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Inequalities and asymptotic formulas for $(1+1/x)^{x+a}$

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Abstract

(i) Let $a \neq \frac{1}{2}$ be a given real number. We give a formula for determining the coefficients $b_j \equiv b_j(a)$ such that $(1+\frac{1}{x})^{x+a} \sim \exp(1+\frac{(a-\frac{1}{2})x}{(x+\sum_{j=0}^{\infty} b_j x^{-j})^2})$, $x \to \infty$. This solves an open problem of Hu and Mortici. (ii) Hu and Mortici presented the following asymptotic representation: $(1+\frac{1}{x})^{x+\frac{1}{2}} \sim \exp(1+\frac{\frac{1}{12}x}{(x+\sum_{j=0}^{\infty} a_j x^{-j})^3})$, $x \to \infty$; the coefficients a_j can be inductively obtained by equating the following relation: $(\sum_{j=3}^{\infty}(-1)^{j-1}\frac{6(j-2)}{j(j-1)x^j})(x+\sum_{j=0}^{\infty}\frac{a_j}{x^j})^3 = 1$. We here provide a recurrence relation for determining the coefficients a_j. The representation using recursive algorithm is better for numerical evaluations. (iii) We present new inequalities and asymptotic formulas for $(1+1/x)^{x+a}$.

MSC: 41A60; 26D20

Keywords: constant e; asymptotic formula; inequality

1 Introduction

The constant e is the base in the natural logarithm. e can be defined by the limit

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828182845904523536028 75\ldots.$$

With the possible exception of π, e is the most important constant in mathematics since it appears in myriad mathematical contexts involving limits and derivatives. It is noted (see, *e.g.*, [1]; see also [2], pp.26-27) that approximations to e were first discovered in the 1600s. For the various inequalities and approximations of the constants e, the reader may be referred to several recent works (see, *e.g.*, [3–9]).

Batir and Cancan (see [3], Theorem 2.5 and Theorem 2.6) proved that for $n \in \mathbb{N} := \{1, 2, \ldots\}$,

$$\exp\left(1-\frac{n}{2(n+c)^2}\right) \le \left(1+\frac{1}{n}\right)^n < \exp\left(1-\frac{n}{2(n+d)^2}\right) \tag{1.1}$$

and

$$\exp\left(1+\frac{1}{2(n+\alpha)}\right) < \left(1+\frac{1}{n}\right)^{n+1} \le \exp\left(1+\frac{1}{2(n+\beta)}\right) \tag{1.2}$$

with the best possible constants

$$c = \frac{1}{\sqrt{2 - \ln 4}} - 1 = 0.27649\ldots, \qquad d = \frac{1}{3},$$

and

$$\alpha = \frac{1}{3}, \qquad \beta = \frac{1}{4\ln 2 - 2} - 1 = 0.2943497\ldots.$$

Inspired by (1.1) and (1.2), Hu and Mortici (see [6], Theorem 1 and Theorem 2) established the following inequalities:

$$\exp\left(1 - \frac{x}{2(x + \frac{1}{3} - \frac{1}{12x})^2}\right) < \left(1 + \frac{1}{x}\right)^x < \exp\left(1 - \frac{x}{2(x + \frac{1}{3} - \frac{1}{12x} + \frac{23}{540x^2})^2}\right) \tag{1.3}$$

and

$$\exp\left(1 + \frac{x}{2(x + \frac{1}{6} - \frac{1}{24x} + \frac{43}{2{,}160x^2})^2}\right) < \left(1 + \frac{1}{x}\right)^{x+1} < \exp\left(1 + \frac{x}{2(x + \frac{1}{6} - \frac{1}{24x})^2}\right) \tag{1.4}$$

for $x \geq 1$.

Also in [6], the authors proved

$$\exp\left(1 + \frac{x}{12(x + \frac{1}{3})^3}\right) < \left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} < \exp\left(1 + \frac{x}{12(x + \frac{1}{3} - \frac{7}{90x})^3}\right), \quad x \geq 1 \tag{1.5}$$

(see [6], Theorem 6), and they proposed as an open problem the following approximation formula:

$$\left(1 + \frac{1}{n}\right)^{n+a} \approx \exp\left(1 + \frac{(a - \frac{1}{2})n}{(n + \frac{3a-2}{6(2a-1)})^2}\right) \tag{1.6}$$

for $a \in [0,1] \setminus \{\frac{1}{2}\}$.

