| = \|S_2v_n - y_{n,1}\| + \beta_{n,1}\|S_1v_n - v_n\|$$

from which $\|S_2v_n - v_n\| \to 0$. Thus by induction $\|S_iv_n - v_n\| \to 0$ for all $i = 2,\dots,N$ since it is enough to observe that

$$\begin{aligned}
\|S_iv_n - v_n\| &\le \|S_iv_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1}v_n\| + \|S_{i-1}v_n - v_n\| \\
&\le \|S_iv_n - y_{n,i-1}\| + (1-\beta_{n,i-1})\|S_{i-1}v_n - y_{n,i-2}\| + \|S_{i-1}v_n - v_n\|.
\end{aligned}$$ □

Remark 3.2 As an example, we consider $M = 1$, $N = 2$, and the sequences:

(a) $\lambda_{1,n} = \eta_1 - \frac{1}{n}$, $\forall n > \frac{1}{\eta_1}$;
(b) $\alpha_n = \frac{1}{\sqrt{n}}$, $r_n = 2 - \frac{1}{n}$, $\forall n > 1$;
(c) $\beta_n = \beta_{n,1} = \frac{1}{2} - \frac{1}{n}$, $\beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\forall n > 2$.

They satisfy the hypotheses on the parameter sequences in Proposition 3.6.

Proposition 3.7 *Let us suppose that $\Omega \neq \emptyset$ and $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose there exists $k \in \{1,\dots,N\}$ such that $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1,\dots,N\}$ be the largest index such that $\beta_{n,k_0} \to 0$ as $n \to \infty$. Suppose that*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;
(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;
(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0,1)$.

Moreover, suppose that (H0), (H7), *and* (H8) *hold. Then* $\lim_{n\to\infty}\|S_iv_n - v_n\| = 0$ *for each* $i = 1,\dots,N$ *provided* $\|Ty_n - y_n\| \to 0$ *as* $n \to \infty$.

Proof First of all we note that if (H7) holds then also (H1)-(H6) are satisfied. So $\{x_n\}$ is asymptotically regular.

Let k_0 be as in the hypotheses. As in Proposition 3.6, for every index $i \in \{1,\dots,N\}$ such that $\beta_{n,i} \to \beta_i \neq 0$ (which leads to $0 < \liminf_{n\to\infty}\beta_{n,i} \le \limsup_{n\to\infty}\beta_{n,i} < 1$), one has $\|S_iv_n - y_{n,i-1}\| \to 0$ as $n \to \infty$.

For all the other indices $i \le k_0$, we can prove that $\|S_i v_n - y_{n,i-1}\| \to 0$ as $n \to \infty$ in a similar manner. By the relation (due to (3.21) and (3.34))

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\|y_n-p\|^2 \\
&\le \beta_n\|x_n-p\|^2 + (1-\beta_n)\Bigg[\alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad + \|x_n-p\|^2 - \beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2\Bigg] \\
&\le \|x_n-p\|^2 + \alpha_n\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\alpha_n\big\|(\gamma f-\mu V)p\big\|\|y_n-p\| \\
&\quad - (1-\beta_n)\beta_{n,i}\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2,
\end{aligned}$$

we immediately obtain

$$\begin{aligned}
&(1-d)\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2 \\
&\quad\le (1-\beta_n)\prod_{k=i}^{N}(1-\beta_{n,k})\|S_i v_n - y_{n,i-1}\|^2 \\
&\quad\le \frac{\alpha_n}{\beta_{n,i}}\big[\mu\bar{\gamma}\|y_{n,N}-p\|^2 + 2\big\|(\gamma f-\mu V)p\big\|\|y_n-p\|\big] \\
&\qquad + \frac{\|x_n-x_{n+1}\|}{\beta_{n,i}}\big(\|x_n-p\| + \|x_{n+1}-p\|\big).
\end{aligned}$$

By Proposition 3.4 or by hypothesis (ii) on the sequences, we have

$$\frac{\|x_n-x_{n+1}\|}{\beta_{n,i}} = \frac{\|x_n-x_{n+1}\|}{\beta_{n,k_0}} \cdot \frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0.$$

So, the conclusion follows. □

Remark 3.3 Let us consider $M = 1$, $N = 3$, and the following sequences:
(a) $\alpha_n = \frac{1}{n^{1/2}}$, $r_n = 2 - \frac{1}{n^2}$, $\forall n > 1$;
(b) $\lambda_{1,n} = \eta_1 - \frac{1}{n^2}$, $\forall n > \frac{1}{\eta_1^{1/2}}$;
(c) $\beta_{n,1} = \frac{1}{n^{1/4}}$, $\beta_n = \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\beta_{n,3} = \frac{1}{n^{1/3}}$, $\forall n > 1$.
It is easy to see that all hypotheses (i)-(iii), (H0), (H7), and (H8) of Proposition 3.7 are satisfied.

