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Hybrid extragradient viscosity method for general system of variational inequalities

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Abstract

In this paper, we introduce a hybrid extragradient viscosity iterative algorithm for finding a common element of the set of solutions of a general mixed equilibrium problem, the set of solutions of a general system of variational inequalities, the set of solutions of a split feasibility problem (SFP), and the set of common fixed points of finitely many nonexpansive mappings and a strict pseudocontraction in a real Hilbert space. The iterative algorithm is based on Korpelevich's extragradient method, viscosity approximation method, Mann's iteration method, hybrid steepest-descent method and gradient-projection method (GPM) with regularization. We derive the strong convergence of the iterative algorithm to a common element of these sets, which also solves some hierarchical variational inequality.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C be a nonempty closed convex subset of H and P_C be the metric projection of H onto C . Let $S : C \rightarrow C$ be a self-mapping on C . We denote by $\text{Fix}(S)$ the set of fixed points of S and by \mathbf{R} the set of all real numbers. A mapping $\mathcal{A} : C \rightarrow H$ is called L -Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$ then \mathcal{A} is called a nonexpansive mapping; if $L \in [0, 1)$ then \mathcal{A} is called a contraction. A mapping $T : C \rightarrow C$ is called ξ -strictly pseudocontractive if there exists a constant $\xi \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \xi \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

In particular, if $\xi = 0$, then T is a nonexpansive mapping.

Let $\mathcal{A} : C \rightarrow H$ be a nonlinear mapping on C . We consider the following variational inequality problem (VIP): find a point $\bar{x} \in C$ such that

$$\langle \mathcal{A}\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution set of VIP (1.1) is denoted by $VI(C, \mathcal{A})$.

VIP (1.1) was first discussed by Lions [1] and now it is well known. Variational inequalities have extensively been investigated; see the monographs [2–6]. It is well known that if \mathcal{A} is a strongly monotone and Lipschitz continuous mapping on C , then VIP (1.1) has a unique solution. In the literature, the recent research work shows that variational inequalities like VIP (1.1) cover several topics, for example, monotone inclusions, convex optimization and quadratic minimization over fixed point sets; see [7–11] for more details.

In 1976, Korpelevich [12] proposed an iterative algorithm for solving VIP (1.1) in the Euclidean space \mathbf{R}^n :

$$\begin{cases} y_n = P_C(x_n - \tau \mathcal{A}x_n), \\ x_{n+1} = P_C(x_n - \tau \mathcal{A}y_n), \end{cases} \quad \forall n \geq 0,$$

with $\tau > 0$ a given number, which is known as the extragradient method. The literature on the VIP is vast, and Korpelevich’s extragradient method has received great attention given by many authors who improved it in various ways; see, *e.g.*, [11, 13–21] and the references therein, to name but a few.

On the other hand, let C and Q be nonempty closed convex subsets of infinite-dimensional real Hilbert spaces H and \mathcal{H} , respectively. The split feasibility problem (SFP) is to find a point x^* with the property

$$x^* \in C \quad \text{and} \quad Ax^* \in Q, \tag{1.2}$$

where $A \in B(H, \mathcal{H})$ and $B(H, \mathcal{H})$ denotes the family of all bounded linear operators from H to \mathcal{H} . We denote by Γ the solution set of the SFP.

In 1994, the SFP was first introduced by Censor and Elfving [22], in finite-dimensional Hilbert spaces, for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. A number of image reconstruction problems can be formulated as the SFP; see, *e.g.*, [23] and the references therein. Recently, it has been found that the SFP can also be applied to study intensity-modulated radiation therapy (IMRT); see, *e.g.*, [24, 25] and the references therein. In the recent past, a wide variety of iterative methods have been used in signal processing and image reconstruction and for solving the SFP; see, *e.g.*, [13, 15, 18, 19, 23–28] and the references therein. A seemingly more popular algorithm that solves the SFP is the CQ algorithm of Byrne [23, 27] which is found to be a gradient-projection method (GPM) in convex minimization. However, it remains a challenge how to implement the CQ algorithm in the case where the projections P_C and/or P_Q fail to have closed-form expressions, though theoretically we can prove the (weak) convergence of the algorithm.

Very recently, Xu [26] gave a continuation of the study on the CQ algorithm and its convergence. He applied Mann’s algorithm to the SFP and proposed an averaged CQ algo-

rithm which was proved to be weakly convergent to a solution of the SFP. He also established the strong convergence result, which shows that the minimum-norm solution can be obtained.

Throughout this paper, assume that the SFP is consistent, that is, the solution set Γ of the SFP is nonempty. Let $f : H \rightarrow \mathbf{R}$ be a continuous differentiable function. The minimization problem

$$\min_{x \in C} f(x) := \frac{1}{2} \|Ax - P_Q Ax\|^2$$

is ill-posed. Therefore, Xu [26] considered the following Tikhonov regularization problem:

$$\min_{x \in C} f_\alpha(x) := \frac{1}{2} \|Ax - P_Q Ax\|^2 + \frac{1}{2} \alpha \|x\|^2,$$

where $\alpha > 0$ is the regularization parameter.

Very recently, by combining the gradient-projection method with regularization and extragradient method due to Nadezhkina and Takahashi [14], Ceng *et al.* [19] proposed a Mann-type extragradient-like algorithm, and proved that the sequences generated by the proposed algorithm converge weakly to a common solution of SFP (1.2) and the fixed point problem of a nonexpansive mapping.

Theorem CAY (see Theorem 3.2 in [19]) *Let $T : C \rightarrow C$ be a nonexpansive mapping such that $\text{Fix}(T) \cap \Gamma \neq \emptyset$. Assume that $0 < \lambda < \frac{2}{\|A\|^2}$, and let $\{x_n\}$ and $\{y_n\}$ be the sequences in C generated by the following Mann-type extragradient-like algorithm:*

$$\begin{cases} x_0 = x \in C & \text{chosen arbitrarily,} \\ y_n = (1 - \beta_n)x_n + \beta_n P_C(x_n - \lambda \nabla f_{\alpha_n}(x_n)), \\ x_{n+1} = \gamma_n x_n + (1 - \gamma_n) T P_C(y_n - \lambda \nabla f_{\alpha_n}(y_n)), & \forall n \geq 0, \end{cases}$$

where the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

- (i) $\sum_{n=0}^\infty \alpha_n < \infty$;
- (ii) $\{\beta_n\} \subset [0, 1]$ and $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;
- (iii) $\{\gamma_n\} \subset [0, 1]$ and $0 < \liminf_{n \rightarrow \infty} \gamma_n \leq \limsup_{n \rightarrow \infty} \gamma_n < 1$.

Then both the sequences $\{x_n\}$ and $\{y_n\}$ converge weakly to an element $z \in \text{Fix}(T) \cap \Gamma$.

In this paper, we consider the following general mixed equilibrium problem (GMEP) (see also [29, 30]) of finding $x \in C$ such that

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C, \tag{1.3}$$

where $\Theta, h : C \times C \rightarrow \mathbf{R}$ are two bi-functions. We denote the set of solutions of GMEP (1.3) by $\text{GMEP}(\Theta, h)$. GMEP (1.3) is very general; for example, it includes the following equilibrium problems as special cases.

As an example, in [16, 31, 32] the authors considered and studied the generalized equilibrium problem (GEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C.$$

The set of solutions of GEP is denoted by $\text{GEP}(\Theta, A)$.

In [29, 33, 34], the authors considered and studied the mixed equilibrium problem (MEP) which is to find $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in C.$$

The set of solutions of MEP is denoted by $\text{MEP}(\Theta, \varphi)$.

In [35–37], the authors considered and studied the equilibrium problem (EP) which is to find $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall y \in C.$$

The set of solutions of EP is denoted by $\text{EP}(\Theta)$. It is worth to mention that the EP is a unified model of several problems, namely, variational inequality problems, optimization problems, saddle point problems, complementarity problems, fixed point problems, Nash equilibrium problems, etc.

Throughout this paper, it is assumed as in [38] that $\Theta : C \times C \rightarrow \mathbf{R}$ is a bi-function satisfying conditions $(\theta 1)$ - $(\theta 3)$ and $h : C \times C \rightarrow \mathbf{R}$ is a bi-function with restrictions (h1)-(h3), where

- ($\theta 1$) $\Theta(x, x) = 0$ for all $x \in C$;
- ($\theta 2$) Θ is monotone (i.e., $\Theta(x, y) + \Theta(y, x) \leq 0, \forall x, y \in C$) and upper hemicontinuous in the first variable, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} \Theta(tz + (1 - t)x, y) \leq \Theta(x, y);$$

- ($\theta 3$) Θ is lower semicontinuous and convex in the second variable;
- (h1) $h(x, x) = 0$ for all $x \in C$;
- (h2) h is monotone and weakly upper semicontinuous in the first variable;
- (h3) h is convex in the second variable.

For $r > 0$ and $x \in H$, let $T_r : H \rightarrow 2^C$ be a mapping defined by

$$T_r x = \left\{ z \in C : \Theta(z, y) + h(z, y) + \frac{1}{r}(y - z, z - x) \geq 0, \forall y \in C \right\}$$

called the resolvent of Θ and h .

Assume that C is the fixed point set of a nonexpansive mapping $T : H \rightarrow H$, i.e., $C = \text{Fix}(T)$. Let $F : H \rightarrow H$ be η -strongly monotone and κ -Lipschitzian with positive constants $\eta, \kappa > 0$. Let $u_0 \in H$ be given arbitrarily and $\{\lambda_n\}_{n=1}^\infty$ be a sequence in $[0, 1]$. The hybrid steepest-descent method introduced by Yamada [39] is the algorithm

$$u_{n+1} := T^{\lambda_{n+1}} u_n = (I - \lambda_{n+1} \mu F) T u_n, \quad \forall n \geq 0, \tag{1.4}$$

where I is the identity mapping on H .

In 2003, Xu and Kim [40] proved the following strong convergence result.

Theorem XK (see Theorem 3.1 in [40]) *Assume that $0 < \mu < 2\eta/\kappa^2$. Assume also that the control conditions hold for $\{\lambda_n\}_{n=1}^\infty$: $\lim_{n \rightarrow \infty} \lambda_n = 0, \sum_{n=1}^\infty \lambda_n = \infty$ and $\lim_{n \rightarrow \infty} \lambda_n/\lambda_{n+1} = 1$*

(or equivalently, $\lim_{n \rightarrow \infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$). Then the sequence $\{u_n\}$ generated by algorithm (1.4) converges strongly to the unique solution u^* in $\text{Fix}(T)$ to the hierarchical VIP:

$$\langle Fu^*, v - u^* \rangle \geq 0, \quad \forall v \in \text{Fix}(T). \tag{1.5}$$

Let $F_1, F_2 : C \rightarrow H$ be two mappings. Consider the following general system of variational inequalities (GSVI) of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 F_1 y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \nu_2 F_2 x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.6}$$

where $\nu_1 > 0$ and $\nu_2 > 0$ are two constants. The solution set of GSVI (1.6) is denoted by $\text{GSVI}(C, F_1, F_2)$.

In particular, if $F_1 = F_2 = \mathcal{A}$, then the GSVI (1.6) reduces to the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \nu_1 \mathcal{A} y^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \nu_2 \mathcal{A} x^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases}$$

which is defined by Verma [41] and it is called a new system of variational inequalities (NSVI). Further, if $x^* = y^*$ additionally, then the NSVI reduces to the classical VIP (1.1). In 2008, Ceng *et al.* [21] transformed GSVI (1.6) into the fixed point problem of the mapping $G = P_C(I - \nu_1 F_1)P_C(I - \nu_2 F_2)$, that is, $Gx^* = x^*$, where $y^* = P_C(I - \nu_2 F_2)x^*$. Throughout this paper, the fixed point set of the mapping G is denoted by \mathcal{E} .

On the other hand, if C is the fixed point set $\text{Fix}(T)$ of a nonexpansive mapping T and S is another nonexpansive mapping (not necessarily with fixed points), then VIP (1.1) becomes the variational inequality problem of finding $x^* \in \text{Fix}(T)$ such that

$$\langle (I - S)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \text{Fix}(T). \tag{1.7}$$

This problem, introduced by Mainge and Moudafi [34, 36], is called the hierarchical fixed point problem. It is clear that if S has fixed points, then they are solutions of VIP (1.7).

If S is a ρ -contraction (i.e., $\|Sx - Sy\| \leq \rho \|x - y\|$ for some $0 \leq \rho < 1$), the solution set of VIP (1.7) is a singleton and it is well known as the viscosity problem. This was previously introduced by Moudafi [7] and also developed by Xu [8]. In this case, it is easy to see that solving VIP (1.7) is equivalent to finding a fixed point of the nonexpansive mapping $P_{\text{Fix}(T)}S$, where $P_{\text{Fix}(T)}$ is the metric projection on the closed and convex set $\text{Fix}(T)$.

In 2012, Marino *et al.* [42] introduced a multi-step iterative scheme

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, & \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1 - \beta_{n,i})y_{n,i-1}, & i = 2, \dots, N, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Ty_{n,N}, \end{cases} \tag{1.8}$$

with $f : C \rightarrow C$ a ρ -contraction and $\{\alpha_n\}, \{\beta_{n,i}\} \subset (0, 1), \{r_n\} \subset (0, \infty)$, that generalizes the two-step iterative scheme in [10] for two nonexpansive mappings to a finite family of nonexpansive mappings $T, S_i : C \rightarrow C, i = 1, \dots, N$, and proved that the proposed scheme (1.8)

converges strongly to a common fixed point of the mappings that is also an equilibrium point of GMEP (1.3).

More recently, Marino, Muglia and Yao’s multi-step iterative scheme (1.8) was extended to develop the following relaxed viscosity iterative algorithm.

Algorithm CKW (see (3.1) in [43]) Let $f : C \rightarrow C$ be a ρ -contraction and $T : C \rightarrow C$ be a ξ -strict pseudocontraction. Let $S_i : C \rightarrow C$ be a nonexpansive mapping for each $i = 1, \dots, N$. Let $F_j : C \rightarrow H$ be ζ_j -inverse strongly monotone with $0 < v_j < \zeta_j$ for each $j = 1, 2$. Let $\Theta : C \times C \rightarrow \mathbf{R}$ be a bi-function satisfying conditions $(\theta 1)$ - $(\theta 3)$ and $h : C \times C \rightarrow \mathbf{R}$ be a bi-function with restrictions (h1)-(h3). Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, & \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1 - \beta_{n,i})y_{n,i-1}, & i = 2, \dots, N, \\ y_n = \alpha_n f(y_{n,N}) + (1 - \alpha_n)Gy_{n,N}, \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, & \forall n \geq 0, \end{cases} \tag{1.9}$$

where $G = P_C(I - v_1F_1)P_C(I - v_2F_2)$, $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$, $\{\gamma_n\}, \{\delta_n\}$ are sequences in $[0,1]$ with $\liminf_{n \rightarrow \infty} \delta_n > 0$ and $\beta_n + \gamma_n + \delta_n = 1, \forall n \geq 0$, $\{\beta_{n,i}\}$ is a sequence in $(0,1)$ for each $i = 1, \dots, N$, $(\gamma_n + \delta_n)\xi \leq \gamma_n, \forall n \geq 0$, and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$.

The authors [43] proved that the proposed scheme (1.9) converges strongly to a common fixed point of the mappings $T, S_i : C \rightarrow C, i = 1, \dots, N$, that is also an equilibrium point of GMEP (1.3) and a solution of GSVI (1.6).

In this paper, we introduce a hybrid extragradient viscosity iterative algorithm for finding a common element of the solution set $\text{GMEP}(\Theta, h)$ of GMEP (1.3), the solution set $\text{GSVI}(C, F_1, F_2)$ (i.e., \mathcal{E}) of GSVI (1.6), the solution set Γ of SFP (1.2), and the common fixed point set $\bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{Fix}(T)$ of finitely many nonexpansive mappings $S_i : C \rightarrow C, i = 1, \dots, N$, and a strictly pseudocontractive mapping $T : C \rightarrow C$, in the setting of the infinite-dimensional Hilbert space. The iterative algorithm is based on Korpelevich’s extragradient method, viscosity approximation method [7] (see also [8]), Mann’s iteration method, hybrid steepest-descent method [40] and gradient-projection method (GPM) with regularization. Our aim is to prove that the iterative algorithm converges strongly to a common element of these sets, which also solves some hierarchical variational inequality. We observe that related results have been derived say in [10, 13, 28, 34, 36, 37, 42–54].