For

$$a \in \left[0, \frac{4 - \sqrt{2}}{7}\right) \cup \left(\frac{1}{2}, \frac{4 + \sqrt{2}}{7}\right) \cup (\theta, 1],$$

where $\theta = 0.82462\ldots$ is the unique zero of $675a^3 - 1{,}098a^2 + 558a - 92$, Hu and Mortici (see [6], Theorems 3 to 5) considered the inequalities for $(1 + 1/x)^{x+a}$. In fact, the authors obtained the following approximation formula:

$$\left(1 + \frac{1}{x}\right)^{x+a} \approx \exp\left(1 + \frac{(a - \frac{1}{2})x}{(x + \frac{3a-2}{6(2a-1)} + \frac{-7a^2+8a-2}{24(2a-1)^2 x})^2}\right), \quad x \to \infty. \tag{1.7}$$

In the case $a = \frac{1}{2}$, Hu and Mortici [6] presented the following asymptotic representation:

$$\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} \sim \exp\left(1 + \frac{\frac{1}{12}x}{(x + \sum_{j=0}^{\infty} a_j x^{-j})^3}\right), \quad x \to \infty. \tag{1.8}$$

Moreover, the authors showed that the coefficients a_j in (1.8) can be inductively obtained by equating the following relation:

$$\left(\sum_{j=3}^{\infty}(-1)^{j-1}\frac{6(j-2)}{j(j-1)x^j}\right)\left(x+\sum_{j=0}^{\infty}\frac{a_j}{x^j}\right)^3 = 1. \tag{1.9}$$

The first few coefficients a_j are

$$a_0 = \frac{1}{3}, \qquad a_1 = -\frac{7}{90}, \qquad a_2 = \frac{16}{405}, \qquad a_3 = -\frac{2,141}{85,050}.$$

We then obtain the following explicit asymptotic expansion:

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} \sim \exp\left(1+\frac{\frac{1}{12}x}{(x+\frac{1}{3}-\frac{7}{90x}+\frac{16}{405x^2}-\frac{2,141}{85,050x^3}+\cdots)^3}\right), \quad x\to\infty. \tag{1.10}$$

In this paper, our tasks are as follows:

(i) We consider the above open problem of Hu and Mortici and develop the approximation formula (1.6) to produce a complete asymptotic expansion. More precisely, we give a formula for determining the coefficients $b_j \equiv b_j(a)$ such that

$$\left(1+\frac{1}{x}\right)^{x+a} \sim \exp\left(1+\frac{(a-\frac{1}{2})x}{(x+\sum_{j=0}^{\infty}b_jx^{-j})^2}\right) \quad \left(x\to\infty; a\neq\frac{1}{2}\right).$$

(ii) We provide a recurrence relation for determining the coefficients a_j in (1.8). The representation using recursive algorithm is better for numerical evaluations.

(iii) We present new inequalities and asymptotic expansions for $(1+1/x)^{x+a}$.

The following lemma is required in the sequel.

Lemma 1.1 (see [10]) *Let $g(x)$ be a function with an asymptotic expansion* ($q_0 = 1$)

$$g(x) \sim \sum_{j=0}^{\infty} q_j x^{-j}, \quad x\to\infty.$$

Then for all real r we have

$$\left[g(x)\right]^r \sim \sum_{j=0}^{\infty} P_j(r)x^{-j}, \quad x\to\infty,$$

where

$$P_0(r) = 1, \qquad P_j(r) = \frac{1}{j}\sum_{k=1}^{j}\left[k(1+r)-j\right]q_kP_{j-k}(r), \quad j\in\mathbb{N}. \tag{1.11}$$

2 Asymptotic formulas

Theorem 2.1 develops the approximation formula (1.6) to produce a complete asymptotic expansion.

