Wang and Si *Journal of Inequalities and Applications* (2015) 2015:104
DOI 10.1186/s13660-015-0627-2

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Necessary and sufficient conditions for the boundedness of rough multilinear fractional operators on Morrey-type spaces

Zhiheng Wang[1] and Zengyan Si[2*]

*Correspondence:
zengyan@hpu.edu.cn
[2]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, 454000, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we study the necessary and sufficient conditions on the parameters for the boundedness of the multilinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha}$ and the multilinear fractional integral operator $\mathcal{I}_{\Omega,\alpha}$ with rough kernels on Morrey spaces and modified Morrey spaces, respectively. This extends some recent results of Guliyev, Hasnov and Zeren; the necessary and sufficient conditions for the boundedness of M_α and I_α on modified spaces are considered.

Keywords: multilinear fractional operators; rough kernels; Morrey-type spaces

1 Introduction

Kenig and Stein [1] studied the boundedness of multilinear fractional integral operator $\mathcal{I}_{\alpha,m}$, $0 < \alpha < mn$, on Lebesgue spaces.

$$\mathcal{I}_{\alpha,m}\vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)f_2(y_2)\cdots f_m(y_m)}{|(x-y_1,x-y_2,\ldots,x-y_m)|^{mn-\alpha}}\,dy_1\cdots dy_m,$$

we denote by $\vec{f}$ the m-tuple $(f_1,f_2,\ldots,f_m)$ and by m, n nonnegative integers with $m \geq 1$, $n \geq 2$. As one of the most important multilinear operators, the multilinear fractional integral operator has been widely studied; we refer the reader to [2–7] for an overview. In this paper, we study the necessary and sufficient conditions on the parameters for boundedness of the multilinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha}$ and the multilinear fractional integrals $\mathcal{I}_{\Omega,\alpha}$ with rough kernels on Morrey spaces and modified Morrey spaces, respectively, whose definitions are given below.

Let $0 < \alpha < mn$, $s > 1$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multilinear fractional integral operator and its corresponding maximal operator are, respectively, defined by

$$\mathcal{I}_{\Omega,\alpha}\vec{f}(x) = \int_{(\mathbb{R}^n)^m} \frac{\Omega(\vec{y})}{|\vec{y}|^{mn-\alpha}} \prod_{i=1}^{m} f_i(x-y_i)\,d\vec{y};$$

$$\mathcal{M}_{\Omega,\alpha}\vec{f}(x) = \sup_{r>0} \frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} \bigl|\Omega(\vec{y})\bigr| \prod_{i=1}^{m} \bigl|f_i(x-y_i)\bigr|\,d\vec{y},$$

where $d\vec{y} = dy_1 \cdots dy_m$. If $m = 1$, $\mathcal{I}_{\Omega,\alpha}$ is the homogeneous fractional integral operators (see [8]). If $m = 1$ and $\Omega \equiv 1$, $\mathcal{I}_{\Omega,\alpha}$ and $\mathcal{M}_{\Omega,\alpha}$ are the Riesz potential I_α and the fractional maximal operator M_α [9, 10] given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}}\,dy, \qquad M_\alpha f(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{|y|\le r} f(x-y)\,dy.$$

In the theory of partial differential equations, Morrey spaces play an important role. Morrey spaces were introduced by Morrey [11] in 1938 in connection with certain problems in elliptic partial differential equations and the calculus of variation.

Definition 1.1 [12, 13] Let $1 \le p < \infty$, $0 \le \lambda \le n$. We denote by $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL^{p,\lambda} = WL^{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the sets of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} \left|f(y)\right|^p dy \right)^{\frac{1}{p}},$$

$$\|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \left|\left\{y \in B(x,t) : \left|f(y)\right| > r\right\}\right| \right)^{\frac{1}{p}},$$

respectively.

Definition 1.2 [14] Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{L}^{p,\lambda} = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, and by $W\widetilde{L}^{p,\lambda} = W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey space, the sets of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms

$$\|f\|_{\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{[t]_1^\lambda} \int_{B(x,t)} \left|f(y)\right|^p dy \right)^{\frac{1}{p}},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{[t]_1^\lambda} \left|\left\{y \in B(x,t) : \left|f(y)\right| > r\right\}\right| \right)^{\frac{1}{p}},$$

respectively.