2 Preliminaries

Throughout this paper, we assume that H is a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H . We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \rightarrow x$ to indicate that the sequence $\{x_n\}$ converges strongly to x . Moreover, we use $\omega_w(x_n)$ to denote the weak ω -limit set of the sequence $\{x_n\}$ and $\omega_s(x_n)$ to denote the strong ω -limit set of the sequence $\{x_n\}$, i.e.,

$$\omega_w(x_n) := \{x \in H : x_{n_i} \rightharpoonup x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}$$

and

$$\omega_s(x_n) := \{x \in H : x_{n_i} \rightarrow x \text{ for some subsequence } \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \rightarrow C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C).$$

The following properties of projections are useful and pertinent to our purpose.

Proposition 2.1 *Given any $x \in H$ and $z \in C$, one has*

- (i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0, \forall y \in C;$
- (ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2, \forall y \in C;$
- (iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2, \forall y \in H$, which hence implies that P_C is nonexpansive and monotone.

Definition 2.1 A mapping $T : H \rightarrow H$ is said to be

- (a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

- (b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H;$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where $S : H \rightarrow H$ is nonexpansive; projections are firmly nonexpansive.

Definition 2.2 A mapping $\mathcal{A} : C \rightarrow H$ is said to be

- (i) monotone if

$$\langle \mathcal{A}x - \mathcal{A}y, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

- (ii) η -strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle \mathcal{A}x - \mathcal{A}y, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C;$$

- (iii) α -inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{A}x - \mathcal{A}y, x - y \rangle \geq \alpha \|\mathcal{A}x - \mathcal{A}y\|^2, \quad \forall x, y \in C.$$

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that the projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

On the other hand, it is obvious that if $\mathcal{A} : C \rightarrow H$ is α -inverse-strongly monotone, then A is monotone and $\frac{1}{\alpha}$ -Lipschitz continuous. Moreover, we also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda\mathcal{A})u - (I - \lambda\mathcal{A})v\|^2 \leq \|u - v\|^2 + \lambda(\lambda - 2\alpha)\|\mathcal{A}u - \mathcal{A}v\|^2. \tag{2.1}$$

So, if $\lambda \leq 2\alpha$, then $I - \lambda\mathcal{A}$ is a nonexpansive mapping from C to H .

In 2008, Ceng *et al.* [21] transformed problem (1.6) into a fixed point problem in the following way.

Proposition 2.2 (see [21]) *For given $\bar{x}, \bar{y} \in C$, (\bar{x}, \bar{y}) is a solution of GSVI (1.6) if and only if \bar{x} is a fixed point of the mapping $G : C \rightarrow C$ defined by*

$$Gx = P_C(I - v_1F_1)P_C(I - v_2F_2)x, \quad \forall x \in C,$$

where $\bar{y} = P_C(I - v_2F_2)\bar{x}$.

In particular, if the mapping $F_j : C \rightarrow H$ is ζ_j -inverse-strongly monotone for $j = 1, 2$, then the mapping G is nonexpansive provided $v_j \in (0, 2\zeta_j]$ for $j = 1, 2$. We denote by \mathcal{E} the fixed point set of the mapping G .

The following result is easy to prove.

Proposition 2.3 (see [18]) *Given $x^* \in H$, the following statements are equivalent:*

- (i) x^* solves the SFP;
- (ii) x^* solves the fixed point equation

$$P_C(I - \lambda\nabla f)x^* = x^*,$$

where $\lambda > 0$, $\nabla f = A^*(I - P_Q)A$ and A^* is the adjoint of A ;

- (iii) x^* solves the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle \nabla f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

It is clear from Proposition 2.1 that

$$\Gamma = \text{Fix}(P_C(I - \lambda\nabla f)) = \text{VI}(C, \nabla f), \quad \forall \lambda > 0.$$

Definition 2.3 A mapping $T : H \rightarrow H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping, that is,

$$T \equiv (1 - \alpha)I + \alpha S,$$

where $\alpha \in (0, 1)$ and $S : H \rightarrow H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α -averaged. Thus firmly nonexpansive mappings (in particular, projections) are $\frac{1}{2}$ -averaged mappings.

Proposition 2.4 (see [55]) *Let $T : H \rightarrow H$ be a given mapping.*

- (i) *T is nonexpansive if and only if the complement $I - T$ is $\frac{1}{2}$ -ism.*
- (ii) *If T is ν -ism, then for $\gamma > 0$, γT is $\frac{\nu}{\gamma}$ -ism.*
- (iii) *T is averaged if and only if the complement $I - T$ is ν -ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α -averaged if and only if $I - T$ is $\frac{1}{2\alpha}$ -ism.*

Proposition 2.5 (see [55, 56]) *Let $S, T, V : H \rightarrow H$ be given operators.*

- (i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*
- (ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*
- (iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*
- (iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^N$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1 -averaged and T_2 is α_2 -averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α -averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.*
- (v) *If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^N \text{Fix}(T_i) = \text{Fix}(T_1 \cdots T_N).$$

The notation $\text{Fix}(T)$ denotes the set of all fixed points of the mapping T , that is, $\text{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 2.1 *Let X be a real inner product space. Then there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \quad \forall x, y \in X.$$

Lemma 2.2 *Let H be a real Hilbert space. Then the following hold:*

- (a) *$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle$ for all $x, y \in H$;*
- (b) *$\|\lambda x + \mu y\|^2 = \lambda \|x\|^2 + \mu \|y\|^2 - \lambda \mu \|x - y\|^2$ for all $x, y \in H$ and $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$;*
- (c) *if $\{x_n\}$ is a sequence in H such that $x_n \rightarrow x$, it follows that*

$$\limsup_{n \rightarrow \infty} \|x_n - y\|^2 = \limsup_{n \rightarrow \infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H.$$

It is clear that, in a real Hilbert space H , $T : C \rightarrow C$ is ξ -strictly pseudocontractive if and only if the following inequality holds:

$$\langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 - \frac{1 - \xi}{2} \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C.$$

This immediately implies that if T is a ξ -strictly pseudocontractive mapping, then $I - T$ is $\frac{1 - \xi}{2}$ -inverse strongly monotone; for further details, we refer to [57] and the references therein. It is well known that the class of strict pseudocontractions strictly includes the

class of nonexpansive mappings and that the class of pseudocontractions strictly includes the class of strict pseudocontractions.

Lemma 2.3 (see Proposition 2.1 in [57]) *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \rightarrow C$ be a mapping.*

- (i) *If T is a ξ -strictly pseudocontractive mapping, then T satisfies the Lipschitzian condition*

$$\|Tx - Ty\| \leq \frac{1 + \xi}{1 - \xi} \|x - y\|, \quad \forall x, y \in C.$$

- (ii) *If T is a ξ -strictly pseudocontractive mapping, then the mapping $I - T$ is semiclosed at 0, that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I - T)x_n \rightarrow 0$, then $(I - T)\tilde{x} = 0$.*
- (iii) *If T is ξ -(quasi-)strict pseudocontraction, then the fixed-point set $\text{Fix}(T)$ of T is closed and convex so that the projection $P_{\text{Fix}(T)}$ is well defined.*

Lemma 2.4 (see [17]) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow C$ be a ξ -strictly pseudocontractive mapping. Let γ and δ be two nonnegative real numbers such that $(\gamma + \delta)\xi \leq \gamma$. Then*

$$\|\gamma(x - y) + \delta(Tx - Ty)\| \leq (\gamma + \delta)\|x - y\|, \quad \forall x, y \in C.$$

Lemma 2.5 (see Demiclosedness principle in [58]) *Let C be a nonempty closed convex subset of a real Hilbert space H . Let S be a nonexpansive self-mapping on C with $\text{Fix}(S) \neq \emptyset$. Then $I - S$ is demiclosed. That is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some y , it follows that $(I - S)x = y$. Here I is the identity operator of H .*

Lemma 2.6 *Let $\mathcal{A} : C \rightarrow H$ be a monotone mapping. In the context of the variational inequality problem, the characterization of the projection (see Proposition 2.1(i)) implies*

$$u \in \text{VI}(C, \mathcal{A}) \iff u = P_C(u - \lambda \mathcal{A}u), \quad \lambda > 0.$$

Let C be a nonempty closed convex subset of a real Hilbert space H . We introduce some notations. Let λ be a number in $(0, 1]$ and let $\mu > 0$. Associating with a nonexpansive mapping $T : C \rightarrow C$, we define the mapping $T^\lambda : C \rightarrow H$ by

$$T^\lambda x := Tx - \lambda \mu F(Tx), \quad \forall x \in C,$$

where $F : C \rightarrow H$ is an operator such that, for some positive constants $\kappa, \eta > 0$, F is κ -Lipschitzian and η -strongly monotone on C ; that is, F satisfies the conditions

$$\|Fx - Fy\| \leq \kappa \|x - y\| \quad \text{and} \quad \langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2$$

for all $x, y \in C$.

Lemma 2.7 (see Lemma 3.1 in [40]) T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$; that is,

$$\|T^\lambda x - T^\lambda y\| \leq (1 - \lambda\tau)\|x - y\|, \quad \forall x, y \in C,$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1]$.

Lemma 2.8 (see Lemma 2.1 in [59]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying

$$a_{n+1} \leq (1 - \beta_n)a_n + \beta_n\gamma_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

- (i) $\{\beta_n\} \subset [0, 1]$ and $\sum_{n=0}^\infty \beta_n = \infty$;
- (ii) either $\limsup_{n \rightarrow \infty} \gamma_n \leq 0$ or $\sum_{n=0}^\infty \beta_n |\gamma_n| < \infty$;
- (iii) $\delta_n \geq 0$ for all $n \geq 0$, and $\sum_{n=1}^\infty \delta_n < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

In the sequel, we will indicate with $\text{GMEP}(\Theta, h)$ the solution set of GMEP (1.3).

Lemma 2.9 (see [38]) Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\Theta : C \times C \rightarrow \mathbf{R}$ be a bi-function satisfying conditions $(\theta 1)$ - $(\theta 3)$ and $h : C \times C \rightarrow \mathbf{R}$ be a bi-function with restrictions $(h 1)$ - $(h 3)$. Moreover, let us suppose that

- (H) for fixed $r > 0$ and $x \in C$, there exist bounded $K \subset C$ and $\hat{x} \in K$ such that for all $z \in C \setminus K$, $-\Theta(\hat{x}, z) + h(z, \hat{x}) + \frac{1}{r}(\hat{x} - z, z - x) < 0$.

For $r > 0$ and $x \in H$, the mapping $T_r : H \rightarrow 2^C$ (i.e., the resolvent of Θ and h) has the following properties:

- (i) $T_r x \neq \emptyset$;
- (ii) $T_r x$ is a singleton;
- (iii) T_r is firmly nonexpansive;
- (iv) $\text{GMEP}(\Theta, h) = \text{Fix}(T_r)$ and it is closed and convex.

Lemma 2.10 (see [38]) Let us suppose that $(\theta 1)$ - $(\theta 3)$, $(h 1)$ - $(h 3)$ and (H) hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then

$$\|T_{r_2} y - T_{r_1} x\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \|T_{r_2} y - y\|.$$

Lemma 2.11 (see [42]) Suppose that the hypotheses of Lemma 2.9 are satisfied. Let $\{r_n\}$ be a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$. Suppose that $\{x_n\}$ is a bounded sequence. Then the following statements are equivalent and true:

- (a) if $\|x_n - T_{r_n} x_n\| \rightarrow 0$ as $n \rightarrow \infty$, each weak cluster point of $\{x_n\}$ satisfies the problem

$$\Theta(x, y) + h(x, y) \geq 0, \quad \forall y \in C,$$

i.e., $\omega_w(x_n) \subseteq \text{GMEP}(\Theta, h)$;

- (b) the demiclosedness principle holds in the sense that if $x_n \rightharpoonup x^*$ and $\|x_n - T_{r_n} x_n\| \rightarrow 0$ as $n \rightarrow \infty$, then $(I - T_{r_k})x^* = 0$ for all $k \geq 1$.

Recall that a set-valued mapping $T : D(T) \subset H \rightarrow 2^H$ is called monotone if for all $x, y \in D(T), f \in Tx$ and $g \in Ty$ imply

$$\langle f - g, x - y \rangle \geq 0.$$

A set-valued mapping T is called maximal monotone if T is monotone and $(I + \lambda T)D(T) = H$ for each $\lambda > 0$, where I is the identity mapping of H . We denote by $G(T)$ the graph of T . It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in G(T)$, $\langle f - g, x - y \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Next we provide an example to illustrate the concept of maximal monotone mapping.

Let $\mathcal{A} : C \rightarrow H$ be a monotone, k -Lipschitz-continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{u \in H : \langle v - p, u \rangle \geq 0, \forall p \in C\}.$$

Define

$$\tilde{T}v = \begin{cases} \mathcal{A}v + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then it is known in [60] that \tilde{T} is maximal monotone and

$$0 \in \tilde{T}v \iff v \in \text{VI}(C, \mathcal{A}). \tag{2.2}$$

3 Main results

We now propose the following hybrid extragradient viscosity iterative scheme:

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_{n,1} = \beta_{n,1} S_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} S_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, \quad i = 2, \dots, N, \\ \tilde{y}_{n,N} = P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})), \\ y_n = P_C[\epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F) G P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}))], \\ x_{n+1} = \beta_n y_n + \gamma_n P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})) + \delta_n T P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})) \end{cases} \tag{3.1}$$

for all $n \geq 0$, where

- $F : C \rightarrow H$ is a κ -Lipschitzian and η -strongly monotone operator with positive constants $\kappa, \eta > 0$ and $V : C \rightarrow C$ is an l -Lipschitzian mapping with constant $l \geq 0$;
- $F_j : C \rightarrow H$ is ζ_j -inverse strongly monotone and $G := P_C(I - v_1 F_1) P_C(I - v_2 F_2)$ with $v_j \in (0, 2\zeta_j)$ for $j = 1, 2$;
- $T : C \rightarrow C$ is a ξ -strict pseudocontraction and $S_i : C \rightarrow C$ is a nonexpansive mapping for each $i = 1, \dots, N$;
- $\Theta, h : C \times C \rightarrow \mathbf{R}$ are two bi-functions satisfying the hypotheses of Lemma 2.9;
- $\{\lambda_n\}$ is a sequence in $(0, \frac{1}{\|A\|^2})$ with $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{1}{\|A\|^2}$;
- $0 < \mu < 2\eta/\kappa^2$ and $0 \leq \gamma l < \tau$ with $\tau := 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)}$;
- $\{\alpha_n\}$ is a sequence in $(0, \infty)$ with $\sum_{n=0}^{\infty} \alpha_n < \infty$;
- $\{\epsilon_n\}, \{\beta_n\}$ are sequences in $(0, 1)$ with $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$;

$\{\gamma_n\}, \{\delta_n\}$ are sequences in $[0, 1]$ with $\beta_n + \gamma_n + \delta_n = 1, \forall n \geq 0$;
 $\{\beta_{n,i}\}_{i=1}^N$ are sequences in $(0, 1)$ and $(\gamma_n + \delta_n)\xi \leq \gamma_n, \forall n \geq 0$;
 $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n \rightarrow \infty} r_n > 0$ and $\liminf_{n \rightarrow \infty} \delta_n > 0$.

We start our main result from the following series of propositions.