Theorem 2.1 *Let $a \neq \frac{1}{2}$ be a given real number. The following asymptotic expansion holds*:

$$\left(1+\frac{1}{x}\right)^{x+a} \sim \exp\left(1+\frac{(a-\frac{1}{2})x}{(x+\sum_{j=0}^{\infty} b_j x^{-j})^2}\right), \quad x\to\infty, \tag{2.1}$$

with the coefficients $b_j \equiv b_j(a)$ given by

$$b_j = P_{j+1}\left(-\frac{1}{2}\right), \quad j \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}, \tag{2.2}$$

where the $P_j(-\frac{1}{2})$ are given by the recurrence relation

$$\begin{aligned} &P_0\left(-\frac{1}{2}\right) = 1, \\ &P_j\left(-\frac{1}{2}\right) = \frac{1}{j}\sum_{k=1}^{j}(-1)^k\left(\frac{k}{2}-j\right)\frac{2((a-1)k+2a-1)}{(2a-1)(k+1)(k+2)}P_{j-k}\left(-\frac{1}{2}\right), \quad j\in\mathbb{N}. \end{aligned} \tag{2.3}$$

Proof To determine the coefficients b_j in (2.1), we write (2.1) as

$$\bigl(g(x)\bigr)^{-1/2} \sim x + \sum_{j=0}^{\infty} b_j x^{-j}, \tag{2.4}$$

where

$$g(x) = \frac{(x+a)\ln(1+\frac{1}{x})-1}{(a-\frac{1}{2})x}.$$

Using the Maclaurin expansion of $\ln(1+t)$ with $t = x^{-1}$ yields

$$g(x) \sim x^{-2}\sum_{k=0}^{\infty} q_k x^{-k}, \quad x\to\infty$$

with

$$q_k = (-1)^k \frac{2((a-1)k+2a-1)}{(2a-1)(k+1)(k+2)}. \tag{2.5}$$

By Lemma 1.1, we have

$$\bigl(g(x)\bigr)^{-1/2} \sim x\left(\sum_{j=0}^{\infty} q_k x^{-k}\right)^{-1/2} \sim x\sum_{j=0}^{\infty} P_j\left(-\frac{1}{2}\right)x^{-j} \tag{2.6}$$

with the coefficients $P_j(-\frac{1}{2})$ given by

$$P_0\left(-\frac{1}{2}\right) = 1, \qquad P_j\left(-\frac{1}{2}\right) = \frac{1}{j}\sum_{k=1}^{j}\left(\frac{k}{2}-j\right)q_k P_{j-k}\left(-\frac{1}{2}\right), \quad j\in\mathbb{N}, \tag{2.7}$$

where the q_k are given by (2.5).

Hence, it follows from (2.4) and (2.6) that

$$x + \sum_{j=0}^{\infty} b_j x^{-j} \sim x \sum_{j=0}^{\infty} P_j\left(-\frac{1}{2}\right) x^{-j},$$

$$\sum_{j=0}^{\infty} b_j x^{-j} \sim \sum_{j=0}^{\infty} P_{j+1}\left(-\frac{1}{2}\right) x^{-j},$$

we then obtain

$$b_j = P_{j+1}\left(-\frac{1}{2}\right), \quad j \in \mathbb{N}_0,$$

where the $P_j(-\frac{1}{2})$ are given in (2.7). The proof of Theorem 2.1 is complete. □

Remark 2.1 Using (2.2) and (2.3), we now show that we easily can determine the b_j in (2.1). The first few coefficients $b_j \equiv b_j(a)$ are

$$b_0 = P_1\left(-\frac{1}{2}\right) = \frac{3a-2}{6(2a-1)} P_0\left(-\frac{1}{2}\right) = \frac{3a-2}{6(2a-1)},$$

$$b_1 = P_2\left(-\frac{1}{2}\right) = \frac{(9a-6)P_1(-\frac{1}{2}) + (3-4a)P_0(-\frac{1}{2})}{12(2a-1)} = -\frac{2-8a+7a^2}{24(2a-1)^2},$$

$$\begin{aligned} b_2 = P_3\left(-\frac{1}{2}\right) &= \frac{(150a-100)P_2(-\frac{1}{2}) + (60-80a)P_1(-\frac{1}{2}) + (45a-36)P_0(-\frac{1}{2})}{108(2a-1)} \\ &= \frac{-92+558a-1{,}098a^2+675a^3}{2{,}160(2a-1)^3}, \end{aligned}$$

$$\begin{aligned} b_3 &= P_4\left(-\frac{1}{2}\right) \\ &= -\frac{(140-210a)P_3(-\frac{1}{2}) + (120a-90)P_2(-\frac{1}{2}) + (60-75a)P_1(-\frac{1}{2}) + (48a-40)P_0(-\frac{1}{2})}{240(2a-1)} \\ &= -\frac{1{,}412-11{,}424a+34{,}308a^2-44{,}928a^3+21{,}249a^4}{51{,}840(2a-1)^4}. \end{aligned}$$