Remark 3.4 Under the hypotheses of Proposition 3.7, analogously to Proposition 3.6, one can see that

$$\lim_{n\to\infty}\|S_i v_n - y_{n,i-1}\| = 0, \quad \forall i \in \{2,\dots,N\}.$$

Corollary 3.1 *Let us suppose that the hypotheses of either Proposition* 3.6 *or Proposition* 3.7 *are satisfied. Then* $\omega_w(x_n) = \omega_w(v_n) = \omega_w(y_{n,1})$, $\omega_s(x_n) = \omega_s(v_n) = \omega_s(y_{n,1})$, *and* $\omega_w(x_n) \subset \Omega$.

Proof By Remark 3.1, we have $\omega_w(x_n) = \omega_w(v_n)$ and $\omega_s(x_n) = \omega_s(v_n)$. Note that by Remark 3.4,

$$\lim_{n\to\infty} \|S_N v_n - y_{n,N-1}\| = 0.$$

In the meantime, it is well known that

$$\lim_{n\to\infty} \|S_N v_n - v_n\| = \lim_{n\to\infty} \|v_n - x_n\| = \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Hence we have

$$\lim_{n\to\infty} \|S_N v_n - y_n\| = 0. \tag{3.35}$$

Furthermore, it follows from (3.1) that

$$\lim_{n\to\infty} \|y_{n,N} - y_{n,N-1}\| = \lim_{n\to\infty} \beta_{n,N}\|S_N v_n - y_{n,N-1}\| = 0,$$

which, together with $\lim_{n\to\infty} \|S_N v_n - y_{n,N-1}\| = 0$, yields

$$\lim_{n\to\infty} \|S_N v_n - y_{n,N}\| = 0. \tag{3.36}$$

Combining (3.35) and (3.36), we conclude that

$$\lim_{n\to\infty} \|y_n - y_{n,N}\| = 0, \tag{3.37}$$

which, together with $\lim_{n\to\infty} \|x_n - y_n\| = 0$, leads to

$$\lim_{n\to\infty} \|x_n - y_{n,N}\| = 0. \tag{3.38}$$

Now we observe that

$$\|x_n - y_{n,1}\| \le \|x_n - v_n\| + \|y_{n,1} - v_n\| = \|x_n - v_n\| + \beta_{n,1}\|S_1 v_n - v_n\|.$$

By Propositions 3.3 and 3.6, $\|x_n - v_n\| \to 0$ and $\|S_1 v_n - v_n\| \to 0$ as $n \to \infty$, and hence

$$\lim_{n\to\infty} \|x_n - y_{n,1}\| = 0.$$

So we get $\omega_w(x_n) = \omega_w(y_{n,1})$ and $\omega_s(x_n) = \omega_s(y_{n,1})$.

Let $p \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Since $p \in \omega_w(v_n)$, by Proposition 3.6 and Lemma 2.5 (demiclosedness principle), we have $p \in \mathrm{Fix}(S_i)$ for each $i = 1, \ldots, N$, *i.e.*, $p \in \bigcap_{i=1}^N \mathrm{Fix}(S_i)$. Taking into account $p \in \omega_w(y_{n,N})$ (due to (3.38)) and $\|y_{n,N} - Gy_{n,N}\| \to 0$ (due to (3.32)), by Lemma 2.5 (demiclosedness principle) we know that $p \in \mathrm{Fix}(G) =: \varXi$. Also, since $p \in \omega_w(y_n)$ (due to $\|x_n - y_n\| \to 0$), in terms of $\|Ty_n - y_n\| \to 0$ and Lemma 2.3 (demiclosedness principle), we get $p \in \mathrm{Fix}(T)$. Moreover, by Lemma 2.11 and Proposition 3.3 we know that $p \in \mathrm{GMEP}(\varTheta, h)$. Next we prove that $p \in \bigcap_{m=1}^M \mathrm{I}(B_m, R_m)$. As a matter of fact, from (3.22) and (3.25) we know that