Proposition 3.1 *Let us suppose that $\Omega = \text{Fix}(T) \cap \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{GMEP}(\Theta, h) \cap \Xi \cap \Gamma \neq \emptyset$. Then the sequences $\{x_n\}, \{y_n\}, \{y_{n,i}\}$ for all $i, \{u_n\}$ are bounded.*

Proof Since $0 < \liminf_{n \rightarrow \infty} \lambda_n \leq \limsup_{n \rightarrow \infty} \lambda_n < \frac{1}{\|A\|^2}$ and $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$, we may assume, without loss of generality, that $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\} \subset [c, d] \subset (0, 1)$. Now, let us show that $P_C(I - \lambda \nabla f_\alpha)$ is σ -averaged for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, where

$$\sigma = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0, 1). \tag{3.2}$$

Indeed, it is easy to see that $\nabla f = A^*(I - P_Q)A$ is $\frac{1}{\|A\|^2}$ -ism, that is,

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq \frac{1}{\|A\|^2} \|\nabla f(x) - \nabla f(y)\|^2. \tag{3.3}$$

Observe that

$$\begin{aligned} & (\alpha + \|A\|^2) \langle \nabla f_\alpha(x) - \nabla f_\alpha(y), x - y \rangle \\ &= (\alpha + \|A\|^2) [\alpha \|x - y\|^2 + \langle \nabla f(x) - \nabla f(y), x - y \rangle] \\ &= \alpha^2 \|x - y\|^2 + \alpha \langle \nabla f(x) - \nabla f(y), x - y \rangle + \alpha \|A\|^2 \|x - y\|^2 \\ &\quad + \|A\|^2 \langle \nabla f(x) - \nabla f(y), x - y \rangle \\ &\geq \alpha^2 \|x - y\|^2 + 2\alpha \langle \nabla f(x) - \nabla f(y), x - y \rangle + \|\nabla f(x) - \nabla f(y)\|^2 \\ &= \|\alpha(x - y) + \nabla f(x) - \nabla f(y)\|^2 \\ &= \|\nabla f_\alpha(x) - \nabla f_\alpha(y)\|^2. \end{aligned} \tag{3.4}$$

Hence, it follows that $\nabla f_\alpha = \alpha I + A^*(I - P_Q)A$ is $\frac{1}{\alpha + \|A\|^2}$ -ism. Thus, $\lambda \nabla f_\alpha$ is $\frac{1}{\lambda(\alpha + \|A\|^2)}$ -ism according to Proposition 2.4(ii). By Proposition 2.4(iii), the complement $I - \lambda \nabla f_\alpha$ is $\frac{\lambda(\alpha + \|A\|^2)}{2}$ -averaged. Therefore, noting that P_C is $\frac{1}{2}$ -averaged and utilizing Proposition 2.5(iv), we know that for each $\lambda \in (0, \frac{2}{\alpha + \|A\|^2})$, $P_C(I - \lambda \nabla f_\alpha)$ is σ -averaged with

$$\sigma = \frac{1}{2} + \frac{\lambda(\alpha + \|A\|^2)}{2} - \frac{1}{2} \cdot \frac{\lambda(\alpha + \|A\|^2)}{2} = \frac{2 + \lambda(\alpha + \|A\|^2)}{4} \in (0, 1). \tag{3.5}$$

This shows that $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$, we have

$$a \leq \inf_{n \geq 0} \lambda_n \leq \sup_{n \geq 0} \lambda_n \leq b < \frac{1}{\|A\|^2} = \lim_{n \rightarrow \infty} \frac{1}{\alpha_n + \|A\|^2}. \tag{3.6}$$

Without loss of generality, we may assume that

$$a \leq \inf_{n \geq 0} \lambda_n \leq \sup_{n \geq 0} \lambda_n \leq b < \frac{1}{\alpha_n + \|A\|^2}, \quad \forall n \geq 0. \tag{3.7}$$

Consequently, it follows that for each integer $n \geq 0$, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is σ_n -averaged with

$$\sigma_n = \frac{1}{2} + \frac{\lambda_n(\alpha_n + \|A\|^2)}{2} - \frac{1}{2} \cdot \frac{\lambda_n(\alpha_n + \|A\|^2)}{2} = \frac{2 + \lambda_n(\alpha_n + \|A\|^2)}{4} \in (0, 1). \tag{3.8}$$

This immediately implies that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 0$.

For simplicity, we write $t_n = P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}))$ and

$$v_n = \epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F) G t_n$$

for all $n \geq 0$. Then $y_n = P_C v_n$ and $x_{n+1} = \beta_n y_n + \gamma_n t_n + \delta_n T t_n$.

First of all, take a fixed $p \in \Omega$ arbitrarily. We observe that

$$\|y_{n,1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

For all from $i = 2$ to $i = N$, by induction, one proves that

$$\|y_{n,i} - p\| \leq \beta_{n,i} \|u_n - p\| + (1 - \beta_{n,i}) \|y_{n,i-1} - p\| \leq \|u_n - p\| \leq \|x_n - p\|.$$

Thus we obtain that for every $i = 1, \dots, N$,

$$\|y_{n,i} - p\| \leq \|u_n - p\| \leq \|x_n - p\|. \tag{3.9}$$

For simplicity, we write $\tilde{p} = P_C(p - v_2 F_2 p)$, $\tilde{t}_n = P_C(t_n - v_2 F_2 t_n)$ and $z_n = P_C(\tilde{t}_n - v_1 F_1 \tilde{t}_n)$ for each $n \geq 0$. Then $z_n = G t_n$ and

$$p = P_C(I - v_1 F_1) \tilde{p} = P_C(I - v_1 F_1) P_C(I - v_2 F_2) p = G p.$$

Since $F_j : C \rightarrow H$ is ζ_j -inverse strongly monotone and $0 < v_j < 2\zeta_j$ for each $j = 1, 2$, we know that for all $n \geq 0$,

$$\begin{aligned} \|z_n - p\|^2 &= \|G t_n - p\|^2 \\ &= \|P_C(I - v_1 F_1) P_C(I - v_2 F_2) t_n - P_C(I - v_1 F_1) P_C(I - v_2 F_2) p\|^2 \\ &\leq \|(I - v_1 F_1) P_C(I - v_2 F_2) t_n - (I - v_1 F_1) P_C(I - v_2 F_2) p\|^2 \\ &= \|[P_C(I - v_2 F_2) t_n - P_C(I - v_2 F_2) p] \\ &\quad - v_1 [F_1 P_C(I - v_2 F_2) t_n - F_1 P_C(I - v_2 F_2) p]\|^2 \\ &\leq \|P_C(I - v_2 F_2) t_n - P_C(I - v_2 F_2) p\|^2 \\ &\quad + v_1 (v_1 - 2\zeta_1) \|F_1 P_C(I - v_2 F_2) t_n - F_1 P_C(I - v_2 F_2) p\|^2 \\ &\leq \|(I - v_2 F_2) t_n - (I - v_2 F_2) p\|^2 + v_1 (v_1 - 2\zeta_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2 \end{aligned}$$

$$\begin{aligned}
 &= \|(t_n - p) - \nu_2(F_2 t_n - F_2 p)\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2 \\
 &\leq \|t_n - p\|^2 + \nu_2(\nu_2 - 2\zeta_2)\|F_2 t_n - F_2 p\|^2 + \nu_1(\nu_1 - 2\zeta_1)\|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2 \\
 &\leq \|t_n - p\|^2.
 \end{aligned}
 \tag{3.10}$$

From (3.1), (3.9) and the nonexpansivity of $P_C(I - \lambda_n \nabla f_{\alpha_n})$, it follows that

$$\begin{aligned}
 \|\tilde{y}_{n,N} - p\| &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N} - P_C(I - \lambda_n \nabla f)p\| \\
 &\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N} - P_C(I - \lambda_n \nabla f_{\alpha_n})p\| \\
 &\quad + \|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \\
 &\leq \|y_{n,N} - p\| + \|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\| \\
 &\leq \|x_n - p\| + \lambda_n \alpha_n \|p\|.
 \end{aligned}
 \tag{3.11}$$

Utilizing Lemma 2.1, we also have

$$\begin{aligned}
 \|\tilde{y}_{n,N} - p\|^2 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N} - P_C(I - \lambda_n \nabla f)p\|^2 \\
 &= \|P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N} - P_C(I - \lambda_n \nabla f_{\alpha_n})p \\
 &\quad + P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\|^2 \\
 &\leq \|P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N} - P_C(I - \lambda_n \nabla f_{\alpha_n})p\|^2 \\
 &\quad + 2\langle P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p, \tilde{y}_{n,N} - p \rangle \\
 &\leq \|y_{n,N} - p\|^2 + 2\|P_C(I - \lambda_n \nabla f_{\alpha_n})p - P_C(I - \lambda_n \nabla f)p\| \|\tilde{y}_{n,N} - p\| \\
 &\leq \|x_n - p\|^2 + 2\|(I - \lambda_n \nabla f_{\alpha_n})p - (I - \lambda_n \nabla f)p\| \|\tilde{y}_{n,N} - p\| \\
 &\leq \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\|.
 \end{aligned}
 \tag{3.12}$$

Furthermore, utilizing Proposition 2.1(ii), we have

$$\begin{aligned}
 \|t_n - p\|^2 &\leq \|y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - p\|^2 - \|y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - t_n\|^2 \\
 &= \|y_{n,N} - p\|^2 - \|y_{n,N} - t_n\|^2 + 2\lambda_n \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), p - t_n \rangle \\
 &= \|y_{n,N} - p\|^2 - \|y_{n,N} - t_n\|^2 + 2\lambda_n (\langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \nabla f_{\alpha_n}(p), p - \tilde{y}_{n,N} \rangle \\
 &\quad + \langle \nabla f_{\alpha_n}(p), p - \tilde{y}_{n,N} \rangle + \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), \tilde{y}_{n,N} - t_n \rangle) \\
 &\leq \|y_{n,N} - p\|^2 - \|y_{n,N} - t_n\|^2 \\
 &\quad + 2\lambda_n (\langle \nabla f_{\alpha_n}(p), p - \tilde{y}_{n,N} \rangle + \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), \tilde{y}_{n,N} - t_n \rangle) \\
 &= \|y_{n,N} - p\|^2 - \|y_{n,N} - t_n\|^2 \\
 &\quad + 2\lambda_n [\langle (\alpha_n I + \nabla f)p, p - \tilde{y}_{n,N} \rangle + \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), \tilde{y}_{n,N} - t_n \rangle] \\
 &\leq \|y_{n,N} - p\|^2 - \|y_{n,N} - t_n\|^2 + 2\lambda_n [\alpha_n \langle p, p - \tilde{y}_{n,N} \rangle + \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), \tilde{y}_{n,N} - t_n \rangle] \\
 &= \|y_{n,N} - p\|^2 - \|y_{n,N} - \tilde{y}_{n,N}\|^2 - 2\langle y_{n,N} - \tilde{y}_{n,N}, \tilde{y}_{n,N} - t_n \rangle - \|\tilde{y}_{n,N} - t_n\|^2 \\
 &\quad + 2\lambda_n [\alpha_n \langle p, p - \tilde{y}_{n,N} \rangle + \langle \nabla f_{\alpha_n}(\tilde{y}_{n,N}), \tilde{y}_{n,N} - t_n \rangle]
 \end{aligned}$$

$$\begin{aligned}
 &= \|y_{n,N} - p\|^2 - \|y_{n,N} - \tilde{y}_{n,N}\|^2 - \|\tilde{y}_{n,N} - t_n\|^2 \\
 &\quad + 2\langle y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \tilde{y}_{n,N}, t_n - \tilde{y}_{n,N} \rangle + 2\lambda_n \alpha_n \langle p, p - \tilde{y}_{n,N} \rangle. \tag{3.13}
 \end{aligned}$$

In the meantime, by Proposition 2.1(i), we have

$$\begin{aligned}
 &\langle y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \tilde{y}_{n,N}, t_n - \tilde{y}_{n,N} \rangle \\
 &= \langle y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N}) - \tilde{y}_{n,N}, t_n - \tilde{y}_{n,N} \rangle \\
 &\quad + \langle \lambda_n \nabla f_{\alpha_n}(y_{n,N}) - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}), t_n - \tilde{y}_{n,N} \rangle \\
 &\leq \langle \lambda_n \nabla f_{\alpha_n}(y_{n,N}) - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}), t_n - \tilde{y}_{n,N} \rangle \\
 &\leq \lambda_n \|\nabla f_{\alpha_n}(y_{n,N}) - \nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \|t_n - \tilde{y}_{n,N}\| \\
 &\leq \lambda_n (\alpha_n + \|A\|^2) \|y_{n,N} - \tilde{y}_{n,N}\| \|t_n - \tilde{y}_{n,N}\|. \tag{3.14}
 \end{aligned}$$

So, from (3.9) and (3.11), we obtain

$$\begin{aligned}
 \|t_n - p\|^2 &\leq \|y_{n,N} - p\|^2 - \|y_{n,N} - \tilde{y}_{n,N}\|^2 - \|\tilde{y}_{n,N} - t_n\|^2 \\
 &\quad + 2\langle y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \tilde{y}_{n,N}, t_n - \tilde{y}_{n,N} \rangle + 2\lambda_n \alpha_n \langle p, p - \tilde{y}_{n,N} \rangle \\
 &\leq \|y_{n,N} - p\|^2 - \|y_{n,N} - \tilde{y}_{n,N}\|^2 - \|\tilde{y}_{n,N} - t_n\|^2 \\
 &\quad + 2\lambda_n (\alpha_n + \|A\|^2) \|y_{n,N} - \tilde{y}_{n,N}\| \|t_n - \tilde{y}_{n,N}\| + 2\lambda_n \alpha_n \langle p, p - \tilde{y}_{n,N} \rangle \\
 &\leq \|y_{n,N} - p\|^2 - \|y_{n,N} - \tilde{y}_{n,N}\|^2 - \|\tilde{y}_{n,N} - t_n\|^2 \\
 &\quad + \lambda_n^2 (\alpha_n + \|A\|^2)^2 \|y_{n,N} - \tilde{y}_{n,N}\|^2 + \|\tilde{y}_{n,N} - t_n\|^2 + 2\lambda_n \alpha_n \langle p, p - \tilde{y}_{n,N} \rangle \\
 &= \|y_{n,N} - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - \tilde{y}_{n,N}\| \\
 &\quad + (\lambda_n^2 (\alpha_n + \|A\|^2)^2 - 1) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\
 &\leq \|y_{n,N} - p\|^2 + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\
 &\leq \|y_{n,N} - p\|^2 + 2\lambda_n \alpha_n \|p\| [\|y_{n,N} - p\| + \lambda_n \alpha_n \|p\|] \\
 &\leq \|y_{n,N} - p\|^2 + 2\sqrt{2}\lambda_n \alpha_n \|p\| \|y_{n,N} - p\| + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \\
 &= (\|y_{n,N} - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 \\
 &\leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2. \tag{3.15}
 \end{aligned}$$

Hence, utilizing Lemma 2.7 we deduce from (3.9) and (3.15) that

$$\begin{aligned}
 \|y_n - p\| &= \|P_C v_n - p\| \\
 &\leq \|\epsilon_n \gamma (V y_{n,N} - V p) + (I - \epsilon_n \mu F) G t_n - (I - \epsilon_n \mu F) p + \epsilon_n (\gamma V - \mu F) p\| \\
 &\leq \epsilon_n \gamma \|V y_{n,N} - V p\| + \|(I - \epsilon_n \mu F) G t_n - (I - \epsilon_n \mu F) p\| + \epsilon_n \|(\gamma V - \mu F) p\| \\
 &\leq \epsilon_n \gamma l \|y_{n,N} - p\| + (1 - \epsilon_n \tau) \|t_n - p\| + \epsilon_n \|(\gamma V - \mu F) p\| \\
 &\leq \epsilon_n \gamma l \|y_{n,N} - p\| + (1 - \epsilon_n \tau) [\|y_{n,N} - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|] + \epsilon_n \|(\gamma V - \mu F) p\| \\
 &\leq (1 - \epsilon_n (\tau - \gamma l)) \|y_{n,N} - p\| + \epsilon_n \|(\gamma V - \mu F) p\| + \sqrt{2}\lambda_n \alpha_n \|p\| \\
 &\leq (1 - \epsilon_n (\tau - \gamma l)) \|x_n - p\| + \epsilon_n \|(\gamma V - \mu F) p\| + \sqrt{2}\lambda_n \alpha_n \|p\|
 \end{aligned}$$

$$\begin{aligned}
 &= (1 - \epsilon_n(\tau - \gamma l))\|x_n - p\| + \epsilon_n(\tau - \gamma l) \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l} + \sqrt{2}\lambda_n\alpha_n\|p\| \\
 &\leq \max\left\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\} + \sqrt{2}\lambda_n\alpha_n\|p\|. \tag{3.16}
 \end{aligned}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.4, we obtain from (3.15) and (3.16) that

$$\begin{aligned}
 \|x_{n+1} - p\| &= \|\beta_n(y_n - p) + \gamma_n(t_n - p) + \delta_n(Tt_n - p)\| \\
 &= \left\| \beta_n(y_n - p) + (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} [\gamma_n(t_n - p) + \delta_n(Tt_n - p)] \right\| \\
 &\leq \beta_n\|y_n - p\| + (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} [\gamma_n(t_n - p) + \delta_n(Tt_n - p)] \right\| \\
 &\leq \beta_n\|y_n - p\| + (\gamma_n + \delta_n)\|t_n - p\| \\
 &\leq \beta_n \left[\max\left\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\} + \sqrt{2}\lambda_n\alpha_n\|p\| \right] \\
 &\quad + (1 - \beta_n)(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|) \\
 &= \beta_n \max\left\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\} + (1 - \beta_n)\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\| \\
 &\leq \max\left\{\|x_n - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\} + \sqrt{2}\lambda_n\alpha_n\|p\|.
 \end{aligned}$$

By induction, we can prove

$$\|x_{n+1} - p\| \leq \max\left\{\|x_0 - p\|, \frac{\|(\gamma V - \mu F)p\|}{\tau - \gamma l}\right\} + \sum_{k=0}^n \sqrt{2}\lambda_k\alpha_k\|p\|, \quad \forall n \geq 0.$$

Since $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ and $\sum_{n=0}^\infty \alpha_n < \infty$, we know that $\{x_n\}$ is bounded, and so are the sequences $\{u_n\}, \{v_n\}, \{t_n\}, \{\tilde{t}_n\}, \{y_n\}, \{\tilde{y}_{n,N}\}, \{y_{n,i}\}$ for each $i = 1, \dots, N$. Since $\|Tt_n - p\| \leq \frac{1+\xi}{1-\xi}\|t_n - p\|$, $\{Tt_n\}$ is also bounded. □

Proposition 3.2 *Let us suppose that $\Omega \neq \emptyset$. Moreover, let us suppose that the following hold:*

- (H0) $\lim_{n \rightarrow \infty} \epsilon_n = 0$ and $\sum_{n=0}^\infty \epsilon_n = \infty$;
- (H1) $\lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n} = 0$ and $\lim_{n \rightarrow \infty} \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} = 0$;
- (H2) $\lim_{n \rightarrow \infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\epsilon_n} = 0$ for each $i = 1, \dots, N$;
- (H3) $\sum_{n=1}^\infty |\epsilon_n - \epsilon_{n-1}| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n} = 0$;
- (H4) $\sum_{n=1}^\infty |r_n - r_{n-1}| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|r_n - r_{n-1}|}{\epsilon_n} = 0$;
- (H5) $\sum_{n=1}^\infty |\beta_n - \beta_{n-1}| < \infty$ or $\lim_{n \rightarrow \infty} \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n} = 0$;
- (H6) $\sum_{n=1}^\infty \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| < \infty$ or $\lim_{n \rightarrow \infty} \frac{1}{\epsilon_n} \left| \frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}} \right| = 0$.