In particular, setting $a = 0$ and $a = 1$ in (2.1), respectively, we have the following complete asymptotic expansions:

$$\left(1+\frac{1}{x}\right)^x \sim \exp\left(1 - \frac{x}{2(x+\frac{1}{3}-\frac{1}{12x}+\frac{23}{540x^2}-\frac{353}{12{,}960x^3}+\cdots)^2}\right) \tag{2.8}$$

and

$$\left(1+\frac{1}{x}\right)^{x+1} \sim \exp\left(1 + \frac{x}{2(x+\frac{1}{6}-\frac{1}{24x}+\frac{43}{2{,}160x^2}-\frac{617}{51{,}840x^3}+\cdots)^2}\right) \tag{2.9}$$

as $x \to \infty$.

Theorem 2.2 provides a recurrence relation for determining the coefficients a_j in (1.8).

Theorem 2.2 *The coefficients a_j in* (1.8) *are given by*

$$a_j = P_{j+1}\left(-\frac{1}{3}\right), \quad j \in \mathbb{N}_0, \tag{2.10}$$

where the $P_j(-\frac{1}{3})$ are given by the recurrence relation

$$\begin{aligned} &P_0\left(-\frac{1}{3}\right) = 1, \\ &P_j\left(-\frac{1}{3}\right) = \frac{1}{j}\sum_{k=1}^{j}(-1)^k\left(\frac{2k}{3} - j\right)\frac{6(k+1)}{(k+2)(k+3)}P_{j-k}\left(-\frac{1}{3}\right), \quad j \in \mathbb{N}. \end{aligned} \tag{2.11}$$

Proof A similar argument to the proof of Theorem 2.1 will establish the result in Theorem 2.2. To determine the coefficients a_j in (1.8), we write (1.8) as

$$\bigl(f(x)\bigr)^{-1/3} \sim x + \sum_{j=0}^{\infty} a_j x^{-j}, \tag{2.12}$$

where

$$f(x) = \frac{(x+\frac{1}{2})\ln(1+\frac{1}{x}) - 1}{\frac{1}{12}x}.$$

Using the Maclaurin expansion of $\ln(1+t)$ with $t = x^{-1}$ yields

$$f(x) \sim x^{-3}\sum_{k=0}^{\infty} p_k x^{-k}, \quad x \to \infty,$$

with

$$p_k = (-1)^k \frac{6(k+1)}{(k+2)(k+3)}. \tag{2.13}$$

By Lemma 1.1, we have

$$\bigl(f(x)\bigr)^{-1/3} \sim x\left(\sum_{k=0}^{\infty} p_k x^{-k}\right)^{-1/3} \sim x\sum_{j=0}^{\infty} P_j\left(-\frac{1}{3}\right)x^{-j} \tag{2.14}$$

with the coefficients $P_j(-\frac{1}{3})$ given by

$$P_0\left(-\frac{1}{3}\right) = 1, \qquad P_j\left(-\frac{1}{3}\right) = \frac{1}{j}\sum_{k=1}^{j}\left(\frac{2k}{3} - j\right)p_k P_{j-k}\left(-\frac{1}{3}\right), \quad j \in \mathbb{N}, \tag{2.15}$$

where the p_k are given by (2.13).