$u_{n_i} \rightharpoonup p$ and $\Lambda_{n_i}^m u_{n_i} \rightharpoonup p$ for each $m = 1, \ldots, M$. Since B_m is η_m-inverse-strongly monotone, B_m is a monotone and Lipschitz-continuous mapping. It follows from Lemma 2.16 that $R_m + B_m$ is maximal monotone. Let $(v, g) \in G(R_m + B_m)$, *i.e.*, $g - B_m v \in R_m v$. Again, since $\Lambda_n^m u_n = J_{R_m,\lambda_{m,n}}(I - \lambda_{m,n} B_m)\Lambda_n^{m-1} u_n$, $n \geq 0$, $m \in \{1, 2, \ldots, N\}$, we have

$$\Lambda_n^{m-1} u_n - \lambda_{m,n} B_m \Lambda_n^{m-1} u_n \in (I + \lambda_{m,n} R_m)\Lambda_n^m u_n,$$

that is,

$$\frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1} u_n - \Lambda_n^m u_n - \lambda_{m,n} B_m \Lambda_n^{m-1} u_n\big) \in R_m \Lambda_n^m u_n.$$

In terms of the monotonicity of R_m, we get

$$\Big\langle v - \Lambda_n^m u_n, g - B_m v - \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1} u_n - \Lambda_n^m u_n - \lambda_{m,n} B_m \Lambda_n^{m-1} u_n\big)\Big\rangle \geq 0$$

and hence

$$\begin{aligned}
&\langle v - \Lambda_n^m u_n, g\rangle \\
&\quad \geq \Big\langle v - \Lambda_n^m u_n, B_m v + \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1} u_n - \Lambda_n^m u_n - \lambda_{m,n} B_m \Lambda_n^{m-1} u_n\big)\Big\rangle \\
&\quad = \Big\langle v - \Lambda_n^m u_n, B_m v - B_m \Lambda_n^m u_n + B_m \Lambda_n^m u_n - B_m \Lambda_n^{m-1} u_n + \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1} u_n - \Lambda_n^m u_n\big)\Big\rangle \\
&\quad \geq \langle v - \Lambda_n^m u_n, B_m \Lambda_n^m u_n - B_m \Lambda_n^{m-1} u_n\rangle + \Big\langle v - \Lambda_n^m u_n, \frac{1}{\lambda_{m,n}}\big(\Lambda_n^{m-1} u_n - \Lambda_n^m u_n\big)\Big\rangle.
\end{aligned}$$

In particular,

$$\begin{aligned}
\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle &\geq \langle v - \Lambda_{n_i}^m u_{n_i}, B_m \Lambda_{n_i}^m u_{n_i} - B_m \Lambda_{n_i}^{m-1} u_{n_i}\rangle \\
&\quad + \Big\langle v - \Lambda_{n_i}^m u_{n_i}, \frac{1}{\lambda_{m,n_i}}\big(\Lambda_{n_i}^{m-1} u_{n_i} - \Lambda_{n_i}^m u_{n_i}\big)\Big\rangle.
\end{aligned}$$

Since $\|\Lambda_n^m u_n - \Lambda_n^{m-1} u_n\| \to 0$ (due to (3.25)) and $\|B_m \Lambda_n^m u_n - B_m \Lambda_n^{m-1} u_n\| \to 0$ (due to the Lipschitz-continuity of B_m), we conclude from $\Lambda_{n_i}^m u_{n_i} \rightharpoonup p$ and $\{\lambda_{m,n}\} \subset [a_m, b_m] \subset (0, 2\eta_m)$ that

$$\lim_{i\to\infty}\langle v - \Lambda_{n_i}^m u_{n_i}, g\rangle = \langle v - p, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B_m + R_m$ that $0 \in (R_m + B_m)w$, *i.e.*, $p \in \mathrm{I}(B_m, R_m)$. Therefore, $p \in \bigcap_{m=1}^M \mathrm{I}(B_m, R_m)$. Consequently, $p \in \mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{m=1}^M \mathrm{I}(B_m, R_m) =: \Omega$. □