If $\|u_n - u_{n-1}\| = o(\epsilon_n)$, then $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, i.e., $\{x_n\}$ is asymptotically regular.

Proof First, it is known that $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\} \subset [c, d] \subset (0, 1)$ as in the proof of Proposition 3.1. Taking into account $\liminf_{n \rightarrow \infty} r_n > 0$, we may assume, without loss of generality, that $\{r_n\} \subset [\bar{r}, \infty)$ for some $\bar{r} > 0$. First, we write $x_n = \beta_{n-1}y_{n-1} + (1 -$

$\beta_{n-1})w_{n-1}, \forall n \geq 1$, where $w_{n-1} = \frac{x_n - \beta_{n-1}y_{n-1}}{1 - \beta_{n-1}}$. It follows that for all $n \geq 1$,

$$\begin{aligned} w_n - w_{n-1} &= \frac{x_{n+1} - \beta_n y_n}{1 - \beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1 - \beta_{n-1}} \\ &= \frac{\gamma_n t_n + \delta_n T t_n}{1 - \beta_n} - \frac{\gamma_{n-1} t_{n-1} + \delta_{n-1} T t_{n-1}}{1 - \beta_{n-1}} \\ &= \frac{\gamma_n(t_n - t_{n-1}) + \delta_n(T t_n - T t_{n-1})}{1 - \beta_n} + \left(\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right) t_{n-1} \\ &\quad + \left(\frac{\delta_n}{1 - \beta_n} - \frac{\delta_{n-1}}{1 - \beta_{n-1}} \right) T t_{n-1}. \end{aligned} \tag{3.17}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.4 we have

$$\| \gamma_n(t_n - t_{n-1}) + \delta_n(T t_n - T t_{n-1}) \| \leq (\gamma_n + \delta_n) \| t_n - t_{n-1} \|. \tag{3.18}$$

Next, we estimate $\|y_n - y_{n-1}\|$. Indeed, according to $\lambda_n(\alpha_n + \|A\|^2) < 1$,

$$\begin{aligned} \|t_n - t_{n-1}\| &\leq \| (y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})) - (y_{n-1,N} - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(\tilde{y}_{n-1,N})) \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + \| \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(\tilde{y}_{n-1,N}) \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) \| \\ &\quad + \lambda_{n-1} \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \nabla f_{\alpha_{n-1}}(\tilde{y}_{n-1,N}) \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) \| \\ &\quad + \lambda_{n-1} (\| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \nabla f_{\alpha_{n-1}}(\tilde{y}_{n,N}) \| + \| \nabla f_{\alpha_{n-1}}(\tilde{y}_{n,N}) - \nabla f_{\alpha_{n-1}}(\tilde{y}_{n-1,N}) \|) \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) \| \\ &\quad + \lambda_{n-1} [|\alpha_n - \alpha_{n-1}| \| \tilde{y}_{n,N} \| + (\alpha_{n-1} + \|A\|^2) \| \tilde{y}_{n,N} - \tilde{y}_{n-1,N} \|] \\ &= \|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) \| \\ &\quad + \lambda_{n-1} |\alpha_n - \alpha_{n-1}| \| \tilde{y}_{n,N} \| + \lambda_{n-1} (\alpha_{n-1} + \|A\|^2) \| \tilde{y}_{n,N} - \tilde{y}_{n-1,N} \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) \| \\ &\quad + \lambda_{n-1} |\alpha_n - \alpha_{n-1}| \| \tilde{y}_{n,N} \| + \| \tilde{y}_{n,N} - \tilde{y}_{n-1,N} \| \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} \| \tilde{y}_{n,N} - \tilde{y}_{n-1,N} \| &= \| P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})) - P_C(y_{n-1,N} - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(y_{n-1,N})) \| \\ &\leq \| P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})) - P_C(y_{n-1,N} - \lambda_n \nabla f_{\alpha_n}(y_{n-1,N})) \| \\ &\quad + \| P_C(y_{n-1,N} - \lambda_n \nabla f_{\alpha_n}(y_{n-1,N})) - P_C(y_{n-1,N} - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(y_{n-1,N})) \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| \\ &\quad + \| (y_{n-1,N} - \lambda_n \nabla f_{\alpha_n}(y_{n-1,N})) - (y_{n-1,N} - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(y_{n-1,N})) \| \\ &= \|y_{n,N} - y_{n-1,N}\| + \| \lambda_n \nabla f_{\alpha_n}(y_{n-1,N}) - \lambda_{n-1} \nabla f_{\alpha_{n-1}}(y_{n-1,N}) \| \\ &\leq \|y_{n,N} - y_{n-1,N}\| + |\lambda_n \alpha_n - \lambda_{n-1} \alpha_{n-1}| \|y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}| \| \nabla f(y_{n-1,N}) \| \end{aligned}$$

$$\begin{aligned} &\leq \|y_{n,N} - y_{n-1,N}\| + \left(\alpha_n |\lambda_n - \lambda_{n-1}| + \lambda_{n-1} |\alpha_n - \alpha_{n-1}|\right) \|y_{n-1,N}\| \\ &\quad + |\lambda_n - \lambda_{n-1}| \|\nabla f(y_{n-1,N})\|. \end{aligned} \tag{3.20}$$

In the meantime, by the definition of $y_{n,i}$ one obtains that, for all $i = N, \dots, 2$,

$$\begin{aligned} \|y_{n,i} - y_{n-1,i}\| &\leq \beta_{n,i} \|u_n - u_{n-1}\| + \|S_i u_{n-1} - y_{n-1,i-1}\| |\beta_{n,i} - \beta_{n-1,i}| \\ &\quad + (1 - \beta_{n,i}) \|y_{n,i-1} - y_{n-1,i-1}\|. \end{aligned} \tag{3.21}$$

In the case $i = 1$, we have

$$\begin{aligned} \|y_{n,1} - y_{n-1,1}\| &\leq \beta_{n,1} \|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}| \\ &\quad + (1 - \beta_{n,1}) \|u_n - u_{n-1}\| \\ &= \|u_n - u_{n-1}\| + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}|. \end{aligned} \tag{3.22}$$

Substituting (3.22) in all (3.21)-type one obtains, for $i = 2, \dots, N$,

$$\begin{aligned} \|y_{n,i} - y_{n-1,i}\| &\leq \|u_n - u_{n-1}\| + \sum_{k=2}^i \|S_k u_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}|, \end{aligned} \tag{3.23}$$

which together with (3.20) implies that

$$\begin{aligned} \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n} &\leq \frac{\|y_{n,N} - y_{n-1,N}\|}{\epsilon_n} + \left(\alpha_n \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} + \lambda_{n-1} \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n}\right) \|y_{n-1,N}\| \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} \|\nabla f(y_{n-1,N})\| \\ &\leq \frac{\|u_n - u_{n-1}\|}{\epsilon_n} + \sum_{k=2}^N \|S_k u_{n-1} - y_{n-1,k-1}\| \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n} \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\| \frac{|\beta_{n,1} - \beta_{n-1,1}|}{\epsilon_n} \\ &\quad + \left(\alpha_n \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} + \lambda_{n-1} \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n}\right) \|y_{n-1,N}\| \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} \|\nabla f(y_{n-1,N})\|. \end{aligned} \tag{3.24}$$

Since $\|u_n - u_{n-1}\| = o(\epsilon_n)$ and the sequences $\{u_n\}, \{y_{n,i}\}_{i=1}^N$ are bounded, we know that

$$\lim_{n \rightarrow \infty} \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n} = 0.$$

On the other hand, we observe that

$$\begin{cases} v_n = \epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F) z_n, \\ v_{n-1} = \epsilon_{n-1} \gamma V y_{n-1,N} + (I - \epsilon_{n-1} \mu F) z_{n-1}, \quad \forall n \geq 1. \end{cases}$$

Simple calculations show that

$$\begin{aligned} v_n - v_{n-1} &= (I - \epsilon_n \mu F)z_n - (I - \epsilon_n \mu F)z_{n-1} \\ &\quad + (\epsilon_n - \epsilon_{n-1})(\gamma Vy_{n-1,N} - \mu Fz_{n-1}) \\ &\quad + \epsilon_n \gamma (Vy_{n,N} - Vy_{n-1,N}). \end{aligned}$$

Then, passing to the norm and using the nonexpansivity of G , we get

$$\begin{aligned} \|y_n - y_{n-1}\| &\leq \|v_n - v_{n-1}\| \\ &\leq \|(I - \epsilon_n \mu F)z_n - (I - \epsilon_n \mu F)z_{n-1}\| + |\epsilon_n - \epsilon_{n-1}| \|\gamma Vy_{n-1,N} - \mu Fz_{n-1}\| \\ &\quad + \epsilon_n \gamma \|Vy_{n,N} - Vy_{n-1,N}\| \\ &\leq (1 - \epsilon_n \tau) \|z_n - z_{n-1}\| + \tilde{M} |\epsilon_n - \epsilon_{n-1}| + \epsilon_n \gamma l \|y_{n,N} - y_{n-1,N}\| \\ &\leq (1 - \epsilon_n \tau) \|t_n - t_{n-1}\| + \tilde{M} |\epsilon_n - \epsilon_{n-1}| + \epsilon_n \gamma l \|y_{n,N} - y_{n-1,N}\|, \end{aligned} \tag{3.25}$$

where $\sup_{n \geq 0} \|\gamma Vy_{n,N} - \mu Fz_n\| \leq \tilde{M}$ for some $\tilde{M} > 0$. Also, it is easy to see from (3.17) and (3.18) that

$$\begin{aligned} \|w_n - w_{n-1}\| &\leq \frac{\|\gamma_n(t_n - t_{n-1}) + \delta_n(Tt_n - Tt_{n-1})\|}{1 - \beta_n} \\ &\quad + \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \|t_{n-1}\| \\ &\quad + \left| \frac{\delta_n}{1 - \beta_n} - \frac{\delta_{n-1}}{1 - \beta_{n-1}} \right| \|Tt_{n-1}\| \\ &\leq \frac{(\gamma_n + \delta_n) \|t_n - t_{n-1}\|}{1 - \beta_n} + \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \|t_{n-1}\| \\ &\quad + \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \|Tt_{n-1}\| \\ &= \|t_n - t_{n-1}\| + \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| (\|t_{n-1}\| + \|Tt_{n-1}\|). \end{aligned} \tag{3.26}$$

Moreover, by Lemma 2.10, we know that

$$\|u_n - u_{n-1}\| \leq \|x_n - x_{n-1}\| + L \left| 1 - \frac{r_{n-1}}{r_n} \right|,$$

where $L = \sup_{n \geq 0} \|u_n - x_n\|$.

Further, we observe that

$$\begin{cases} x_{n+1} = \beta_n y_n + (1 - \beta_n) w_n, \\ x_n = \beta_{n-1} y_{n-1} + (1 - \beta_{n-1}) w_{n-1}, \quad \forall n \geq 1. \end{cases}$$

Simple calculations show that

$$x_{n+1} - x_n = (1 - \beta_n)(w_n - w_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - w_{n-1}) + \beta_n(y_n - y_{n-1}).$$

Consequently, passing to the norm we get from (3.19), (3.23) and (3.25)-(3.26)

$$\begin{aligned}
 & \|x_{n+1} - x_n\| \\
 & \leq (1 - \beta_n)\|w_n - w_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \beta_n\|y_n - y_{n-1}\| \\
 & \leq (1 - \beta_n)\left[\|t_n - t_{n-1}\| + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|)\right] \\
 & \quad + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \beta_n[(1 - \epsilon_n\tau)\|t_n - t_{n-1}\| \\
 & \quad + \tilde{M}|\epsilon_n - \epsilon_{n-1}| + \epsilon_n\gamma l\|y_{n,N} - y_{n-1,N}\|] \\
 & \leq (1 - \beta_n\epsilon_n\tau)\|t_n - t_{n-1}\| + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|) \\
 & \quad + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \tilde{M}|\epsilon_n - \epsilon_{n-1}| + \beta_n\epsilon_n\gamma l\|y_{n,N} - y_{n-1,N}\| \\
 & \leq (1 - \beta_n\epsilon_n\tau)[\|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}|\|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \\
 & \quad + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|] \\
 & \quad + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|) \\
 & \quad + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \tilde{M}|\epsilon_n - \epsilon_{n-1}| + \beta_n\epsilon_n\gamma l\|y_{n,N} - y_{n-1,N}\| \\
 & \leq (1 - \beta_n\epsilon_n(\tau - \gamma l))\|y_{n,N} - y_{n-1,N}\| + |\lambda_n - \lambda_{n-1}|\|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \\
 & \quad + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| \\
 & \quad + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|) \\
 & \quad + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \tilde{M}|\epsilon_n - \epsilon_{n-1}| \\
 & \leq (1 - \beta_n\epsilon_n(\tau - \gamma l))\left[\|u_n - u_{n-1}\| + \sum_{k=2}^N \|S_k u_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \right. \\
 & \quad \left. + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}|\right] + |\lambda_n - \lambda_{n-1}|\|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \\
 & \quad + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| \\
 & \quad + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|) \\
 & \quad + |\beta_n - \beta_{n-1}|\|y_{n-1} - w_{n-1}\| + \tilde{M}|\epsilon_n - \epsilon_{n-1}| \\
 & \leq (1 - \beta_n\epsilon_n(\tau - \gamma l))\left[\|x_n - x_{n-1}\| + L\left|1 - \frac{r_{n-1}}{r_n}\right| \right. \\
 & \quad \left. + \sum_{k=2}^N \|S_k u_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \right. \\
 & \quad \left. + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}|\right] + |\lambda_n - \lambda_{n-1}|\|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \\
 & \quad + \lambda_{n-1}|\alpha_n - \alpha_{n-1}|\|\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| + \left|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}\right|(\|t_{n-1}\| + \|Tt_{n-1}\|)
 \end{aligned}$$