Hence, it follows from (2.12) and (2.14) that

$$x + \sum_{j=0}^{\infty} a_j x^{-j} \sim x\sum_{j=0}^{\infty} P_j\left(-\frac{1}{3}\right)x^{-j},$$

$$\sum_{j=0}^{\infty} a_j x^{-j} \sim \sum_{j=0}^{\infty} P_{j+1}\left(-\frac{1}{3}\right) x^{-j},$$

we then obtain

$$a_j = P_{j+1}\left(-\frac{1}{3}\right), \quad j \in \mathbb{N}_0,$$

where the $P_j(-\frac{1}{3})$ are given in (2.15). The proof of Theorem 2.2 is complete. □

Remark 2.2 Using (2.10) and (2.11), we now show how easily we can determine the a_j in (1.8). The first few coefficients a_j are

$$\begin{aligned}
a_0 &= P_1\left(-\frac{1}{3}\right) = \frac{1}{3}P_0\left(-\frac{1}{3}\right) = \frac{1}{3}, \\
a_1 &= P_2\left(-\frac{1}{3}\right) = \frac{2}{3}P_1\left(-\frac{1}{3}\right) - \frac{3}{10}P_0\left(-\frac{1}{3}\right) = -\frac{7}{90}, \\
a_2 &= P_3\left(-\frac{1}{3}\right) = \frac{7}{9}P_2\left(-\frac{1}{3}\right) - \frac{1}{2}P_1\left(-\frac{1}{3}\right) + \frac{4}{15}P_0\left(-\frac{1}{3}\right) = \frac{16}{405}, \\
a_3 &= P_4\left(-\frac{1}{3}\right) = \frac{5}{6}P_3\left(-\frac{1}{3}\right) - \frac{3}{5}P_2\left(-\frac{1}{3}\right) + \frac{2}{5}P_1\left(-\frac{1}{3}\right) - \frac{5}{21}P_0\left(-\frac{1}{3}\right) = -\frac{2{,}141}{85{,}050}.
\end{aligned}$$

We note that the values of a_j (for $j = 0, 1, 2, 3$) here are equal to the coefficients of $1/x^j$ (for $j = 0, 1, 2, 3$) in (1.10), respectively.

Theorems 2.3 and 2.4 present new asymptotic expansions for $(1 + 1/x)^{x+\alpha}$ (with $\alpha \neq \frac{1}{2}$) and $(1 + 1/x)^{x+1/2}$, respectively. As the proofs of Theorems 2.3 and 2.4 are similar to the proof of Theorem 2.1, we omit them.

Theorem 2.3 *Let $\alpha \neq \frac{1}{2}$ be a given real number. The following asymptotic expansion holds*:

$$\left(1 + \frac{1}{x}\right)^{x+\alpha} \sim \exp\left(1 + \frac{\alpha - \frac{1}{2}}{x + \sum_{j=0}^{\infty} \beta_j x^{-j}}\right), \quad x \to \infty, \tag{2.16}$$

with the coefficients $\beta_j \equiv \beta_j(\alpha)$ given by

$$\beta_j = P_{j+1}(-1), \quad j \in \mathbb{N}_0, \tag{2.17}$$

where the $P_j(-1)$ are given by the recurrence relation

$$P_0(-1) = 1, \qquad P_j(-1) = \sum_{k=1}^{j} (-1)^{k-1} \frac{2((\alpha - 1)k + 2\alpha - 1)}{(2\alpha - 1)(k+1)(k+2)} P_{j-k}(-1), \quad j \in \mathbb{N}. \tag{2.18}$$

Remark 2.3 The first few coefficients $\beta_j \equiv \beta_j(\alpha)$ are

$$\beta_0 = \frac{3\alpha - 2}{3(2\alpha - 1)},$$

$$\beta_1 = -\frac{6\alpha^2 - 6\alpha + 1}{18(2\alpha - 1)^2},$$

$$\beta_2 = \frac{90\alpha^3 - 132\alpha^2 + 57\alpha - 8}{270(2\alpha - 1)^3},$$

$$\beta_3 = -\frac{684\alpha^4 - 1{,}332\alpha^3 + 918\alpha^2 - 276\alpha + 31}{1{,}620(2\alpha - 1)^4}.$$

In particular, setting $\alpha = 0$ and $\alpha = 1$ in (2.16), respectively, we have the following complete asymptotic expansions:

$$\left(1 + \frac{1}{x}\right)^x \sim \exp\left(1 - \frac{\frac{1}{2}}{x + \frac{2}{3} - \frac{1}{18x} + \frac{4}{135x^2} - \frac{31}{1{,}620x^3} + \cdots}\right) \tag{2.19}$$

and

$$\left(1 + \frac{1}{x}\right)^{x+1} \sim \exp\left(1 + \frac{\frac{1}{2}}{x + \frac{1}{3} - \frac{1}{18x} + \frac{7}{270x^2} - \frac{5}{324x^3} + \cdots}\right) \tag{2.20}$$

as $x \to \infty$.