Theorem 3.1 *Let us suppose that* $\Omega \neq \emptyset$. *Let* $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \ldots, N$, *be sequences in* $(0,1)$ *such that* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for each index i. Moreover, let us suppose that* (H0)-(H6) *hold. Then the sequences* $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, *and* $\{y_n\}$ *defined by scheme* (3.1), *all converge strongly to* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *if and only if* $\|y_n - Ty_n\| \to 0$ *as* $n \to \infty$, *where* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^* \rangle \le 0, \quad \forall x \in \Omega, \tag{3.39}$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x \in \Omega} \frac{\mu}{2} \langle Vx, x \rangle - \Psi(x), \tag{3.40}$$

where Ψ *is a potential function for* γf.

Proof First of all, we note that V is a $\bar{\gamma}$-strongly positive bounded linear operator on H and $f : H \to H$ is an l-Lipschitz-continuous mapping with $0 \le \gamma l < \mu \bar{\gamma}$. It is clear that

$$\langle (\mu V - \gamma f)x - (\mu V - \gamma f)y, x - y \rangle \ge (\mu \bar{\gamma} - \gamma l)\|x - y\|^2, \quad \forall x, y \in H.$$

Hence we deduce that $\mu V - \gamma f$ is $(\mu \bar{\gamma} - \gamma l)$-strongly monotone. In the meantime, it is easy to see that $\mu V - \gamma f$ is $(\mu \|V\| + \gamma l)$-Lipschitz-continuous with constant $\mu \|V\| + \gamma l > 0$. Thus, there exists a unique solution x^* in Ω to the VIP (3.39). Equivalently, x^* is the unique solution of the minimization problem (3.40).

Now, observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle (\gamma f - \mu V)x^*, x_n - x^* \rangle = \lim_{i \to \infty} \langle (\gamma f - \mu V)x^*, x_{n_i} - x^* \rangle. \tag{3.41}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $p \in H$. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup p$. Then by Corollary 3.1, we get $p \in \omega_w(x_n) \subset \Omega$. Hence, from (3.39) and (3.41), we have

$$\limsup_{n \to \infty} \langle (\gamma f - \mu V)x^*, x_n - x^* \rangle = \langle (\gamma f - \mu V)x^*, p - x^* \rangle \le 0. \tag{3.42}$$

Since (H1)-(H6) hold, the sequence $\{x_n\}$ is asymptotically regular (according to Proposition 3.2). In terms of (3.33) and Proposition 3.3, $\|x_n - y_n\| \to 0$ and $\|x_n - v_n\| \to 0$ as $n \to \infty$.

Let us show that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Indeed, putting $p = x^*$, we deduce from (3.4), (3.20), and (3.21) that

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\quad \le \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\|y_n - x^*\|^2 \\
&\quad \le \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\biggl[\alpha_n \mu \bar{\gamma} \frac{(\gamma l)^2}{(\mu \bar{\gamma})^2} \|y_{n,N} - x^*\|^2 + (1 - \alpha_n \mu \bar{\gamma})\|z_n - x^*\|^2 \\
&\qquad + 2\alpha_n \langle (\gamma f - \mu V)x^*, y_n - x^* \rangle \biggr] \\
&\quad \le \beta_n \|x_n - x^*\|^2 + (1 - \beta_n)\biggl[\alpha_n \frac{(\gamma l)^2}{\mu \bar{\gamma}} \|x_n - x^*\|^2 + (1 - \alpha_n \mu \bar{\gamma})\|x_n - x^*\|^2
\end{aligned}$$