$$\begin{aligned}
 & + |\beta_n - \beta_{n-1}| \|y_{n-1} - w_{n-1}\| + \tilde{M} |\epsilon_n - \epsilon_{n-1}| \\
 \leq & (1 - c\epsilon_n(\tau - \gamma l)) \|x_n - x_{n-1}\| + L \frac{|r_n - r_{n-1}|}{\bar{r}} + \sum_{k=2}^N \|S_k u_{n-1} - y_{n-1,k-1}\| |\beta_{n,k} - \beta_{n-1,k}| \\
 & + \|S_1 u_{n-1} - u_{n-1}\| |\beta_{n,1} - \beta_{n-1,1}| + |\lambda_n - \lambda_{n-1}| \|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| \\
 & + \lambda_{n-1} |\alpha_n - \alpha_{n-1}| \|\tilde{y}_{n,N}\| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| + \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| (\|t_{n-1}\| + \|Tt_{n-1}\|) \\
 & + |\beta_n - \beta_{n-1}| \|y_{n-1} - w_{n-1}\| + \tilde{M} |\epsilon_n - \epsilon_{n-1}| \\
 \leq & (1 - \epsilon_n(\tau - \gamma l)c) \|x_n - x_{n-1}\| + \tilde{M}_0 \frac{|r_n - r_{n-1}|}{\bar{r}} \\
 & + \tilde{M}_0 \sum_{k=2}^N |\beta_{n,k} - \beta_{n-1,k}| + \tilde{M}_0 |\beta_{n,1} - \beta_{n-1,1}| \\
 & + \tilde{M}_0 |\lambda_n - \lambda_{n-1}| + \tilde{M}_0 |\alpha_n - \alpha_{n-1}| + \tilde{M}_0 \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| + \tilde{M}_0 \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \\
 & + \tilde{M}_0 |\beta_n - \beta_{n-1}| + \tilde{M}_0 |\epsilon_n - \epsilon_{n-1}| \\
 = & (1 - \epsilon_n(\tau - \gamma l)c) \|x_n - x_{n-1}\| + \tilde{M}_0 \left\{ \frac{|r_n - r_{n-1}|}{\bar{r}} + \sum_{k=1}^N |\beta_{n,k} - \beta_{n-1,k}| \right. \\
 & + |\lambda_n - \lambda_{n-1}| + |\alpha_n - \alpha_{n-1}| + \|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\| + \left. \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \right. \\
 & \left. + |\beta_n - \beta_{n-1}| + |\epsilon_n - \epsilon_{n-1}| \right\} \\
 = & (1 - \epsilon_n(\tau - \gamma l)c) \|x_n - x_{n-1}\| \\
 & + \epsilon_n(\tau - \gamma l)c \cdot \frac{\tilde{M}_0}{(\tau - \gamma l)c} \left\{ \frac{|r_n - r_{n-1}|}{\epsilon_n \bar{r}} + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n} \right. \\
 & + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n} + \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n} \\
 & \left. + \frac{\left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right|}{\epsilon_n} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n} \right\}, \tag{3.27}
 \end{aligned}$$

where

$$\begin{aligned}
 \sup_{n \geq 0} & \left\{ 1 + L + \tilde{M} + \sum_{k=2}^N \|S_k u_n - y_{n,k-1}\| + \|S_1 u_n - u_n\| \right. \\
 & \left. + \|\nabla f_{\alpha_n}(\tilde{y}_{n,N})\| + b \|\tilde{y}_{n,N}\| + \|t_n\| + \|Tt_n\| + \|y_n - w_n\| \right\} \leq \tilde{M}_0
 \end{aligned}$$

for some $\tilde{M}_0 > 0$. Noticing $\lim_{n \rightarrow \infty} \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n} = 0$ and using hypotheses (H0)-(H6) and Lemma 2.8, we obtain the claim. \square

Proposition 3.3 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $\{x_n\}$ is asymptotically regular. Then $\|x_n - u_n\| = \|x_n - T_{r_n} x_n\| \rightarrow 0$ and $\|y_{n,N} - \tilde{y}_{n,N}\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof Take fixed $p \in \Omega$ arbitrarily. We recall that, by the firm nonexpansivity of T_{r_n} , a standard calculation (see [44]) shows that for $p \in \text{GMEP}(\Theta, h)$,

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2. \tag{3.28}$$

Utilizing Lemmas 2.1 and 2.7, we obtain from $0 \leq \gamma l < \tau$, (3.1) and (3.10) that

$$\begin{aligned} & \|y_n - p\|^2 \\ &= \|\epsilon_n \gamma (Vy_{n,N} - Vp) + (I - \epsilon_n \mu F)z_n - (I - \epsilon_n \mu F)p + \epsilon_n (\gamma V - \mu F)p\|^2 \\ &\leq \|\epsilon_n \gamma (Vy_{n,N} - Vp) + (I - \epsilon_n \mu F)z_n - (I - \epsilon_n \mu F)p\|^2 + 2\epsilon_n \langle (\gamma V - \mu F)p, y_n - p \rangle \\ &\leq [\epsilon_n \gamma \|Vy_{n,N} - Vp\| + \|(I - \epsilon_n \mu F)z_n - (I - \epsilon_n \mu F)p\|]^2 + 2\epsilon_n \langle (\gamma V - \mu F)p, y_n - p \rangle \\ &\leq [\epsilon_n \gamma l \|y_{n,N} - p\| + (1 - \epsilon_n \tau) \|z_n - p\|]^2 + 2\epsilon_n \langle (\gamma V - \mu F)p, y_n - p \rangle \\ &= \left[\epsilon_n \tau \frac{\gamma l}{\tau} \|y_{n,N} - p\| + (1 - \epsilon_n \tau) \|z_n - p\| \right]^2 + 2\epsilon_n \langle (\gamma V - \mu F)p, y_n - p \rangle \\ &\leq \epsilon_n \tau \frac{(\gamma l)^2}{\tau^2} \|y_{n,N} - p\|^2 + (1 - \epsilon_n \tau) \|z_n - p\|^2 + 2\epsilon_n \langle (\gamma V - \mu F)p, y_n - p \rangle \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + \|t_n - p\|^2 - \nu_2 (2\zeta_2 - \nu_2) \|F_2 t_n - F_2 p\|^2 \\ &\quad - \nu_1 (2\zeta_1 - \nu_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|. \end{aligned} \tag{3.29}$$

Since $(\gamma_n + \delta_n)\xi \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 2.4 we have from (3.1), (3.9), (3.15), (3.28) and (3.29) that

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ &= \|\beta_n (y_n - p) + \gamma_n (t_n - p) + \delta_n (Tt_n - p)\|^2 \\ &= \left\| \beta_n (y_n - p) + (\gamma_n + \delta_n) \frac{1}{\gamma_n + \delta_n} [\gamma_n (t_n - p) + \delta_n (Tt_n - p)] \right\|^2 \\ &\leq \beta_n \|y_n - p\|^2 + (\gamma_n + \delta_n) \left\| \frac{1}{\gamma_n + \delta_n} [\gamma_n (t_n - p) + \delta_n (Tt_n - p)] \right\|^2 \\ &\leq \beta_n \|y_n - p\|^2 + (\gamma_n + \delta_n) \|t_n - p\|^2 \\ &= \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|t_n - p\|^2 \\ &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|t_n - p\|^2 - \nu_2 (2\zeta_2 - \nu_2) \|F_2 t_n - F_2 p\|^2 \\ &\quad - \nu_1 (2\zeta_1 - \nu_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] + (1 - \beta_n) \|t_n - p\|^2 \\ &\leq \|t_n - p\|^2 - \beta_n [\nu_2 (2\zeta_2 - \nu_2) \|F_2 t_n - F_2 p\|^2 + \nu_1 (2\zeta_1 - \nu_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\ &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\leq \|y_{n,N} - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - \tilde{y}_{n,N}\| + (\lambda_n^2 (\alpha_n + \|A\|^2)^2 - 1) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\ &\quad - \beta_n [\nu_2 (2\zeta_2 - \nu_2) \|F_2 t_n - F_2 p\|^2 + \nu_1 (2\zeta_1 - \nu_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\ &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \end{aligned}$$

$$\begin{aligned}
 &\leq \|u_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - \tilde{y}_{n,N}\| + (\lambda_n^2 (\alpha_n + \|A\|^2)^2 - 1) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\
 &\quad - \beta_n [v_2(2\zeta_2 - v_2) \|F_2 t_n - F_2 p\|^2 + v_1(2\zeta_1 - v_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\
 &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 &\leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2\lambda_n \alpha_n \|p\| \|p - \tilde{y}_{n,N}\| \\
 &\quad + (\lambda_n^2 (\alpha_n + \|A\|^2)^2 - 1) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\
 &\quad - \beta_n [v_2(2\zeta_2 - v_2) \|F_2 t_n - F_2 p\|^2 + v_1(2\zeta_1 - v_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\
 &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|. \tag{3.30}
 \end{aligned}$$

So, we deduce from $\{\beta_n\} \subset [c, d] \subset (0, 1)$ and $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ that

$$\begin{aligned}
 &\|x_n - u_n\|^2 + (1 - b^2 (\alpha_n + \|A\|^2)^2) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\
 &\quad + c [v_2(2\zeta_2 - v_2) \|F_2 t_n - F_2 p\|^2 + v_1(2\zeta_1 - v_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\
 &\leq \|x_n - u_n\|^2 + (1 - \lambda_n^2 (\alpha_n + \|A\|^2)^2) \|y_{n,N} - \tilde{y}_{n,N}\|^2 \\
 &\quad + \beta_n [v_2(2\zeta_2 - v_2) \|F_2 t_n - F_2 p\|^2 + v_1(2\zeta_1 - v_1) \|F_1 \tilde{t}_n - F_1 \tilde{p}\|^2] \\
 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - \tilde{y}_{n,N}\| \\
 &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 &\leq \|x_n - x_{n+1}\| (\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n b \|p\| \|p - \tilde{y}_{n,N}\| \\
 &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|.
 \end{aligned}$$

By Propositions 3.1 and 3.2 we know that the sequences $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$ and $\{\tilde{y}_{n,N}\}$ are bounded and that $\{x_n\}$ is asymptotically regular. Therefore, from $\alpha_n \rightarrow 0$ and $\epsilon_n \rightarrow 0$ we obtain that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = \lim_{n \rightarrow \infty} \|F_2 t_n - F_2 p\| = \lim_{n \rightarrow \infty} \|F_1 \tilde{t}_n - F_1 \tilde{p}\| = \lim_{n \rightarrow \infty} \|y_{n,N} - \tilde{y}_{n,N}\| = 0. \tag{3.31}$$

□

Remark 3.1 By the last proposition we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$, i.e., the sets of strong/weak cluster points of $\{x_n\}$ and $\{u_n\}$ coincide.

Of course, if $\beta_{n,i} \rightarrow \beta_i \neq 0$ as $n \rightarrow \infty$, for all indices i , the assumptions of Proposition 3.2 are enough to assure that

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,i}} = 0, \quad \forall i \in \{1, \dots, N\}.$$

In the next proposition, we estimate the case in which at least one sequence $\{\beta_{n,k_0}\}$ is a null sequence.

Proposition 3.4 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that (H0) holds. Moreover, for an index $k_0 \in \{1, \dots, N\}$, $\lim_{n \rightarrow \infty} \beta_{n,k_0} = 0$ and the following hold:*

(H7) for each index $i \in \{1, \dots, N\}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|\beta_{n,i} - \beta_{n-1,i}|}{\epsilon_n \beta_{n,k_0}} &= \lim_{n \rightarrow \infty} \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} = \lim_{n \rightarrow \infty} \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} = \lim_{n \rightarrow \infty} \frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0}} \\ &= \lim_{n \rightarrow \infty} \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} = \lim_{n \rightarrow \infty} \frac{1}{\epsilon_n \beta_{n,k_0}} \left| \frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}} \right| \\ &= \lim_{n \rightarrow \infty} \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} = 0; \end{aligned}$$

(H8) there exists a constant $\delta > 0$ such that $\frac{1}{\epsilon_n} \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| < \delta$ for all $n \geq 1$.

If $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$, then

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.$$

Proof It is clear from (3.24) that

$$\begin{aligned} \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} &\leq \frac{\|u_n - u_{n-1}\|}{\epsilon_n \beta_{n,k_0}} + \sum_{k=2}^N \|S_k u_{n-1} - y_{n-1,k-1}\| \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} \\ &\quad + \|S_1 u_{n-1} - u_{n-1}\| \frac{|\beta_{n,1} - \beta_{n-1,1}|}{\epsilon_n \beta_{n,k_0}} \\ &\quad + \left(\alpha_n \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \lambda_{n-1} \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} \right) \|y_{n-1,N}\| \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} \|\nabla f(y_{n-1,N})\|. \end{aligned}$$

According to (H7) and $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$, we get

$$\lim_{n \rightarrow \infty} \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} = 0. \tag{3.32}$$

Consider (3.27). Dividing both the terms by β_{n,k_0} , we have

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} &\leq (1 - \epsilon_n(\tau - \gamma l)c) \frac{\|x_n - x_{n-1}\|}{\beta_{n,k_0}} \\ &\quad + \epsilon_n(\tau - \gamma l)c \cdot \frac{\tilde{M}_0}{(\tau - \gamma l)c} \left\{ \frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0} \bar{r}} + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} \right. \\ &\quad + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} \\ &\quad \left. + \frac{|\frac{\gamma_n}{1 - \beta_n} - \frac{\gamma_{n-1}}{1 - \beta_{n-1}}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} \right\}. \end{aligned}$$

So, by (H8) we have

$$\begin{aligned} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} &\leq (1 - \epsilon_n(\tau - \gamma l)c) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} \\ &\quad + (1 - \epsilon_n(\tau - \gamma l)c) \|x_n - x_{n-1}\| \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| \end{aligned}$$

$$\begin{aligned}
 & + \epsilon_n(\tau - \gamma l)c \cdot \frac{\tilde{M}_0}{(\tau - \gamma l)c} \left\{ \frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0} \bar{r}} \right. \\
 & + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} \\
 & + \left. \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} \right\} \\
 \leq & (1 - \epsilon_n(\tau - \gamma l)c) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \|x_n - x_{n-1}\| \left| \frac{1}{\beta_{n,k_0}} - \frac{1}{\beta_{n-1,k_0}} \right| \\
 & + \epsilon_n(\tau - \gamma l)c \cdot \frac{\tilde{M}_0}{(\tau - \gamma l)c} \left\{ \frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0} \bar{r}} \right. \\
 & + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} \\
 & + \left. \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} \right\} \\
 \leq & (1 - \epsilon_n(\tau - \gamma l)c) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} + \epsilon_n \delta \|x_n - x_{n-1}\| \\
 & + \epsilon_n(\tau - \gamma l)c \cdot \frac{\tilde{M}_0}{(\tau - \gamma l)c} \left\{ \frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0} \bar{r}} \right. \\
 & + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} \\
 & + \left. \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} \right\} \\
 = & (1 - \epsilon_n(\tau - \gamma l)c) \frac{\|x_n - x_{n-1}\|}{\beta_{n-1,k_0}} \\
 & + \epsilon_n(\tau - \gamma l)c \cdot \frac{1}{(\tau - \gamma l)c} \left\{ \delta \|x_n - x_{n-1}\| \right. \\
 & + \tilde{M}_0 \left[\frac{|r_n - r_{n-1}|}{\epsilon_n \beta_{n,k_0} \bar{r}} + \sum_{k=1}^N \frac{|\beta_{n,k} - \beta_{n-1,k}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\lambda_n - \lambda_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\alpha_n - \alpha_{n-1}|}{\epsilon_n \beta_{n,k_0}} \right. \\
 & + \left. \left. \frac{|\beta_n - \beta_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\frac{\gamma_n}{1-\beta_n} - \frac{\gamma_{n-1}}{1-\beta_{n-1}}|}{\epsilon_n \beta_{n,k_0}} + \frac{|\epsilon_n - \epsilon_{n-1}|}{\epsilon_n \beta_{n,k_0}} + \frac{\|\tilde{y}_{n,N} - \tilde{y}_{n-1,N}\|}{\epsilon_n \beta_{n,k_0}} \right] \right\}.
 \end{aligned}$$