It is easy to see that $L^{p,0}(\mathbb{R}^n) = \widetilde{L}^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, $WL^{p,0}(\mathbb{R}^n) = W\widetilde{L}^{p,0}(\mathbb{R}^n) = WL^p(\mathbb{R}^n)$. If $\lambda < 0$ or $\lambda > n$, then $\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. In addition, from [14], we know

$$\widetilde{L}^{p,\lambda}\left(\mathbb{R}^n\right) \subset_{\succ} L^{p,\lambda}\left(\mathbb{R}^n\right) \cap L^p\left(\mathbb{R}^n\right), \qquad \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\right\} \le \|f\|_{\widetilde{L}^{p,\lambda}}.$$

We list two remarkable results on Morrey spaces for I_α.

Theorem A [13] *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$, and $\mu/q = \lambda/p$. Then for $p > 1$, the operator I_α is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem B [12, 14] *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) *If $p > 1$, then the condition $1/p - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If $p = 1$, then the condition $1 - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.*

Motivated by these two results above, we study the necessary and sufficient conditions on the parameters for the boundedness of the multilinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha}$ and the multilinear fractional integral operator $\mathcal{I}_{\Omega,\alpha}$ with rough kernels on Morrey spaces and modified Morrey spaces, respectively. This extends a recent result of [14]; the necessary and sufficient conditions for the boundedness of M_α and I_α on modified spaces are considered. If we denote by p, q the harmonic mean of $p_1, \dots, p_m > 1$ and $q_1, \dots, q_m > 1$, then our results can be stated as follows.

Theorem 1.1 *Let $0 < \alpha < mn$, $1 < s < \infty$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$. Suppose $\frac{\lambda}{p} = \sum_{j=1}^m \frac{\lambda_j}{p_j}$, $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m(n-\lambda_j)}$ and $0 \le \lambda_j < n - \frac{\alpha p_j}{m}$.*

(i) *If $p > s'$ and $\frac{\lambda}{q} = \sum_{j=1}^m \frac{\lambda_j}{q_j}$, then the condition $1/p - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If $p = s'$ and $\lambda \sum_{j=1}^m \frac{1}{p_j q_j} = \sum_{j=1}^m \frac{\lambda_j}{p_j q_j}$, then the condition $1/p - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.*

Moreover, the corresponding estimates for $\mathcal{I}_{\Omega,\alpha}$ hold.

Theorem 1.2 *Let α, Ω, s, p_j, λ_j, p and λ be as in Theorem* 1.1.

(i) *If $p > s'$ and $\frac{\lambda}{q} = \sum_{j=1}^m \frac{\lambda_j}{q_j}$, then the condition $\alpha/n \le 1/p - 1/q \le \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha}$ from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If $p = s'$ and $\lambda \sum_{j=1}^m \frac{1}{p_j q_j} = \sum_{j=1}^m \frac{\lambda_j}{p_j q_j}$, then the condition $\alpha/n \le 1/p - 1/q \le \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha}$ from $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $W\widetilde{L}^{q,\lambda}(\mathbb{R}^n)$.*

Moreover, the corresponding estimates for $\mathcal{I}_{\Omega,\alpha}$ hold.

The organization of this paper is as follows: We will give the boundedness of $\mathcal{M}_{\Omega,\alpha}$ and $\mathcal{I}_{\Omega,\alpha}$ on Morrey spaces and on modified Morrey spaces in Section 2 and Section 3, respectively. In Section 4, some applications are given.

2 Boundedness on Morrey spaces

In this section we study the boundedness of $\mathcal{M}_{\Omega,\alpha}$ and $\mathcal{I}_{\Omega,\alpha}$ on Morrey spaces. The following lemmas play an important role in the proof of Theorem 1.1.