$$\begin{aligned}
&\quad + 2\alpha_n\big\langle(\gamma f - \mu V)x^*, y_n - x^*\big\rangle\Big] \\
&= \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left[\left(1 - \alpha_n\frac{(\mu\bar{\gamma})^2 - (\gamma l)^2}{\mu\bar{\gamma}}\right)\|x_n - x^*\|^2\right. \\
&\quad \left. + 2\alpha_n\big\langle(\gamma f - \mu V)x^*, y_n - x^*\big\rangle\right] \\
&= \left(1 - \alpha_n(1-\beta_n)\frac{(\mu\bar{\gamma})^2 - (\gamma l)^2}{\mu\bar{\gamma}}\right)\|x_n - x^*\|^2 + 2\alpha_n(1-\beta_n)\big\langle(\gamma f - \mu V)x^*, y_n - x^*\big\rangle \\
&\le \left(1 - \alpha_n(1-\beta_n)\frac{(\mu\bar{\gamma})^2 - (\gamma l)^2}{\mu\bar{\gamma}}\right)\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1-\beta_n)\frac{(\mu\bar{\gamma})^2 - (\gamma l)^2}{\mu\bar{\gamma}} \cdot \frac{2\mu\bar{\gamma}}{(\mu\bar{\gamma})^2 - (\gamma l)^2}\big\langle(\gamma f - \mu V)x^*, y_n - x^*\big\rangle. \qquad (3.43)
\end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, $\{\beta_n\} \subset [c,d] \subset (0,1)$ and $\|x_n - y_n\| \to 0$, we obtain $\sum_{n=0}^{\infty}\alpha_n(1-\beta_n)\frac{(\mu\bar{\gamma})^2-(\gamma l)^2}{\mu\bar{\gamma}} \ge \sum_{n=0}^{\infty}\alpha_n(1-d)\frac{(\mu\bar{\gamma})^2-(\gamma l)^2}{\mu\bar{\gamma}} = \infty$ and

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{2\mu\bar{\gamma}}{(\mu\bar{\gamma})^2 - (\gamma l)^2}\big\langle(\gamma f - \mu V)x^*, y_n - x^*\big\rangle \\
&\quad = \limsup_{n\to\infty}\frac{2\mu\bar{\gamma}}{(\mu\bar{\gamma})^2 - (\gamma l)^2}\big(\big\langle(\gamma f - \mu V)x^*, x_n - x^*\big\rangle + \big\langle(\gamma f - \mu V)x^*, y_n - x_n\big\rangle\big) \\
&\quad = \limsup_{n\to\infty}\frac{2\mu\bar{\gamma}}{(\mu\bar{\gamma})^2 - (\gamma l)^2}\big\langle(\gamma f - \mu V)x^*, x_n - x^*\big\rangle \le 0
\end{aligned}$$

(due to (3.42)). Applying Lemma 2.8 to (3.43), we infer that the sequence $\{x_n\}$ converges strongly to x^*. This completes the proof. □

In a similar way, we can conclude to another theorem, as follows.

Theorem 3.2 *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1,\dots,N$, be sequences in* $(0,1)$ *such that $\beta_{n,i} \to \beta_i$ for each index i as $n \to \infty$. Suppose that there exists $k \in \{1,\dots,N\}$ for which $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1,\dots,N\}$ the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that* (H0), (H7), *and* (H8) *hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;

(ii) *if $i \le k_0$ and $\beta_{n,i} \to \beta_i$ then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;

(iii) *if $\beta_{n,i} \to \beta_i \neq 0$ then β_i lies in* $(0,1)$.

Then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, and $\{y_n\}$ defined by scheme (3.1) *all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ if and only if $\|y_n - Ty_n\| \to 0$ as $n \to \infty$, where $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ is the unique solution of the VIP*

$$\big\langle(\gamma f - \mu V)x^*, x - x^*\big\rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Remark 3.5 According to the above argument process for Theorems 3.1 and 3.2, we can readily see that if in scheme (3.1), the iterative step $y_n = \alpha_n \gamma f(y_{n,N}) + (I - \alpha_n \mu V)Gy_{n,N}$ is replaced by the iterative one $y_n = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu V)Gy_{n,N}$, then Theorems 3.1 and 3.2 remain valid.

Remark 3.6 Theorems 3.1 and 3.2 improve, extend, supplement, and develop [13], Theorems 3.1 and 3.2 and [33], Theorems 3.12 and 3.13 in the following aspects.

(i) The multi-step iterative scheme (3.1) of [13] is extended to develop our relaxed extragradient iterative scheme (3.1) by virtue of Korpelevich's extragradient method and the strongly positive bounded linear operator approach. The iterative scheme (3.1) is based on Korpelevich's extragradient method, the viscosity approximation method [42] (see also [43]), Mann's iterative method, and the strongly positive bounded linear operator approach.