Therefore, utilizing Lemma 2.8, from (3.32), (H0), (H7) and the asymptotical regularity of $\{x_n\}$ (due to Proposition 3.2), we deduce that

$$\lim_{n \rightarrow \infty} \frac{\|x_{n+1} - x_n\|}{\beta_{n,k_0}} = 0.$$

□

Proposition 3.5 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that (H0)-(H6) hold. If $\|u_n - u_{n-1}\| = o(\epsilon_n)$, then $\|z_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof Let $p \in \Omega$. In terms of the firm nonexpansivity of P_C and the ζ_j -inverse strong monotonicity of F_j for $j = 1, 2$, we obtain from $v_j \in (0, 2\zeta_j), j = 1, 2$ and (3.10) that

$$\begin{aligned} \|\tilde{t}_n - \tilde{p}\|^2 &= \|P_C(I - v_2F_2)t_n - P_C(I - v_2F_2)p\|^2 \\ &\leq \langle (I - v_2F_2)t_n - (I - v_2F_2)p, \tilde{t}_n - \tilde{p} \rangle \\ &= \frac{1}{2} [\|(I - v_2F_2)t_n - (I - v_2F_2)p\|^2 + \|\tilde{t}_n - \tilde{p}\|^2 \\ &\quad - \|(I - v_2F_2)t_n - (I - v_2F_2)p - (\tilde{t}_n - \tilde{p})\|^2] \\ &\leq \frac{1}{2} [\|t_n - p\|^2 + \|\tilde{t}_n - \tilde{p}\|^2 - \|(t_n - \tilde{t}_n) - v_2(F_2t_n - F_2p) - (p - \tilde{p})\|^2] \\ &= \frac{1}{2} [\|t_n - p\|^2 + \|\tilde{t}_n - \tilde{p}\|^2 - \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\ &\quad + 2v_2\langle (t_n - \tilde{t}_n) - (p - \tilde{p}), F_2t_n - F_2p \rangle - v_2^2\|F_2t_n - F_2p\|^2] \end{aligned}$$

and

$$\begin{aligned} \|z_n - p\|^2 &= \|P_C(I - v_1F_1)\tilde{t}_n - P_C(I - v_1F_1)\tilde{p}\|^2 \\ &\leq \langle (I - v_1F_1)\tilde{t}_n - (I - v_1F_1)\tilde{p}, z_n - p \rangle \\ &= \frac{1}{2} [\|(I - v_1F_1)\tilde{t}_n - (I - v_1F_1)\tilde{p}\|^2 + \|z_n - p\|^2 \\ &\quad - \|(I - v_1F_1)\tilde{t}_n - (I - v_1F_1)\tilde{p} - (z_n - p)\|^2] \\ &\leq \frac{1}{2} [\|\tilde{t}_n - \tilde{p}\|^2 + \|z_n - p\|^2 - \|(\tilde{t}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2v_1\langle F_1\tilde{t}_n - F_1\tilde{p}, (\tilde{t}_n - z_n) + (p - \tilde{p}) \rangle - v_1^2\|F_1\tilde{t}_n - F_1\tilde{p}\|^2] \\ &\leq \frac{1}{2} [\|t_n - p\|^2 + \|z_n - p\|^2 - \|(\tilde{t}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2v_1\langle F_1\tilde{t}_n - F_1\tilde{p}, (\tilde{t}_n - z_n) + (p - \tilde{p}) \rangle]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\tilde{t}_n - \tilde{p}\|^2 &\leq \|t_n - p\|^2 - \|(\tilde{t}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2v_2\langle (t_n - \tilde{t}_n) - (p - \tilde{p}), F_2t_n - F_2p \rangle - v_2^2\|F_2t_n - F_2p\|^2 \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} \|z_n - p\|^2 &\leq \|t_n - p\|^2 - \|(\tilde{t}_n - z_n) + (p - \tilde{p})\|^2 \\ &\quad + 2v_1\|F_1\tilde{t}_n - F_1\tilde{p}\| \|(\tilde{t}_n - z_n) + (p - \tilde{p})\|. \end{aligned} \tag{3.34}$$

Consequently, from (3.10), (3.15), (3.29), (3.30) and (3.33), it follows that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \beta_n\|y_n - p\|^2 + (1 - \beta_n)\|t_n - p\|^2 \end{aligned}$$

$$\begin{aligned}
 &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|\tilde{t}_n - \tilde{p}\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|t_n - p\|^2 - \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_2 \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| \|F_2 t_n - F_2 p\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] \\
 &\quad + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \|t_n - p\|^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 - \beta_n \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_2 \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| \|F_2 t_n - F_2 p\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 &\leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 - \beta_n \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_2 \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| \|F_2 t_n - F_2 p\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|,
 \end{aligned}$$

which yields

$$\begin{aligned}
 &c \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\
 &\leq \beta_n \|(t_n - \tilde{t}_n) - (p - \tilde{p})\|^2 \\
 &\leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 - \|x_{n+1} - p\|^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 \\
 &\quad + 2\nu_2 \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| \|F_2 t_n - F_2 p\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 &\leq (\|x_n - x_{n+1}\| + \sqrt{2}\lambda_n \alpha_n \|p\|) (\|x_n - p\| + \|x_{n+1} - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|) \\
 &\quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\nu_2 \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| \|F_2 t_n - F_2 p\| \\
 &\quad + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \epsilon_n = 0$, $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, and $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, $\{t_n\}$ and $\{\tilde{t}_n\}$ are bounded, we deduce from (3.31) that

$$\lim_{n \rightarrow \infty} \|(t_n - \tilde{t}_n) - (p - \tilde{p})\| = 0. \tag{3.35}$$

Furthermore, from (3.15), (3.29), (3.30) and (3.34), it follows that

$$\begin{aligned}
 &\|x_{n+1} - p\|^2 \\
 &\leq \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \beta_n [\epsilon_n \tau \|y_{n,N} - p\|^2 + \|t_n - p\|^2 - \|\tilde{t}_n - z_n + (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_1 \|F_1 \tilde{t}_n - F_1 \tilde{p}\| \|\tilde{t}_n - z_n + (p - \tilde{p})\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|] \\
 &\quad + (1 - \beta_n) \|t_n - p\|^2 \\
 &\leq \|t_n - p\|^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 - \beta_n \|\tilde{t}_n - z_n + (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_1 \|F_1 \tilde{t}_n - F_1 \tilde{p}\| \|\tilde{t}_n - z_n + (p - \tilde{p})\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 &\leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 - \beta_n \|\tilde{t}_n - z_n + (p - \tilde{p})\|^2 \\
 &\quad + 2\nu_1 \|F_1 \tilde{t}_n - F_1 \tilde{p}\| \|\tilde{t}_n - z_n + (p - \tilde{p})\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|,
 \end{aligned}$$

which leads to

$$\begin{aligned}
 & c \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\|^2 \\
 & \leq \beta_n \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\|^2 \\
 & \leq \left(\|x_n - p\| + \sqrt{2} \lambda_n \alpha_n \|p\| \right)^2 - \|x_{n+1} - p\|^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 \\
 & \quad + 2\nu_1 \|F_1 \tilde{t}_n - F_1 \tilde{p}\| \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\| + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\
 & \leq \left(\|x_n - x_{n+1}\| + \sqrt{2} \lambda_n \alpha_n \|p\| \right) \left(\|x_n - p\| + \|x_{n+1} - p\| + \sqrt{2} \lambda_n \alpha_n \|p\| \right) \\
 & \quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\nu_1 \|F_1 \tilde{t}_n - F_1 \tilde{p}\| \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\| \\
 & \quad + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\|.
 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\lim_{n \rightarrow \infty} \epsilon_n = 0$, $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$, and $\{x_n\}$, $\{y_n\}$, $\{y_{n,N}\}$, $\{z_n\}$ and $\{\tilde{t}_n\}$ are bounded, we deduce from (3.31) that

$$\lim_{n \rightarrow \infty} \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\| = 0. \tag{3.36}$$

Note that

$$\|t_n - z_n\| \leq \left\| (t_n - \tilde{t}_n) - (p - \tilde{p}) \right\| + \left\| (\tilde{t}_n - z_n) + (p - \tilde{p}) \right\|.$$

Hence from (3.35) and (3.36) we get

$$\lim_{n \rightarrow \infty} \|t_n - z_n\| = \lim_{n \rightarrow \infty} \|t_n - Gt_n\| = 0. \tag{3.37}$$

□

Proposition 3.6 *Let us suppose that $\Omega \neq \emptyset$. Let us suppose that $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$ for each $i = 1, \dots, N$. Moreover, suppose that $\|u_n - u_{n-1}\| = o(\epsilon_n)$ and (H0)-(H6) are satisfied. Then $\lim_{n \rightarrow \infty} \|S_i u_n - u_n\| = 0$ for each $i = 1, \dots, N$ provided $\|Ty_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.*

Proof First of all, it is clear that

$$\begin{aligned}
 \|t_n - \tilde{y}_{n,N}\| &= \left\| P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})) - P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})) \right\| \\
 &\leq \left\| (y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})) - (y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})) \right\| \\
 &= \lambda_n \left\| \nabla f_{\alpha_n}(\tilde{y}_{n,N}) - \nabla f_{\alpha_n}(y_{n,N}) \right\| \\
 &\leq \lambda_n (\alpha_n + \|A\|^2) \|\tilde{y}_{n,N} - y_{n,N}\| \\
 &\leq \|\tilde{y}_{n,N} - y_{n,N}\|.
 \end{aligned}$$

By Proposition 3.3, we get

$$\lim_{n \rightarrow \infty} \|t_n - \tilde{y}_{n,N}\| = 0,$$

which together with (3.31) implies that

$$\lim_{n \rightarrow \infty} \|t_n - y_{n,N}\| = 0. \tag{3.38}$$

Note that

$$\begin{aligned} \|y_n - t_n\| &\leq \|\epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F) z_n - t_n\| \\ &\leq \epsilon_n \|\gamma V y_{n,N} - \mu F z_n\| + \|z_n - t_n\|. \end{aligned}$$

From Proposition 3.5 and $\epsilon_n \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \|y_n - t_n\| = 0. \tag{3.39}$$

Also, observe that

$$\begin{aligned} x_{n+1} - x_n + x_n - y_n &= x_{n+1} - y_n \\ &= \gamma_n(t_n - y_n) + \delta_n(Tt_n - y_n) \\ &= \gamma_n(t_n - y_n) + \delta_n(Tt_n - Ty_n) + \delta_n(Ty_n - y_n). \end{aligned}$$

By Proposition 3.2 we know that $\{x_n\}$ is asymptotically regular. Utilizing Lemma 2.4 we have from $(\gamma_n + \delta_n)\xi \leq \gamma_n$ that

$$\begin{aligned} \|y_n - x_n\| &= \|x_{n+1} - x_n - \gamma_n(t_n - y_n) - \delta_n(Tt_n - Ty_n) - \delta_n(Ty_n - y_n)\| \\ &\leq \|x_{n+1} - x_n\| + \|\gamma_n(t_n - y_n) - \delta_n(Tt_n - Ty_n)\| + \delta_n \|Ty_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + (\gamma_n + \delta_n) \|t_n - y_n\| + \delta_n \|Ty_n - y_n\| \\ &\leq \|x_{n+1} - x_n\| + \|t_n - y_n\| + \|Ty_n - y_n\|, \end{aligned}$$

which together with (3.39) and $\|Ty_n - y_n\| \rightarrow 0$ leads to

$$\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0. \tag{3.40}$$

Let us show that for each $i \in \{1, \dots, N\}$, one has $\|S_i u_n - y_{n,i-1}\| \rightarrow 0$ as $n \rightarrow \infty$. Let $p \in \Omega$. When $i = N$, by Lemma 2.2(b) we have from (3.9), (3.10), (3.15) and (3.29)

$$\begin{aligned} \|y_n - p\|^2 &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + \|z_n - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + \|t_n - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| + \|y_{n,N} - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \beta_{n,N} \|S_N u_n - p\|^2 + (1 - \beta_{n,N}) \|y_{n,N-1} - p\|^2 - \beta_{n,N} (1 - \beta_{n,N}) \|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \beta_{n,N} \|u_n - p\|^2 + (1 - \beta_{n,N}) \|u_n - p\|^2 - \beta_{n,N} (1 - \beta_{n,N}) \|S_N u_n - y_{n,N-1}\|^2 \\ &= \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \|u_n - p\|^2 - \beta_{n,N} (1 - \beta_{n,N}) \|S_N u_n - y_{n,N-1}\|^2 \end{aligned}$$

$$\begin{aligned} &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \|x_n - p\|^2 - \beta_{n,N}(1 - \beta_{n,N}) \|S_N u_n - y_{n,N-1}\|^2. \end{aligned}$$

So, we have

$$\begin{aligned} &\beta_{n,N}(1 - \beta_{n,N}) \|S_N u_n - y_{n,N-1}\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

Since $\alpha_n \rightarrow 0, \epsilon_n \rightarrow 0, 0 < \liminf_{n \rightarrow \infty} \beta_{n,N} \leq \limsup_{n \rightarrow \infty} \beta_{n,N} < 1$ and $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ (due to (3.40)), it is known that $\{\|S_N u_n - y_{n,N-1}\|\}$ is a null sequence.

Let $i \in \{1, \dots, N - 1\}$. Then one has

$$\begin{aligned} &\|y_n - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| + \|y_{n,N} - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \beta_{n,N} \|S_N u_n - p\|^2 + (1 - \beta_{n,N}) \|y_{n,N-1} - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \beta_{n,N} \|x_n - p\|^2 + (1 - \beta_{n,N}) \|y_{n,N-1} - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \beta_{n,N} \|x_n - p\|^2 + (1 - \beta_{n,N}) [\beta_{n,N-1} \|S_{N-1} u_n - p\|^2 + (1 - \beta_{n,N-1}) \|y_{n,N-2} - p\|^2] \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + (\beta_{n,N} + (1 - \beta_{n,N})\beta_{n,N-1}) \|x_n - p\|^2 + \prod_{k=N-1}^N (1 - \beta_{n,k}) \|y_{n,N-2} - p\|^2, \end{aligned}$$

and so, after $(N - i + 1)$ -iterations,

$$\begin{aligned} &\|y_n - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \left(\beta_{n,N} + \sum_{j=i+2}^N \left(\prod_{l=j}^N (1 - \beta_{n,l}) \right) \beta_{n,j-1} \right) \|x_n - p\|^2 + \prod_{k=i+1}^N (1 - \beta_{n,k}) \|y_{n,i} - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \left(\beta_{n,N} + \sum_{j=i+2}^N \left(\prod_{l=j}^N (1 - \beta_{n,l}) \right) \beta_{n,j-1} \right) \|x_n - p\|^2 + \prod_{k=i+1}^N (1 - \beta_{n,k}) [\beta_{n,i} \|S_i u_n - p\|^2 \\ &\quad + (1 - \beta_{n,i}) \|y_{n,i-1} - p\|^2 - \beta_{n,i}(1 - \beta_{n,i}) \|S_i u_n - y_{n,i-1}\|^2] \end{aligned}$$

$$\begin{aligned} &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ &\quad + \|x_n - p\|^2 - \beta_{n,i} \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2. \end{aligned} \tag{3.41}$$

Again we obtain that

$$\begin{aligned} &\beta_{n,i} \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\quad + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ &\quad + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| + \|x_n - y_n\| (\|x_n - p\| + \|y_n - p\|). \end{aligned}$$

Since $\alpha_n \rightarrow 0$, $\epsilon_n \rightarrow 0$, $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$ for each $i = 1, \dots, N - 1$, and $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ (due to (3.40)), it is known that

$$\lim_{n \rightarrow \infty} \|S_i u_n - y_{n,i-1}\| = 0.$$

Obviously for $i = 1$, we have $\|S_1 u_n - u_n\| \rightarrow 0$.

To conclude, we have that

$$\|S_2 u_n - u_n\| \leq \|S_2 u_n - y_{n,1}\| + \|y_{n,1} - u_n\| = \|S_2 u_n - y_{n,1}\| + \beta_{n,1} \|S_1 u_n - u_n\|$$

from which $\|S_2 u_n - u_n\| \rightarrow 0$. Thus by induction $\|S_i u_n - u_n\| \rightarrow 0$ for all $i = 2, \dots, N$ since it is enough to observe that

$$\begin{aligned} \|S_i u_n - u_n\| &\leq \|S_i u_n - y_{n,i-1}\| + \|y_{n,i-1} - S_{i-1} u_n\| + \|S_{i-1} u_n - u_n\| \\ &\leq \|S_i u_n - y_{n,i-1}\| + (1 - \beta_{n,i-1}) \|S_{i-1} u_n - y_{n,i-2}\| + \|S_{i-1} u_n - u_n\|. \end{aligned} \quad \square$$

Remark 3.2 As an example, we consider $N = 2$ and the sequences:

- (a) $\beta_n = \frac{1}{2} + \frac{2}{n}$, $\gamma_n = \delta_n = \frac{1}{4} - \frac{1}{n}$, $\forall n > 4$;
- (b) $\lambda_n = \frac{1}{2\|A\|^2} - \frac{1}{2n}$, $\forall n > \|A\|^2$;
- (c) $\alpha_n = \frac{1}{n^2}$, $\epsilon_n = \frac{1}{\sqrt{n}}$, $r_n = 2 - \frac{1}{n}$, $\forall n > 1$;
- (d) $\beta_{n,1} = \frac{1}{2} - \frac{1}{n}$, $\beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}$, $\forall n > 2$.

Then they satisfy the hypotheses on the parameter sequences in Proposition 3.6.

Proposition 3.7 *Let us suppose that $\Omega \neq \emptyset$ and $\beta_{n,i} \rightarrow \beta_i$ for all i as $n \rightarrow \infty$. Suppose that there exists $k \in \{1, \dots, N\}$ such that $\beta_{n,k} \rightarrow 0$ as $n \rightarrow \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index such that $\beta_{n,k_0} \rightarrow 0$ as $n \rightarrow \infty$. Suppose that*

- (i) $\frac{\alpha_n + \epsilon_n}{\beta_{n,k_0}} \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) if $i \leq k_0$ and $\beta_{n,i} \rightarrow 0$, then $\frac{\beta_{n,k_0}}{\beta_{n,i}} \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) if $\beta_{n,i} \rightarrow \beta_i \neq 0$, then β_i lies in $(0, 1)$.