Lemma 2.1 [12, 14] *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.*

(i) *If $p > 1$, then the condition $1/p - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator M_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If $p = 1$, then the condition $1 - 1/q = \alpha/(n - \lambda)$ is necessary and sufficient for the boundedness of the operator M_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.*

Lemma 2.2 [15] *Let $0 < \alpha < mn$, and let $f_j \in L^{p_j}(\mathbb{R}^n)$ with $1 < p_j < \infty$ for $j = 1, 2, \dots, m$. For any $0 < \epsilon < \min\{\alpha, mn - \alpha\}$, there exists a constant $C < \infty$ such that for any $x \in \mathbb{R}^n$,*

$$\bigl|\mathcal{I}_{\Omega,\alpha}\vec{f}(x)\bigr| \le C\bigl[\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)\bigr]^{\frac{1}{2}}\bigl[\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)\bigr]^{\frac{1}{2}}.$$

Lemma 2.3 *Let $0 < \alpha < mn$, $1 \le s' < \frac{mn}{\alpha}$, and let $f_j \in L^{p_j}(\mathbb{R}^n)$ with $1 < p_j < \infty$ for $j = 1, 2, \ldots, m$. Then there exists a constant $C < \infty$ such that for any $x \in \mathbb{R}^n$,*

$$\mathcal{M}_{\Omega,\alpha}\vec{f}(x) \le C \prod_{i=1}^{m} \big[M_{\frac{\alpha s'}{m}} f_j^{s'}\big]^{\frac{1}{s'}}(x).$$

Proof Since $\Omega \in L^s(\mathbb{S}^{mn-1})$, using the Hölder inequality, we obtain

$$\begin{aligned}
&\frac{1}{r^{mn-\alpha}} \int_{|\vec{y}|<r} \big|\Omega(\vec{y})\big| \prod_{j=1}^{m} \big|f_j(x-y_j)\big|\, d\vec{y} \\
&\quad \le \frac{1}{r^{mn-\alpha}} \left(\int_{|\vec{y}|<r} \big|\Omega(\vec{y})\big|^s \, d\vec{y} \right)^{\frac{1}{s}} \left(\int_{|\vec{y}|<r} \prod_{j=1}^{m} \big|f_j(x-y_j)\big|^{s'} \, d\vec{y} \right)^{\frac{1}{s'}} \\
&\quad \le C \sup_{r>0} \frac{1}{r^{mn(1-\frac{1}{s})-\alpha}} \left(\int_{|\vec{y}|<r} \prod_{j=1}^{m} \big|f_j(x-y_j)\big|^{s'} \, d\vec{y} \right)^{\frac{1}{s'}} \\
&\quad \le C \sup_{r>0} \left(\frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}|<r} \prod_{j=1}^{m} \big|f_j(x-y_j)\big|^{s'} \, d\vec{y} \right)^{\frac{1}{s'}} \\
&\quad \le C \left(\sup_{r>0} \frac{1}{r^{mn-\alpha s'}} \int_{|\vec{y}|<r} \prod_{j=1}^{m} \big|f_j(x-y_j)\big|^{s'} \, d\vec{y} \right)^{\frac{1}{s'}} \\
&\quad \le C \left(\sup_{r>0} \frac{1}{r^{mn-\alpha s'}} \int_{|y_1|<r} \cdots \int_{|y_m|<r} \prod_{j=1}^{m} \big|f_j(x-y_j)\big|^{s'} \, d\vec{y} \right)^{\frac{1}{s'}} \\
&\quad \le C \prod_{j=1}^{m} \left(\sup_{r>0} \frac{1}{r^{n-\alpha s'/m}} \int_{|y_j|<r} \big|f_j(x-y_j)\big|^{s'} \, dy_j \right)^{\frac{1}{s'}} \\
&\quad = C \prod_{j=1}^{m} \big[M_{\frac{\alpha s'}{m}} f_j^{s'}\big]^{\frac{1}{s'}}(x).
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Theorem 1.1 We first prove Theorem 1.1 is true for $\mathcal{M}_{\Omega,\alpha}$; then the proof for $\mathcal{I}_{\Omega,\alpha}$ follows.