(ii) The argument techniques in our Theorems 3.1 and 3.2 are very different from the techniques in [13], Theorems 3.1 and 3.2 and [33], Theorems 3.12 and 3.13, because we make use of the properties of strict pseudocontractions (see Lemmas 2.3 and 2.4), the ones of resolvent operators and maximal monotone mappings (see Remark 2.1 and Lemmas 2.12-2.16), the ones of the resolvent operator associated with Θ and h (see Lemmas 2.9-2.11), the fixed point problem $x^* = Gx^*$ ($\Leftrightarrow$ GSVI (1.6)) (see Proposition 2.2), and the ones of strongly positive boundedness linear operators (see Lemma 2.7).

(iii) The problem of finding an element of $\mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \cap \Xi$ in our Theorems 3.1 and 3.2 is more general and more subtle than the one of finding an element of $\mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h)$ in [33], Theorems 3.12 and 3.13 (where T is a nonexpansive mapping) and the one of finding an element of $\mathrm{Fix}(T) \cap \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{GMEP}(\Theta, h) \cap \Xi$ in [13], Theorems 3.1 and 3.2 (where T is a strict pseudocontraction).

(iv) Our Theorems 3.1 and 3.2 generalizes from the nonexpansive mapping T to the strict pseudocontraction T and extend [33], Theorems 3.12 and 3.13 to the setting of GSVI (1.6), hierarchical minimization (3.40) and finitely many variational inclusions for maximal monotone and inverse-strongly monotone mappings. In the meantime, our Theorems 3.1 and 3.2 extend [13], Theorems 3.1 and 3.2 to the setting of hierarchical minimization (3.40) and finitely many variational inclusions for maximal monotone and inverse-strongly monotone mappings.

4 Applications

For a given nonlinear mapping $A : C \to H$, we consider the variational inequality problem (VIP) of finding $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

We will indicate by $\mathrm{VI}(C, A)$ the set of solutions of the VIP (4.1).

Recall that if u is a point in C, then the following relation holds:

$$u \in \mathrm{VI}(C, A) \quad \Leftrightarrow \quad u = P_C(I - \lambda A)u, \quad \forall \lambda > 0.$$

In the meantime, it is easy to see that the following relation holds:

$$\text{GSVI (1.6) with } F_2 = 0 \quad \Leftrightarrow \quad \text{VIP (4.1) with } A = F_1. \tag{4.2}$$

An operator $A : C \to H$ is said to be an α-inverse-strongly monotone operator if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

As an example, we recall that the α-inverse-strongly monotone operators are firmly nonexpansive mappings if $\alpha \geq 1$ and that every α-inverse-strongly monotone operator is also $\frac{1}{\alpha}$-Lipschitz-continuous (see [45]).

Let us observe also that, if A is α-inverse-strongly monotone, the mappings $P_C(I - \lambda A)$ are nonexpansive for all $\lambda \in (0, 2\alpha]$ since they are compositions of nonexpansive mappings (see p.419 in [45]).

Let us consider $\tilde{S}_1, \ldots, \tilde{S}_K$ to be a finite number of nonexpansive self-mappings on C and $A_1, \ldots, A_N$ to be a finite number of α-inverse-strongly monotone operators. Let $T : H \to H$ be a ξ-strict pseudocontraction on C with fixed points. Let us consider the mixed problem of finding $x^* \in \operatorname{Fix}(T) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k)$ such that

$$\begin{cases} \langle (I - \tilde{S}_1)x^*, y - x^*\rangle \geq 0, \quad \forall y \in \operatorname{Fix}(T) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k), \\ \langle (I - \tilde{S}_2)x^*, y - x^*\rangle \geq 0, \quad \forall y \in \operatorname{Fix}(T) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k), \\ \cdots \\ \langle (I - \tilde{S}_K)x^*, y - x^*\rangle \geq 0, \quad \forall y \in \operatorname{Fix}(T) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k), \\ \langle A_1 x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \\ \langle A_2 x^*, y - x^*\rangle \geq 0, \quad \forall y \in C, \\ \cdots \\ \langle A_N x^*, y - x^*\rangle \geq 0, \quad \forall y \in C. \end{cases} \tag{4.3}$$

Let us call (SVI) the set of solutions of the $(N + K)$-system. This problem is equivalent to finding a common fixed point of T, $\{P_{\operatorname{Fix}(T)\cap\operatorname{GMEP}(\Theta,h)\cap\Xi\cap\bigcap_{k=1}^M \mathrm{I}(B_k,R_k)}\tilde{S}_i\}_{i=1}^K$, $\{P_C(I - \lambda A_i)\}_{i=1}^N$. So we claim that the following holds.