Moreover, suppose that $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$ and (H0), (H7) and (H8) hold. Then $\lim_{n \rightarrow \infty} \|S_i u_n - u_n\| = 0$ for each $i = 1, \dots, N$ provided $\|T y_n - y_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof First of all we note that if (H7) holds then also (H1)-(H6) are satisfied. So $\{x_n\}$ is asymptotically regular.

Let k_0 be as in the hypotheses. As in Proposition 3.6, for every index $i \in \{1, \dots, N\}$ such that $\beta_{n,i} \rightarrow \beta_i \neq 0$ (which leads to $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$), one has $\|S_i u_n - y_{n,i-1}\| \rightarrow 0$ as $n \rightarrow \infty$.

For all the other indices $i \leq k_0$, we can prove that $\|S_i u_n - y_{n,i-1}\| \rightarrow 0$ as $n \rightarrow \infty$ in a similar manner. By the relation (due to (3.15), (3.30) and (3.41))

$$\begin{aligned} & \|x_{n+1} - p\|^2 \\ & \leq \beta_n \|y_n - p\|^2 + (1 - \beta_n) \|t_n - p\|^2 \\ & \leq \beta_n \left[\epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \right. \\ & \quad \left. + \|x_n - p\|^2 - \beta_{n,i} \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \right] \\ & \quad + (1 - \beta_n) (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 \\ & \leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| \\ & \quad + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| - \beta_n \beta_{n,i} \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2, \end{aligned}$$

we immediately obtain that

$$\begin{aligned} & c \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ & \leq \beta_n \prod_{k=i}^N (1 - \beta_{n,k}) \|S_i u_n - y_{n,i-1}\|^2 \\ & \leq (\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|)^2 - \|x_{n+1} - p\|^2 + \epsilon_n \tau \|y_{n,N} - p\|^2 \\ & \quad + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\| \\ & \leq \frac{\|x_n - x_{n+1}\| + \sqrt{2}\lambda_n \alpha_n \|p\|}{\beta_{n,i}} (\|x_n - p\| + \|x_{n+1} - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|) \\ & \quad + \epsilon_n \tau \|y_{n,N} - p\|^2 + 2\epsilon_n \|(\gamma V - \mu F)p\| \|y_n - p\| + 2\lambda_n \alpha_n \|p\| \|\tilde{y}_{n,N} - p\|. \end{aligned}$$

By Proposition 3.4 or by hypothesis (ii) on the sequences, we have

$$\frac{\|x_n - x_{n+1}\|}{\beta_{n,i}} = \frac{\|x_n - x_{n+1}\|}{\beta_{n,k_0}} \cdot \frac{\beta_{n,k_0}}{\beta_{n,i}} \rightarrow 0.$$

So, the conclusion follows. □

Remark 3.3 Let us consider $N = 3$ and the following sequences:

- (a) $\beta_n = \frac{1}{2} + \frac{2}{n^2}, \gamma_n = \delta_n = \frac{1}{4} - \frac{1}{n^2}, \forall n > 2;$
- (b) $\lambda_n = \frac{1}{2\|A\|^2} - \frac{1}{2n^2}, \forall n > \|A\|;$

- (c) $\alpha_n = \frac{1}{n^2}, \alpha_n = \frac{1}{n^{1/2}}, r_n = 2 - \frac{1}{n^2}, \forall n > 1;$
- (d) $\beta_{n,1} = \frac{1}{n^{1/4}}, \beta_{n,2} = \frac{1}{2} - \frac{1}{n^2}, \beta_{n,3} = \frac{1}{n^{1/3}}, \forall n > 1.$

It is easy to see that all hypotheses (i)-(iii), (H0), (H7) and (H8) of Proposition 3.7 are satisfied.

Remark 3.4 Under the hypotheses of Proposition 3.7, analogously to Proposition 3.6, one can see that

$$\lim_{n \rightarrow \infty} \|S_i u_n - y_{n,i-1}\| = 0, \quad \forall i \in \{2, \dots, N\}.$$

Corollary 3.1 *Let us suppose that the hypotheses of either Proposition 3.6 or Proposition 3.7 are satisfied. Then $\omega_w(x_n) = \omega_w(u_n) = \omega_w(y_{n,1}), \omega_s(x_n) = \omega_s(u_n) = \omega_s(y_{n,1})$ and $\omega_w(x_n) \subset \Omega$.*

Proof By Remark 3.1, we have $\omega_w(x_n) = \omega_w(u_n)$ and $\omega_s(x_n) = \omega_s(u_n)$. Note that by Remark 3.4,

$$\lim_{n \rightarrow \infty} \|S_N u_n - y_{n,N-1}\| = 0.$$

In the meantime, it is known that

$$\lim_{n \rightarrow \infty} \|S_N u_n - u_n\| = \lim_{n \rightarrow \infty} \|u_n - x_n\| = \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Hence we have

$$\lim_{n \rightarrow \infty} \|S_N u_n - y_n\| = 0. \tag{3.42}$$

Furthermore, it follows from (3.1) that

$$\lim_{n \rightarrow \infty} \|y_{n,N} - y_{n,N-1}\| = \lim_{n \rightarrow \infty} \beta_{n,N} \|S_N u_n - y_{n,N-1}\| = 0,$$

which together with $\lim_{n \rightarrow \infty} \|S_N u_n - y_{n,N-1}\| = 0$ yields

$$\lim_{n \rightarrow \infty} \|S_N u_n - y_{n,N}\| = 0. \tag{3.43}$$

Combining (3.42) and (3.43), we conclude that

$$\lim_{n \rightarrow \infty} \|y_n - y_{n,N}\| = 0, \tag{3.44}$$

which together with $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$ leads to

$$\lim_{n \rightarrow \infty} \|x_n - y_{n,N}\| = 0. \tag{3.45}$$

Now we observe that

$$\|x_n - y_{n,1}\| \leq \|x_n - u_n\| + \|y_{n,1} - u_n\| = \|x_n - u_n\| + \beta_{n,1} \|S_1 u_n - u_n\|.$$

By Propositions 3.3 and 3.6, $\|x_n - u_n\| \rightarrow 0$ and $\|S_1 u_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$, and hence

$$\lim_{n \rightarrow \infty} \|x_n - y_{n,1}\| = 0.$$

So we get $\omega_w(x_n) = \omega_w(y_{n,1})$ and $\omega_s(x_n) = \omega_s(y_{n,1})$.

Let $p \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Since $p \in \omega_w(u_n)$, by Proposition 3.6 and Lemma 2.5 (demiclosedness principle), we have $p \in \text{Fix}(S_i)$ for each $i = 1, \dots, N$, i.e., $p \in \bigcap_{i=1}^N \text{Fix}(S_i)$. Combining (3.38) and (3.45), we obtain $\|x_n - t_n\| \rightarrow 0$ as $n \rightarrow \infty$. Taking into account $p \in \omega_w(t_n)$ and $\|t_n - Gt_n\| \rightarrow 0$ (due to (3.37)), by Lemma 2.5 (demiclosedness principle) we know that $p \in \text{Fix}(G) =: \mathcal{E}$. Also, since $p \in \omega_w(y_n)$ (due to (3.40)), in terms of $\|Ty_n - y_n\| \rightarrow 0$ and Lemma 2.3 (demiclosedness principle), we get $p \in \text{Fix}(T)$. Moreover, by Lemma 2.11 and Proposition 3.3 we know that $p \in \text{GMEP}(\Theta, h)$. Next we prove that $p \in \Gamma$. As a matter of fact, from (3.31) and (3.45) we know that $y_{n_i} \rightharpoonup p$ and $\tilde{y}_{n_i,N} \rightharpoonup p$. Let

$$\tilde{T}v = \begin{cases} \nabla f(v) + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases}$$

where $N_C v = \{u \in H : \langle v - p, u \rangle \geq 0, \forall p \in C\}$. Then \tilde{T} is maximal monotone and $0 \in \tilde{T}v$ if and only if $v \in \text{VI}(C, \nabla f)$; see [60] for more details. Let $(v, u) \in G(\tilde{T})$. Since $u - \nabla f(v) \in N_C v$ and $\tilde{y}_{n,N} \in C$, we have

$$\langle v - \tilde{y}_{n,N}, u - \nabla f(v) \rangle \geq 0.$$

On the other hand, from $\tilde{y}_{n,N} = P_C(I - \lambda_n \nabla f_{\alpha_n})y_{n,N}$ and $v \in C$, we have

$$\langle v - \tilde{y}_{n,N}, \tilde{y}_{n,N} - (y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})) \rangle \geq 0,$$

and hence

$$\left\langle v - \tilde{y}_{n,N}, \frac{\tilde{y}_{n,N} - y_{n,N}}{\lambda_n} + \nabla f_{\alpha_n}(y_{n,N}) \right\rangle \geq 0.$$

Therefore we have

$$\begin{aligned} & \langle v - \tilde{y}_{n_i,N}, u \rangle \\ & \geq \langle v - \tilde{y}_{n_i,N}, \nabla f(v) \rangle \\ & \geq \langle v - \tilde{y}_{n_i,N}, \nabla f(v) \rangle - \left\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\lambda_{n_i}} + \nabla f_{\alpha_{n_i}}(y_{n_i,N}) \right\rangle \\ & = \langle v - \tilde{y}_{n_i,N}, \nabla f(v) \rangle - \left\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\lambda_{n_i}} + \nabla f(y_{n_i,N}) \right\rangle - \alpha_{n_i} \langle v - \tilde{y}_{n_i,N}, y_{n_i,N} \rangle \\ & = \langle v - \tilde{y}_{n_i,N}, \nabla f(v) - \nabla f(\tilde{y}_{n_i,N}) \rangle + \langle v - \tilde{y}_{n_i,N}, \nabla f(\tilde{y}_{n_i,N}) - \nabla f(y_{n_i,N}) \rangle \\ & \quad - \left\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\lambda_{n_i}} \right\rangle - \alpha_{n_i} \langle v - \tilde{y}_{n_i,N}, y_{n_i,N} \rangle \\ & \geq \langle v - \tilde{y}_{n_i,N}, \nabla f(\tilde{y}_{n_i,N}) - \nabla f(y_{n_i,N}) \rangle - \left\langle v - \tilde{y}_{n_i,N}, \frac{\tilde{y}_{n_i,N} - y_{n_i,N}}{\lambda_{n_i}} \right\rangle - \alpha_{n_i} \langle v - \tilde{y}_{n_i,N}, y_{n_i,N} \rangle. \end{aligned}$$

From (3.31) and since ∇f is Lipschitz continuous, we obtain that $\lim_{n \rightarrow \infty} \|\nabla f(\tilde{y}_{n_iN}) - \nabla f(y_{n_iN})\| = 0$. From $\tilde{y}_{n_iN} \rightharpoonup p$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ and (3.31), we have

$$\langle v - p, u \rangle \geq 0.$$

Since \tilde{T} is maximal monotone, we have $p \in \tilde{T}^{-1}0$ and hence $p \in \text{VI}(C, \nabla f)$, which implies $p \in \Gamma$. Consequently, it is known that $p \in \text{Fix}(T) \cap \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma =: \Omega$. □

Theorem 3.1 *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$ for each index i . Moreover, let us suppose that (H0)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ defined by scheme (3.1) all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ if and only if $\lim_{n \rightarrow \infty} \|y_n - Ty_n\| = 0$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n)$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is the unique solution of the hierarchical VIP*

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega. \tag{3.46}$$

Proof First of all, we note that $F : C \rightarrow H$ is η -strongly monotone and κ -Lipschitzian on C and $f : C \rightarrow C$ is an l -Lipschitz continuous mapping with $0 \leq \gamma l < \tau$. Observe that

$$\begin{aligned} \mu\eta \geq \tau &\Leftrightarrow \mu\eta \geq 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \\ &\Leftrightarrow \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \geq 1 - \mu\eta \\ &\Leftrightarrow 1 - 2\mu\eta + \mu^2\kappa^2 \geq 1 - 2\mu\eta + \mu^2\eta^2 \\ &\Leftrightarrow \kappa^2 \geq \eta^2 \\ &\Leftrightarrow \kappa \geq \eta. \end{aligned}$$

It is clear that

$$\langle (\mu F - \gamma f)x - (\mu F - \gamma f)y, x - y \rangle \geq (\mu\eta - \gamma l)\|x - y\|^2, \quad \forall x, y \in C.$$

Hence we deduce that $\mu F - \gamma f$ is $(\mu\eta - \gamma l)$ -strongly monotone. In the meantime, it is easy to see that $\mu F - \gamma f$ is $(\mu\kappa + \gamma l)$ -Lipschitz continuous with constant $\mu\kappa + \gamma l > 0$. Thus, there exists a unique solution x^* in Ω to VIP (3.46).

Now, observe that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu F)x^*, x_n - x^* \rangle = \lim_{i \rightarrow \infty} \langle (\gamma f - \mu F)x^*, x_{n_i} - x^* \rangle. \tag{3.47}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to some $p \in H$. Without loss of generality, we may assume that $x_{n_{i_j}} \rightharpoonup p$. Then, by Corollary 3.1, we get $p \in \omega_w(x_n) \subset \Omega$. Hence, from (3.46) and (3.47), we have

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - \mu F)x^*, x_n - x^* \rangle = \langle (\gamma f - \mu F)x^*, p - x^* \rangle \leq 0. \tag{3.48}$$

Since (H1)-(H6) hold, the sequence $\{x_n\}$ is asymptotically regular (according to Proposition 3.2). In terms of (3.40) and Proposition 3.3, $\|x_n - y_n\| \rightarrow 0$ and $\|x_n - u_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let us show that $\|x_n - x^*\| \rightarrow 0$ as $n \rightarrow \infty$. Indeed, putting $p = x^*$, we deduce from (3.9), (3.10), (3.15), (3.29) and (3.30) that

$$\begin{aligned}
 & \|x_{n+1} - x^*\|^2 \\
 & \leq \beta_n \|y_n - x^*\|^2 + (1 - \beta_n) \|t_n - x^*\|^2 \\
 & \leq \beta_n \left[\epsilon_n \tau \frac{(\gamma l)^2}{\tau^2} \|y_{n,N} - x^*\|^2 + (1 - \epsilon_n \tau) \|z_n - x^*\|^2 + 2\epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \right] \\
 & \quad + (1 - \beta_n) \|t_n - x^*\|^2 \\
 & \leq \beta_n \left[\epsilon_n \frac{(\gamma l)^2}{\tau} \|y_{n,N} - x^*\|^2 + (1 - \epsilon_n \tau) \|t_n - x^*\|^2 + 2\epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \right] \\
 & \quad + (1 - \beta_n) \|t_n - x^*\|^2 \\
 & = (1 - \beta_n \epsilon_n \tau) \|t_n - x^*\|^2 + \beta_n \epsilon_n \frac{(\gamma l)^2}{\tau} \|y_{n,N} - x^*\|^2 + 2\beta_n \epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & \leq (1 - \beta_n \epsilon_n \tau) (\|x_n - x^*\| + \sqrt{2} \lambda_n \alpha_n \|x^*\|)^2 + \beta_n \epsilon_n \frac{(\gamma l)^2}{\tau} \|x_n - x^*\|^2 \\
 & \quad + 2\beta_n \epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & = (1 - \beta_n \epsilon_n \tau) [\|x_n - x^*\|^2 + 2\lambda_n \alpha_n \|x^*\| (\sqrt{2} \|x_n - x^*\| + \lambda_n \alpha_n \|x^*\|)] \\
 & \quad + \beta_n \epsilon_n \frac{(\gamma l)^2}{\tau} \|x_n - x^*\|^2 + 2\beta_n \epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & \leq \left(1 - \beta_n \epsilon_n \frac{\tau^2 - (\gamma l)^2}{\tau} \right) \|x_n - x^*\|^2 + 2\beta_n \epsilon_n \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & \quad + 2\lambda_n \alpha_n \|x^*\| (\sqrt{2} \|x_n - x^*\| + \lambda_n \alpha_n \|x^*\|) \\
 & = \left(1 - \beta_n \epsilon_n \frac{\tau^2 - (\gamma l)^2}{\tau} \right) \|x_n - x^*\|^2 \\
 & \quad + \beta_n \epsilon_n \frac{\tau^2 - (\gamma l)^2}{\tau} \cdot \frac{2\tau}{\tau^2 - (\gamma l)^2} \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & \quad + 2\lambda_n \alpha_n \|x^*\| (\sqrt{2} \|x_n - x^*\| + \lambda_n \alpha_n \|x^*\|). \tag{3.49}
 \end{aligned}$$

Since $\sum_{n=0}^\infty \alpha_n < \infty$, $\sum_{n=0}^\infty \epsilon_n = \infty$, $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{\|A\|^2})$ and $\{\beta_n\} \subset [c, d] \subset (0, 1)$, we conclude from (3.48) that $\sum_{n=0}^\infty 2\lambda_n \alpha_n \|x^*\| (\sqrt{2} \|x_n - x^*\| + \lambda_n \alpha_n \|x^*\|) < \infty$,

$$\sum_{n=0}^\infty \beta_n \epsilon_n \frac{\tau^2 - (\gamma l)^2}{\tau} \geq \sum_{n=0}^\infty c \epsilon_n \frac{\tau^2 - (\gamma l)^2}{\tau} = \infty$$

and

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \frac{2\tau}{\tau^2 - (\gamma l)^2} \langle (\gamma V - \mu F)x^*, y_n - x^* \rangle \\
 & = \limsup_{n \rightarrow \infty} \frac{2\tau}{\tau^2 - (\gamma l)^2} (\langle (\gamma V - \mu F)x^*, x_n - x^* \rangle + \langle (\gamma V - \mu F)x^*, y_n - x_n \rangle)
 \end{aligned}$$

$$\begin{aligned}
 &= \limsup_{n \rightarrow \infty} \frac{2\tau}{\tau^2 - (\gamma l)^2} \langle (\gamma V - \mu F)x^*, x_n - x^* \rangle \\
 &\leq 0.
 \end{aligned}$$

Applying Lemma 2.8 to (3.49), we infer that the sequence $\{x_n\}$ converges strongly to x^* . This completes the proof. \square

In a similar way, we can conclude another theorem as follows.