(i) *Sufficiency*. The case $p > s'$. Since each $p_j > s'$, by the Hölder inequality and Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} \big|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\big|^q \, dy \right)^{\frac{1}{q}} \\
&\le C \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} \left| \prod_{j=1}^{m} \big[M_{\frac{\alpha s'}{m}} f_j^{s'}(y)\big]^{\frac{1}{s'}} \right|^q dy \right)^{\frac{1}{q}} \\
&\le C \prod_{j=1}^{m} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^{\lambda_j}} \int_{\mathbb{R}^n} \big|M_{\frac{\alpha s'}{m}} f_j^{s'}(y)\big|^{\frac{q_j}{s'}} \, dy \right)^{\frac{1}{q_j}}
\end{aligned}$$

$$\begin{aligned}
&\leq C\prod_{j=1}^{m}\|f_j^{s'}\|_{L^{p_j/s',\lambda_j}(\mathbb{R}^n)}^{1/s'}\\
&= C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)},
\end{aligned}$$

where $\frac{1}{q_j} = \frac{1}{p_j} - \frac{\alpha}{m(n-\lambda_j)}$.

Necessity. Suppose that $\mathcal{M}_{\Omega,\alpha}$ is bounded from $L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$. Let $\vec{f}_\epsilon(x) = (f_1(\epsilon x),\ldots,f_m(\epsilon x))$ for all $\epsilon > 0$. Then by changing of the variables, we see that

$$\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon(y) = \epsilon^{-\alpha}\mathcal{M}_{\Omega,\alpha}\vec{f}(\epsilon y). \tag{2.1}$$

Thus

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{L^{q,\lambda}} &= \epsilon^{-\alpha}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{B(x,t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(\epsilon y)\right|^q dy\right)^{1/q}\\
&= \epsilon^{-\alpha-n/q}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{B(x,\epsilon t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\right|^q dy\right)^{1/q}\\
&= \epsilon^{-\alpha-n/q+\lambda/q}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{(\epsilon t)^\lambda}\int_{B(x,\epsilon t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\right|^q dy\right)^{1/q}\\
&= \epsilon^{-\alpha-(n-\lambda)/q}\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}}.
\end{aligned}$$

Since $\mathcal{M}_{\Omega,\alpha}$ is bounded from $L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$, we have

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}} &= \epsilon^{\alpha+(n-\lambda)/q}\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{L^{q,\lambda}}\\
&\leq C\epsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^{m}\|f_j(\epsilon\cdot)\|_{L^{p_j,\lambda_j}}\\
&= C\epsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^{m}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^{\lambda_j}}\int_{B(x,t)}\left|f_j(\epsilon y)\right|^{p_j} dy\right)^{1/p_j}\\
&= C\epsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^{m}\epsilon^{-n/p_j}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^{\lambda_j}}\int_{B(x,\epsilon t)}\left|f_j(y)\right|^{p_j} dy\right)^{1/p_j}\\
&= C\epsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^{m}\epsilon^{(\lambda_j-n)/p_j}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{(\epsilon t)^{\lambda_j}}\int_{B(x,\epsilon t)}\left|f_j(y)\right|^{p_j} dy\right)^{1/p_j}\\
&= C\epsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ϵ.

If $1/p < 1/q + \alpha/(n-\lambda)$, then for all $\vec{f} \in L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}} = 0$ as $\epsilon \to 0$.

Also, if $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\vec{f} \in L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}} = 0$ as $\epsilon \to \infty$.

Therefore we get $1/p = 1/q + \alpha/(n-\lambda)$.

(ii) *Sufficiency*. The case $p = s'$. We apply the Hölder inequality to Lemma 2.3 to obtain the fact

$$\mathcal{M}_{\Omega,\alpha}\vec{f}(x) \le C\prod_{j=1}^{m}\big[M_{\frac{\alpha s'}{m}}f_j^{s'}\big]^{\frac{1}{s'}}(x) \le C\prod_{j=1}^{m}\big[M_{\frac{\alpha p_j s'}{mp}}f_j^{\frac{p_j s'}{p}}\big]^{\frac{p}{p_j s'}}(x) = C\prod_{j=1}^{m}\big[M_{\frac{\alpha p_j}{m}}f_j^{p_j}\big]^{\frac{1}{p_j}}(x).$$