Theorem 4.1 *Let us suppose that* $\Omega = \operatorname{Fix}(T) \cap (\mathrm{SVI}) \cap \operatorname{GMEP}(\Theta, h) \cap \Xi \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \neq \emptyset$. *Fix* $\lambda > 0$. *Let* $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \ldots, (K + N)$, *be sequences in* $(0,1)$ *such that* $0 < \liminf_{n\to\infty} \beta_{n,i} \leq \limsup_{n\to\infty} \beta_{n,i} < 1$ *for all indices i. Moreover, let us suppose that* (H0)-(H6) *hold. Then the sequences* $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, *and* $\{y_n\}$ *explicitly defined by scheme*

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ v_n = J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}B_M)J_{R_{M-1},\lambda_{M-1,n}}(I - \lambda_{M-1,n}B_{M-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ y_{n,1} = \beta_{n,1}P_{\operatorname{Fix}(T)\cap\operatorname{GMEP}(\Theta,h)\cap\Xi\cap\bigcap_{k=1}^M \mathrm{I}(B_k,R_k)}\tilde{S}_1 v_n + (1 - \beta_{n,1})v_n, \\ y_{n,i} = \beta_{n,i}P_{\operatorname{Fix}(T)\cap\operatorname{GMEP}(\Theta,h)\cap\Xi\cap\bigcap_{k=1}^M \mathrm{I}(B_k,R_k)}\tilde{S}_i v_n + (1 - \beta_{n,i})y_{n,i-1}, \quad i = 2, \ldots, K, \\ y_{n,K+j} = \beta_{n,K+j}P_C(I - \lambda A_j)v_n + (1 - \beta_{n,K+j})y_{n,K+j-1}, \quad j = 1, \ldots, N, \\ y_n = \alpha_n \gamma f(y_{n,K+N}) + (I - \alpha_n \mu V)Gy_{n,K+N}; \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \end{cases} \tag{4.4}$$

all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *if and only if* $\|y_n - Ty_n\| \to 0$ *as* $n \to \infty$, *where* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^* \rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2} \langle Vx, x\rangle - \Psi(x),$$

where Ψ *is a potential function for* γf.

Theorem 4.2 *Let us suppose that* $\Omega \neq \emptyset$. *Fix* $\lambda > 0$. *Let* $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \dots, (K+N)$, *be sequences in* $(0,1)$ *and* $\beta_{n,i} \to \beta_i$ *for all* i *as* $n \to \infty$. *Suppose that there exists* $k \in \{1, \dots, K + N\}$ *such that* $\beta_{n,k} \to 0$ *as* $n \to \infty$. *Let* $k_0 \in \{1, \dots, K+N\}$ *be the largest index for which* $\beta_{n,k_0} \to 0$. *Moreover, let us suppose that* (H0), (H7), *and* (H8) *hold and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as* $n \to \infty$;
(ii) *if* $i \le k_0$ *and* $\beta_{n,i} \to 0$ *then* $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ *as* $n \to \infty$;
(iii) *if* $\beta_{n,i} \to \beta_i \neq 0$ *then* β_i *lies in* $(0,1)$.

Then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, *and* $\{y_n\}$ *explicitly defined by scheme* (4.4) *all converge strongly to* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *if and only if* $\|y_n - Ty_n\| \to 0$ *as* $n \to \infty$, *where* $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$ *is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^* \rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2} \langle Vx, x\rangle - \Psi(x),$$

where Ψ *is a potential function for* γf.

Remark 4.1 If in system (4.3), $F_1 = F_2 = A_1 = \cdots = A_N = 0$, $B_1 = \cdots = B_M = R_1 = \cdots = R_M = 0$, and T is a nonexpansive mapping, we obtain a system of hierarchical fixed point problems introduced by Mainge and Moudafi [26, 28].

On the other hand, if $S : C \to C$ is a κ-strictly pseudocontractive mapping, that is, there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C,$$

then $A = I - S$ is $\frac{1-\kappa}{2}$-inverse-strongly monotone; see [52].