Theorem 2.4 *The following asymptotic expansion holds*:

$$\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} \sim \exp\left(1 + \frac{\frac{1}{12}}{x^2 + x + \frac{1}{10} + \sum_{j=1}^{\infty} c_j x^{-j}}\right), \quad x \to \infty, \tag{2.21}$$

with the coefficients c_j given by

$$c_j = P_{j+2}, \quad j \in \mathbb{N}, \tag{2.22}$$

where the P_j are given by the recurrence relation

$$P_0 = 1, \qquad P_j = \sum_{k=1}^{j} (-1)^{k-1} \frac{6(k+1)}{(k+2)(k+3)} P_{j-k}, \quad j \in \mathbb{N}. \tag{2.23}$$

Remark 2.4 Here, from (2.21), we give the following explicit asymptotic expansion:

$$\left(1 + \frac{1}{x}\right)^{x+\frac{1}{2}} \sim \exp\left(1 + \frac{\frac{1}{12}}{x^2 + x + \frac{1}{10} - \frac{3}{700x^2} + \frac{3}{700x^3} - \frac{79}{21{,}000x^4} + \cdots}\right), \quad x \to \infty. \tag{2.24}$$

Here, we show the superiority of our new approximation formulas over Hu and Mortici's approximation formulas.

It follows from (2.8) and (2.19) that

$$\left(1 + \frac{1}{x}\right)^x \approx \exp\left(1 - \frac{\frac{1}{2}x}{(x + \frac{1}{3} - \frac{1}{12x} + \frac{23}{540x^2} - \frac{353}{12{,}960x^3})^2}\right) := f_1(x) \tag{2.25}$$

and

$$\left(1 + \frac{1}{x}\right)^x \approx \exp\left(1 - \frac{\frac{1}{2}}{x + \frac{2}{3} - \frac{1}{18x} + \frac{4}{135x^2} - \frac{31}{1{,}620x^3}}\right) := g_1(x). \tag{2.26}$$

Moreover, we find by the Maple software that, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^x = f_1(x) + O\bigl(x^{-6}\bigr) \quad \text{and} \quad \left(1+\frac{1}{x}\right)^x = g_1(x) + O\bigl(x^{-6}\bigr).$$

We now prove that

$$f_1(x) < g_1(x) < \left(1+\frac{1}{x}\right)^x, \quad x \geq 1. \tag{2.27}$$

Elementary calculation shows that

$$\frac{f_1(x)}{g_1(x)} = \exp\left(-\frac{810x^3(4{,}242{,}240x^3 - 1{,}067{,}184x^2 + 389{,}712x - 124{,}609)}{A(x)}\right)$$

with

$$\begin{aligned} A(x) &= \bigl(12{,}960x^4 + 4{,}320x^3 - 1{,}080x^2 + 552x - 353\bigr)^2 \\ &\quad \times \bigl(1{,}620x^4 + 1{,}080x^3 - 90x^2 + 48x - 31\bigr). \end{aligned}$$

Hence, we have $f_1(x) < g_1(x)$ for $x \geq 1$.

The second inequality in (2.27) is obtained by considering the function $F(x)$ defined for $x \geq 1$ by

$$\begin{aligned} F(x) &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\ln g_1(x) \\ &= \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\left(1 - \frac{\frac{1}{2}}{x + \frac{2}{3} - \frac{1}{18x} + \frac{4}{135x^2} - \frac{31}{1{,}620x^3}}\right). \end{aligned}$$

Differentiation yields

$$F'(x) = -\frac{119{,}951 + 364{,}668(x-1) + 369{,}576(x-1)^2 + 125{,}820(x-1)^3}{x^2(x+1)(1{,}620x^4 + 1{,}080x^3 - 90x^2 + 48x - 31)^2} < 0, \quad x \geq 1.$$

Hence, $F(x)$ is strictly decreasing for $x \geq 1$, and we have

$$F(x) > \lim_{t\to\infty} F(t) = 0, \quad x \geq 1.$$

This means that the second inequality in (2.27) holds for $x \geq 1$.