Theorem 3.2 *Let us suppose that $\Omega \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $\beta_{n,i} \rightarrow \beta_i$ for each index i as $n \rightarrow \infty$. Suppose that there exists $k \in \{1, \dots, N\}$ for which $\beta_{n,k} \rightarrow 0$ as $n \rightarrow \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index for which $\beta_{n,k_0} \rightarrow 0$. Moreover, let us suppose that (H0), (H7) and (H8) hold and*

- (i) $\frac{\alpha_n + \epsilon_n}{\beta_{n,k_0}} \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) if $i \leq k_0$ and $\beta_{n,i} \rightarrow \beta_i$, then $\frac{\beta_{n,k_0}}{\beta_{n,i}} \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) if $\beta_{n,i} \rightarrow \beta_i \neq 0$, then β_i lies in $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ defined by scheme (3.1) all converge strongly to $x^ = P_\Omega(I - (\mu F - \gamma f))x^*$ if and only if $\lim_{n \rightarrow \infty} \|y_n - Ty_n\| = 0$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$, where $x^* = P_\Omega(I - (\mu F - \gamma f))x^*$ is the unique solution of the hierarchical VIP*

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Remark 3.5 According to the above argument process for Theorems 3.1 and 3.2, we can readily see that if in scheme (3.1) the iterative step $y_n = P_C[\epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F) G P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}))]$ is replaced by the iterative one $y_n = P_C[\epsilon_n \gamma V x_n + (I - \epsilon_n \mu F) G P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N}))]$, then Theorems 3.1 and 3.2 remain valid.

Remark 3.6 Theorems 3.1 and 3.2 improve, extend, supplement and develop Theorems 3.1 and 3.2 in [43] and Theorems 3.12 and 3.13 in [42] in the following aspects.

(i) The multi-step iterative scheme (3.1) of [43] is extended to develop a hybrid extragradient viscosity iterative scheme (3.1) by virtue of Korpelevich’s extragradient method, hybrid steepest-descent method [40] and gradient-projection method (GPM) with regularization. The iterative scheme (3.1) is based on Korpelevich’s extragradient method, viscosity approximation method [7] (see also [8]), Mann’s iteration method, hybrid steepest-descent method [40] and gradient-projection method (GPM) with regularization.

(ii) The argument techniques in our Theorems 3.1 and 3.2 are very different from those techniques in Theorems 3.1 and 3.2 in [43] and Theorems 3.12 and 3.13 in [42] because we make use of the properties of strict pseudocontractions (see Lemmas 2.3 and 2.4), the ones of the resolvent operator associated with Θ and h (see Lemmas 2.9-2.11), the fixed point problem $x^* = Gx^*$ (\Leftrightarrow GSVI (1.6)) (see Proposition 2.2), the equivalence of inclusion problem $0 \in \tilde{T}v$ to the VIP $v \in VI(C, \nabla f)$ for maximal monotone operator \tilde{T} (see (2.2)) and the contractive coefficient estimates for the contractions T^λ associating with nonexpansive mappings (see Lemma 2.7).

(iii) The problem of finding an element of $\text{Fix}(T) \cap \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma$ in our Theorems 3.1 and 3.2 is more general and more subtle than the one of finding an element of $\text{Fix}(T) \cap \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{GMEP}(\Theta, h)$ in Theorems 3.12 and 3.13 in [42] (where

T is a nonexpansive mapping) and the one of finding an element of $\text{Fix}(T) \cap \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E}$ in Theorems 3.1 and 3.2 in [43] (where T is a strict pseudocontraction).

(iv) Our Theorems 3.1 and 3.2 generalize Theorems 3.12 and 3.13 in [42] from the nonexpansive mapping T to the strict pseudocontraction T and extend them to the setting of GSVI (1.6), hierarchical VIP (3.46) and SFP (1.2). In the meantime, our Theorems 3.1 and 3.2 extend Theorems 3.1 and 3.2 in [43] to the setting of hierarchical VIP (3.46) and SFP (1.2).

4 Applications

For a given nonlinear mapping $\mathcal{A} : C \rightarrow H$, we consider the variational inequality problem (VIP) of finding $\bar{x} \in C$ such that

$$\langle \mathcal{A}\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{4.1}$$

We will indicate with $\text{VI}(C, \mathcal{A})$ the set of solutions of VIP (4.1).

Recall that if u is a point in C , then the following relation holds:

$$u \in \text{VI}(C, \mathcal{A}) \iff u = P_C(I - \lambda \mathcal{A})u, \quad \forall \lambda > 0.$$

In the meantime, it is easy to see that the following relation holds:

$$\text{GSVI (1.6) with } F_2 = 0 \iff \text{VIP (4.1) with } \mathcal{A} = F_1. \tag{4.2}$$

An operator $\mathcal{A} : C \rightarrow H$ is said to be an α -inverse strongly monotone operator if there exists a constant $\alpha > 0$ such that

$$\langle \mathcal{A}x - \mathcal{A}y, x - y \rangle \geq \alpha \|\mathcal{A}x - \mathcal{A}y\|^2, \quad \forall x, y \in C.$$

As an example, we recall that the α -inverse strongly monotone operators are firmly nonexpansive mappings if $\alpha \geq 1$ and that every α -inverse strongly monotone operator is also $\frac{1}{\alpha}$ -Lipschitz continuous (see [45]).

Let us observe also that if \mathcal{A} is α -inverse strongly monotone, the mappings $P_C(I - \lambda \mathcal{A})$ are nonexpansive for all $\lambda \in (0, 2\alpha]$ since they are compositions of nonexpansive mappings (see p.419 in [45]).

Let us consider $\tilde{S}_1, \dots, \tilde{S}_M$ be a finite number of nonexpansive self-mappings on C and A_1, \dots, A_N be a finite number of α -inverse strongly monotone operators. Let $T : C \rightarrow C$ be a ξ -strict pseudocontraction with fixed points. Let us consider the following mixed problem of finding $x^* \in \text{Fix}(T) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma$ such that

$$\left\{ \begin{array}{l} \langle (I - \tilde{S}_1)x^*, y - x^* \rangle \geq 0, \quad \forall y \in \text{Fix}(T) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma, \\ \langle (I - \tilde{S}_2)x^*, y - x^* \rangle \geq 0, \quad \forall y \in \text{Fix}(T) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma, \\ \dots, \\ \langle (I - \tilde{S}_M)x^*, y - x^* \rangle \geq 0, \quad \forall y \in \text{Fix}(T) \cap \text{GMEP}(\Theta, h) \cap \mathcal{E} \cap \Gamma, \\ \langle A_1x^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \\ \langle A_2x^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \\ \dots, \\ \langle A_Nx^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \end{array} \right. \tag{4.3}$$

Let us call (SVI) the set of solutions of the $(M + N)$ -system. This problem is equivalent to finding a common fixed point of $T, \{P_{\text{Fix}(T) \cap \text{GMPEP}(\theta, h) \cap \mathcal{E} \cap \Gamma} \tilde{S}_i\}_{i=1}^M, \{P_C(I - \lambda A_i)\}_{i=1}^N$. So we claim that the following holds.

Theorem 4.1 *Let us suppose that $\Omega = \text{Fix}(T) \cap (\text{SVI}) \cap \text{GMPEP}(\theta, h) \cap \mathcal{E} \cap \Gamma \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, (M + N)$, be sequences in $(0, 1)$ such that $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$ for all indices i . Moreover, let us suppose that (H0)-(H6) hold. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme*

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, & \forall y \in C, \\ y_{n,1} = \beta_{n,1} P_{\text{Fix}(T) \cap \text{GMPEP}(\theta, h) \cap \mathcal{E} \cap \Gamma} \tilde{S}_1 u_n + (1 - \beta_{n,1}) u_n, \\ y_{n,i} = \beta_{n,i} P_{\text{Fix}(T) \cap \text{GMPEP}(\theta, h) \cap \mathcal{E} \cap \Gamma} \tilde{S}_i u_n + (1 - \beta_{n,i}) y_{n,i-1}, & i = 2, \dots, M, \\ y_{n,M+j} = \beta_{n,M+j} P_C(I - \lambda A_j) u_n + (1 - \beta_{n,M+j}) y_{n,M+j-1}, & j = 1, \dots, N, \\ \tilde{y}_{n,M+N} = P_C(y_{n,M+N} - \lambda_n \nabla f_{\alpha_n}(y_{n,M+N})), \\ y_n = P_C[\epsilon_n \gamma V y_{n,M+N} + (I - \epsilon_n \mu F) G P_C(y_{n,M+N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,M+N}))], \\ x_{n+1} = \beta_n y_n + \gamma_n P_C(y_{n,M+N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,M+N})) + \delta_n T P_C(y_{n,M+N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,M+N})), \end{cases} \tag{4.4}$$

all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$ if and only if $\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n)$, where $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$ is the unique solution of the hierarchical VIP

$$\langle (\gamma V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Theorem 4.2 *Let us suppose that $\Omega \neq \emptyset$. Fix $\lambda > 0$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, (M + N)$, be sequences in $(0, 1)$ and $\beta_{n,i} \rightarrow \beta_i$ for all i as $n \rightarrow \infty$. Suppose that there exists $k \in \{1, \dots, M + N\}$ such that $\beta_{n,k} \rightarrow 0$ as $n \rightarrow \infty$. Let $k_0 \in \{1, \dots, M + N\}$ be the largest index for which $\beta_{n,k_0} \rightarrow 0$. Moreover, let us suppose that (H0), (H7) and (H8) hold and*

- (i) $\frac{\alpha_n + \epsilon_n}{\beta_{n,k_0}} \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) if $i \leq k_0$ and $\beta_{n,i} \rightarrow 0$, then $\frac{\beta_{n,k_0}}{\beta_{n,i}} \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) if $\beta_{n,i} \rightarrow \beta_i \neq 0$, then β_i lies in $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ explicitly defined by scheme (4.4) all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$ if and only if $\lim_{n \rightarrow \infty} \|y_n - T y_n\| = 0$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$, where $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$ is the unique solution of the VIP

$$\langle (\gamma V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Remark 4.1 If in system (4.3), $F_1 = F_2 = A_1 = \dots = A_N = 0$ and T is a nonexpansive mapping, we obtain a system of hierarchical fixed point problems introduced by Mainge and Moudafi [34, 36].

On the other hand, recall that a mapping $S : C \rightarrow C$ is called ζ -strictly pseudocontractive if there exists a constant $\zeta \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \zeta \|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

If $\zeta = 0$, then S is nonexpansive. Put $\mathcal{A} = I - S$, where $S : C \rightarrow C$ is a ζ -strictly pseudocontractive mapping. Then \mathcal{A} is $\frac{1-\zeta}{2}$ -inverse strongly monotone; see [57].

Utilizing Theorems 3.1 and 3.2, we also give two strong convergence theorems for finding a common element of the solution set $\text{GMEP}(\Theta, h)$ of GMEP (1.3), the solution set Γ of SFP (1.2) and the common fixed point set $\bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{Fix}(S)$ of finitely many non-expansive mappings $S_i : C \rightarrow C, i = 1, \dots, N$, and a ζ -strictly pseudocontractive mapping $S : C \rightarrow C$.

Theorem 4.3 *Let $v_1 \in (0, 1 - \zeta)$. Let us suppose that $\Omega = \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{Fix}(S) \cap \text{GMEP}(\Theta, h) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $0 < \liminf_{n \rightarrow \infty} \beta_{n,i} \leq \limsup_{n \rightarrow \infty} \beta_{n,i} < 1$ for all indices i . Moreover, let us suppose that there hold (H0)-(H6) with $\gamma_n = 0, \forall n \geq 0$. Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ generated explicitly by*

$$\begin{cases} \Theta(u_n, y) + h(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - x_n) \geq 0, & \forall y \in C, \\ y_{n,1} = \beta_{n,1}S_1u_n + (1 - \beta_{n,1})u_n, \\ y_{n,i} = \beta_{n,i}S_iu_n + (1 - \beta_{n,i})y_{n,i-1}, & i = 2, \dots, N, \\ \tilde{y}_{n,N} = P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(y_{n,N})), \\ t_n = P_C(y_{n,N} - \lambda_n \nabla f_{\alpha_n}(\tilde{y}_{n,N})), \\ y_n = P_C[\epsilon_n \gamma V y_{n,N} + (I - \epsilon_n \mu F)((1 - v_1)t_n + v_1 S t_n)], \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)t_n, & \forall n \geq 0, \end{cases} \tag{4.5}$$

all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n)$, which is the unique solution of the VIP

$$\langle (\gamma V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Proof In Theorem 3.1, put $F_1 = \mathcal{A} = I - S$ and $F_2 = 0$. Then \mathcal{A} is $\frac{1-\zeta}{2}$ -inverse strongly monotone. Hence we deduce that $\text{Fix}(S) = \text{VI}(C, \mathcal{A}) = \mathcal{E}$ and

$$\begin{aligned} Gt_n &= P_C(I - v_1 F_1)P_C(I - v_2 F_2)t_n \\ &= P_C(I - v_1 F_1)t_n \\ &= (1 - v_1)t_n + v_1 S t_n. \end{aligned}$$

Thus, in terms of Theorem 3.1, we obtain the desired result. □

Theorem 4.4 *Let $v_1 \in (0, 1 - \zeta)$. Let us suppose that $\Omega = \bigcap_{i=1}^N \text{Fix}(S_i) \cap \text{Fix}(S) \cap \text{GMEP}(\Theta, h) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_{n,i}\}, i = 1, \dots, N$, be sequences in $(0, 1)$ such that $\beta_{n,i} \rightarrow \beta_i$ for all i as $n \rightarrow \infty$. Suppose that there exists $k \in \{1, \dots, N\}$ for which $\beta_{n,k} \rightarrow 0$ as $n \rightarrow \infty$. Let $k_0 \in \{1, \dots, N\}$ be the largest index for which $\beta_{n,k_0} \rightarrow 0$. Moreover, let us suppose that there hold (H0), (H7) and (H8) with $\gamma_n = 0, \forall n \geq 0$ and*

- (i) $\frac{\alpha_n + \epsilon_n}{\beta_{n,k_0}} \rightarrow 0$ as $n \rightarrow \infty$;
- (ii) if $i \leq k_0$ and $\beta_{n,i} \rightarrow 0$, then $\frac{\beta_{n,k_0}}{\beta_{n,i}} \rightarrow 0$ as $n \rightarrow \infty$;
- (iii) if $\beta_{n,i} \rightarrow \beta_i \neq 0$, then β_i lies in $(0, 1)$.

Then the sequences $\{x_n\}, \{y_n\}$ and $\{u_n\}$ generated explicitly by (4.5) all converge strongly to $x^* = P_\Omega(I - (\mu F - \gamma V))x^*$, provided $\|u_n - u_{n-1}\| = o(\epsilon_n \beta_{n,k_0})$, which is the unique solution of the hierarchical VIP

$$\langle (\gamma V - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \Omega.$$

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

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