Utilizing Theorems 3.1 and 3.2, we also give two strong convergence theorems for finding a common element of the solution set GMEP(Θ, h) of GMEP (1.2), the solution set $\bigcap_{k=1}^M \mathrm{I}(B_k, R_k)$ of finitely many variational inclusions and the common fixed point set $\bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{Fix}(S)$ of finitely many nonexpansive mappings $S_i : C \to C$, $i = 1, \dots, N$, and a κ-strictly pseudocontractive mapping S.

Theorem 4.3 *Let $\nu_1 \in (0, 1-\kappa)$. Let us suppose that $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{Fix}(S) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \dots, N$, be sequences in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_{n,i} \le \limsup_{n\to\infty} \beta_{n,i} < 1$ for all indices i. Moreover, let us suppose that we have* (H0)-(H6) *with $\gamma_n = 0$, $\forall n \ge 0$. Then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, and $\{y_n\}$ generated explicitly by*

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ v_n = J_{R_M,\lambda_{M,n}}(I - \lambda_{M,n}B_M)J_{R_{M-1},\lambda_{M-1,n}}(I - \lambda_{M-1,n}B_{M-1}) \cdots J_{R_1,\lambda_{1,n}}(I - \lambda_{1,n}B_1)u_n, \\ y_{n,1} = \beta_{n,1}S_1 v_n + (1-\beta_{n,1})v_n, \\ y_{n,i} = \beta_{n,i}S_i v_n + (1-\beta_{n,i})y_{n,i-1}, \quad i = 2, \dots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1-\alpha_n)((1-\nu_1)y_{n,N} + \nu_1 S y_{n,N}), \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad \forall n \ge 0, \end{cases} \tag{4.5}$$

all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$, which is the unique solution of the VIP

$$\langle (\gamma f - \mu V)x^*, x - x^*\rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega} \frac{\mu}{2}\langle Vx, x\rangle - \Psi(x),$$

where Ψ is a potential function for γf.

Proof In Theorem 3.1, put $F_1 = A = I - S$ and $F_2 = 0$. Then A is $\frac{1-\kappa}{2}$-inverse-strongly monotone. Hence we deduce that $\mathrm{Fix}(S) = \mathrm{VI}(C, A) = \Gamma$ and

$$\begin{aligned} Gy_{n,N} &= P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)y_{n,N} \\ &= P_C(I - \nu_1 F_1)y_{n,N} \\ &= (1-\nu_1)y_{n,N} + \nu_1 S y_{n,N}. \end{aligned}$$

Thus, in terms of Theorem 3.1, we obtain the desired result. □

Theorem 4.4 *Let $\nu_1 \in (0, 1-\kappa)$. Let us suppose that $\Omega = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \cap \mathrm{Fix}(S) \cap \mathrm{GMEP}(\Theta, h) \cap \bigcap_{k=1}^M \mathrm{I}(B_k, R_k) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_{n,i}\}$, $i = 1, \dots, N$, be sequences in $(0,1)$ such that $\beta_{n,i} \to \beta_i$ for all i as $n \to \infty$. Suppose that there exists $k \in \{1, \dots, N\}$ for which $\beta_{n,k} \to 0$ as $n \to \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index for which $\beta_{n,k_0} \to 0$. Moreover, let us suppose that we have* (H0), (H7), *and* (H8) *with $\gamma_n = 0$, $\forall n \ge 0$, and*

(i) $\frac{\alpha_n}{\beta_{n,k_0}} \to 0$ *as $n \to \infty$;*
(ii) *if $i \le k_0$ and $\beta_{n,i} \to 0$ then $\frac{\beta_{n,k_0}}{\beta_{n,i}} \to 0$ as $n \to \infty$;*
(iii) *if $\beta_{n,i} \to \beta_i \neq 0$ then β_i lies in* $(0,1)$.

Then the sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$, and $\{y_n\}$, generated explicitly by (4.5), *all converge strongly to $x^* = P_\Omega(I - (\mu V - \gamma f))x^*$, which is the unique solution of the VIP*

$$\langle (\gamma f - \mu V)x^*, x - x^*\rangle \le 0, \quad \forall x \in \Omega,$$

or, equivalently, the unique solution of the minimization problem

$$\min_{x\in\Omega}\frac{\mu}{2}\langle Vx,x\rangle-\Psi(x),$$

where Ψ is a potential function for γf.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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