The double inequality (2.27) shows that (2.26) is better than (2.25).

It follows from (2.9) and (2.20) that

$$\left(1+\frac{1}{x}\right)^{x+1} \approx \exp\left(1 + \frac{\frac{1}{2}x}{(x + \frac{1}{6} - \frac{1}{24x} + \frac{43}{2{,}160x^2} - \frac{617}{51{,}840x^3})^2}\right) := f_2(x) \tag{2.28}$$

and

$$\left(1+\frac{1}{x}\right)^{x+1} \approx \exp\left(1 + \frac{\frac{1}{2}}{x + \frac{1}{3} - \frac{1}{18x} + \frac{7}{270x^2} - \frac{5}{324x^3}}\right) := g_2(x). \tag{2.29}$$

Moreover, we find by the Maple software that, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^{x+1} = f_2(x) + O\left(x^{-6}\right) \quad \text{and} \quad \left(1+\frac{1}{x}\right)^{x+1} = g_2(x) + O\left(x^{-6}\right).$$

In fact, the following double inequality holds:

$$\left(1+\frac{1}{x}\right)^{x+1} < g_2(x) < f_2(x), \quad x \geq 1. \tag{2.30}$$

As the proofs of the inequalities (2.30) are similar to the proofs of the inequalities (2.27), we omit them. The double inequality (2.30) shows that (2.29) is better than (2.28).

It follows from (1.10) and (2.24) that

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} \approx \exp\left(1+\frac{\frac{1}{12}x}{(x+\frac{1}{3}-\frac{7}{90x}+\frac{16}{405x^2}-\frac{2{,}141}{85{,}050x^3})^3}\right) := f_3(x) \tag{2.31}$$

and

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} \approx \exp\left(1+\frac{\frac{1}{12}}{x^2+x+\frac{1}{10}-\frac{3}{700x^2}}\right) := g_3(x). \tag{2.32}$$

Moreover, we find by the Maple software that, as $x \to \infty$,

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} = f_3(x) + O\left(x^{-7}\right) \quad \text{and} \quad \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} = g_3(x) + O\left(x^{-7}\right).$$

In fact, the following double inequality holds:

$$\left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} < g_3(x) < f_3(x), \quad x \geq 1. \tag{2.33}$$

As the proofs of the inequalities (2.33) are similar to the proofs of the inequalities (2.27), we omit them. The double inequality (2.33) shows that (2.32) is better than (2.31).

3 Inequalities

Equations (2.19), (2.20), and (2.24) motivated us to observe Theorem 3.1.

Theorem 3.1 (i) *The following inequality holds*:

$$\exp\left(1-\frac{\frac{1}{2}}{x+\frac{2}{3}-\frac{1}{18x}}\right) < \left(1+\frac{1}{x}\right)^{x} < \exp\left(1-\frac{\frac{1}{2}}{x+\frac{2}{3}-\frac{1}{18x}+\frac{4}{135x^2}}\right). \tag{3.1}$$

The first inequality in (3.1) *is valid provided* $x \geq 1$, *while the second inequality in* (3.1) *holds for* $x > 0$.

(ii) *The following inequality holds*:

$$\exp\left(1+\frac{\frac{1}{2}}{x+\frac{1}{3}-\frac{1}{18x}+\frac{7}{270x^2}}\right) < \left(1+\frac{1}{x}\right)^{x+1} < \exp\left(1+\frac{\frac{1}{2}}{x+\frac{1}{3}-\frac{1}{18x}}\right). \tag{3.2}$$

The first inequality in (3.2) *holds for* $x > 0$, *while the second inequality in* (3.2) *is valid provided* $x \geq 1$.

(iii) *The following inequality holds*:

$$\exp\left(1+\frac{\frac{1}{12}}{x^2+x+\frac{1}{10}}\right) < \left(1+\frac{1}{x}\right)^{x+\frac{1}{2}} < \exp\left(1+\frac{\frac{1}{12}}{x^2+x+\frac{1}{10}-\frac{3}{700x^2}}\right). \tag{3.3}$$

The first inequality in (3.3) *holds for* $x > 0$, *while the second inequality in* (3.3) *is valid provided* $x \geq 1$.