For any $\beta > 0$, let $\varepsilon_0 = \beta, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m-1} > 0$ and $\varepsilon_m = 1$ such that

$$\left(\frac{\varepsilon_j}{\varepsilon_{j-1}}\right)^{p_j q_j} = \frac{[\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}]^q}{\beta^q\|f_j\|_{L^{p_j,\lambda_j}}^{p_j}}, \quad j = 1, 2, \dots, m,$$

where q_j is given by $1 - \frac{1}{q_j} = \frac{\alpha p_j}{m(n-\lambda_j)}$. Hence, we have

$$\big\{y \in B(x,t) : \big|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\big| > C\beta\big\} \subset \bigcup_{j=1}^{m}\left\{y \in B(x,t) : \big[M_{\frac{\alpha p_j}{m}}f_j^{p_j}\big]^{\frac{1}{p_j}}(y) > \frac{\varepsilon_{j-1}}{\varepsilon_j t^{(\lambda-\lambda_j)/p_j q_j}}\right\}.$$

Then, by Lemma 2.1, we have

$$\begin{aligned}
&\big|\big\{y \in B(x,t) : \big|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\big| > \beta\big\}\big| \\
&\quad\le C\sum_{j=1}^{m}\left|\left\{y \in B(x,t) : \big[M_{\frac{\alpha p_j}{m}}f_j^{p_j}\big]^{\frac{1}{p_j}}(y) > \frac{\varepsilon_{j-1}}{\varepsilon_j t^{(\lambda-\lambda_j)/p_j q_j}}\right\}\right| \\
&\quad\le C\sum_{j=1}^{m}\left|\left\{y \in B(x,t) : M_{\frac{\alpha p_j}{m}}f_j^{p_j}(y) > \left(\frac{\varepsilon_{j-1}}{\varepsilon_j t^{(\lambda-\lambda_j)/p_j q_j}}\right)^{p_j}\right\}\right| \\
&\quad\le C\sum_{j=1}^{m}t^{\lambda_j}\left(\frac{\varepsilon_j t^{(\lambda-\lambda_j)/p_j q_j}}{\varepsilon_{j-1}}\right)^{p_j q_j}\big\|f_j^{p_j}\big\|_{L^{1,\lambda_j}}^{q_j} \\
&\quad= C\sum_{j=1}^{m}t^{\lambda}\left(\frac{\varepsilon_j}{\varepsilon_{j-1}}\right)^{p_j q_j}\|f_j\|_{L^{p_j,\lambda_j}}^{p_j} \\
&\quad= Ct^{\lambda}\left(\frac{1}{\beta}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\right)^q.
\end{aligned}$$

Hence, we obtain the following inequality:

$$\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}} = \sup_{\beta>0}\beta\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^{\lambda}}\big|\big\{y \in B(x,t) : \big|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\big| > \beta\big\}\big|\right)^{\frac{1}{q}} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Necessity. Let $\mathcal{M}_{\Omega,\alpha}$ be bounded from $L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$. By using (2.1), we obtain

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{WL^{q,\lambda}} &= \sup_{r>0} r\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^{\lambda}}\int_{\{y\in B(x,t):|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon(y)|>r\}}dy\right)^{\frac{1}{q}} \\
&= \sup_{r>0} r\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^{\lambda}}\int_{\{y\in B(x,t):|\mathcal{M}_{\Omega,\alpha}\vec{f}(\epsilon y)|>r\epsilon^{\alpha}\}}dy\right)^{\frac{1}{q}}
\end{aligned}$$

$$= \epsilon^{-n/q} \sup_{r>0} r \sup_{x\in\mathbb{R}^n,t>0} \left(\frac{1}{t^\lambda} \int_{\{y\in B(x,\epsilon t):|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)|>r\epsilon^\alpha\}} dy \right)^{\frac{1}{q}}$$

$$= \epsilon^{-\alpha-n/q+\lambda/q} \sup_{r>0} r\epsilon^\alpha \sup_{x\in\mathbb{R}^n,t>0} \left(\frac{1}{(\epsilon t)^\lambda} \int_{\{y\in B(x,\epsilon t):|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)|>r\epsilon^\alpha\}} dy \right)^{\frac{1}{q}}$$

$$= \epsilon^{-\alpha-(n-\lambda)/q} \|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}}.$$

By the boundedness of $\mathcal{M}_{\Omega,\alpha}$ from $L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$, we have

$$\begin{aligned} \|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}} &= \epsilon^{\alpha+(n-\lambda)/q} \|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{WL^{q,\lambda}} \\ &\le C\epsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \big\|f_j(\epsilon\cdot)\big\|_{L^{p_j,\lambda_j}} \\ &\le C\epsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}, \end{aligned}$$

where C is independent of ϵ.