Proof In order to prove the inequalities (3.1), (3.2) and (3.3), it suffices to show that

$$L_1(x) := \ln\left(1+\frac{1}{x}\right) - \frac{1}{x}\left(1-\frac{\frac{1}{2}}{x+\frac{2}{3}-\frac{1}{18x}}\right) > 0, \quad x \geq 1,$$

$$U_1(x) := \frac{1}{x}\left(1-\frac{\frac{1}{2}}{x+\frac{2}{3}-\frac{1}{18x}+\frac{4}{135x^2}}\right) - \ln\left(1+\frac{1}{x}\right) > 0, \quad x > 0,$$

$$L_2(x) := \ln\left(1+\frac{1}{x}\right) - \frac{1}{x+1}\left(1+\frac{\frac{1}{2}}{x+\frac{1}{3}-\frac{1}{18x}+\frac{7}{270x^2}}\right) > 0, \quad x > 0,$$

$$U_2(x) := \frac{1}{x+1}\left(1+\frac{\frac{1}{2}}{x+\frac{1}{3}-\frac{1}{18x}}\right) - \ln\left(1+\frac{1}{x}\right) > 0, \quad x \geq 1,$$

$$L_3(x) := \ln\left(1+\frac{1}{x}\right) - \left(x+\frac{1}{2}\right)^{-1}\left(1+\frac{\frac{1}{12}}{x^2+x+\frac{1}{10}}\right) > 0, \quad x > 0,$$

and

$$U_3(x) := \left(x+\frac{1}{2}\right)^{-1}\left(1+\frac{\frac{1}{12}}{x^2+x+\frac{1}{10}-\frac{3}{700x^2}}\right) - \ln\left(1+\frac{1}{x}\right) > 0, \quad x \geq 1.$$

Differentiation yields

$$L_1'(x) = -\frac{24x-1}{x^2(x+1)(18x^2+12x-1)^2} < 0, \quad x \geq 1,$$

$$U_1'(x) = -\frac{4{,}185(x-\frac{8}{279})^2+\frac{5{,}632}{93}}{x^2(x+1)(270x^3+180x^2-15x+8)^2} < 0, \quad x > 0,$$

$$L_2'(x) = -\frac{3{,}375(x-\frac{7}{225})^2+\frac{686}{15}}{x(x+1)^2(270x^3+90x^2-15x+7)^2} < 0, \quad x > 0,$$

$$U_2'(x) = -\frac{21x-1}{x(x+1)^2(18x^2+6x-1)^2} < 0, \quad x \geq 1,$$

$$L_3'(x) = -\frac{1}{x(x+1)(10x^2+10x+1)^2(2x+1)^2} < 0, \quad x > 0,$$

$$U_3'(x) = -\frac{6{,}711+19{,}040(x-1)+17{,}920(x-1)^2+5{,}600(x-1)^3}{x(700x^4+700x^3+70x^2-3)^2(2x+1)^2(x+1)} < 0, \quad x \geq 1.$$

We then obtain

$$L_1(x) > \lim_{t\to\infty} L_1(t) = 0, \quad x \geq 1, \qquad U_1(x) > \lim_{t\to\infty} U_1(t) = 0, \quad x > 0,$$

$$L_2(x) > \lim_{t\to\infty} L_2(t) = 0, \quad x > 0, \qquad U_2(x) > \lim_{t\to\infty} U_2(t) = 0, \quad x \geq 1,$$

$$L_3(x) > \lim_{t\to\infty} L_3(t) = 0, \quad x > 0, \qquad U_3(x) > \lim_{t\to\infty} U_3(t) = 0, \quad x \geq 1.$$

The proof of Theorem 3.1 is complete. □

Remark 3.1 The inequalities (3.1) are sharper than the inequalities (1.3). The inequalities (3.2) are sharper than the inequalities (1.4). The inequalities (3.3) are sharper than the inequalities (1.5).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made equal contributions to each part of this paper. All authors have read and approved the final manuscript.

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