If $1/p < 1/q + \alpha/(n-\lambda)$, then for all $\vec{f} \in L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}} = 0$ as $\epsilon \to 0$.

Also, if $1/p > 1/q+\alpha/(n-\lambda)$, then for all $\vec{f} \in L^{p_1,\lambda_1} \times \cdots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}} = 0$ as $\epsilon \to \infty$.

Consequently, we get $1/p = 1/q + \alpha/(n-\lambda)$.

Now we prove the corresponding estimates for $\mathcal{I}_{\Omega,\alpha}$ hold. By the same arguments as above we can get the necessity parts of Theorem 1.1(i) and (ii) for $\mathcal{I}_{\Omega,\alpha}$. So we just give the sufficiency parts, respectively.

First we study the sufficiency of the condition in Theorem 1.1(i) for $\mathcal{I}_{\Omega,\alpha}$.

Following the method used in [16], we choose a small positive number ϵ with $0 < \epsilon < \min\{\alpha, \frac{m(n-\lambda_j)}{p_j} - \alpha, \frac{n-\lambda}{p} - \alpha\}$. One can then see from the condition of Theorem 1.1 that $1 \le s' < p_j < \frac{m(n-\lambda_j)}{\alpha+\epsilon}$ and $1 \le s' < p_j < \frac{m(n-\lambda_j)}{\alpha-\epsilon}$, and we let

$$\frac{1}{\tilde{q}_1} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha+\epsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha+\epsilon}{n-\lambda},$$

and

$$\frac{1}{\tilde{q}_2} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_m} - \frac{\alpha-\epsilon}{n-\lambda} = \frac{1}{p} - \frac{\alpha-\epsilon}{n-\lambda}.$$

Now if each $p_j > s'$, then Theorem 1.1(i) implies that

$$\|\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}\|_{L^{\tilde{q}_1,\lambda}(\mathbb{R}^n)} \le \|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)}, \qquad \|\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}\|_{L^{\tilde{q}_2,\lambda}(\mathbb{R}^n)} \le \|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)}.$$

A simple calculation yields $\frac{q}{2\tilde{q}_1} + \frac{q}{2\tilde{q}_2} = 1$. Hence, using Lemma 2.2, the Hölder inequality and the above inequalities, we have

$$\begin{aligned} &\|\mathcal{I}_{\Omega,\alpha}\vec{f}\|_{L^{q,\lambda}(\mathbb{R}^n)} \\ &\quad = \sup_{x\in\mathbb{R}^n,t>0} \left(\frac{1}{t^\lambda} \int_{B(x,t)} \big|\mathcal{I}_{\Omega,\alpha}\vec{f}(y)\big|^q \, dy \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
&\le C \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\mathbb{R}^n}\left[\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)\right]^{\frac{q}{2}}\left[\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)\right]^{\frac{q}{2}}\,dx\right)^{\frac{1}{q}}\\
&\le C \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\mathbb{R}^n}\left[\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)\right]^{\tilde{q}_1}\,dx\right)^{\frac{1}{2\tilde{q}_1}} \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\mathbb{R}^n}\left[\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)\right]^{\tilde{q}_2}\,dx\right)^{\frac{1}{2\tilde{q}_2}}\\
&\le C\|\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}\|^{1/2}_{L^{\tilde{q}_1,\lambda}(\mathbb{R}^n)}\|\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}\|^{1/2}_{L^{\tilde{q}_2,\lambda}(\mathbb{R}^n)}\\
&\le C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)}.
\end{aligned}$$

Now we study the sufficiency of the condition in Theorem 1.1(ii) for $\mathcal{I}_{\Omega,\alpha}$.

For any $\beta > 0$, we denote $\mu^2 = \beta^{2-\frac{q}{\tilde{q}_2}}(\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}})^{\frac{q}{\tilde{q}_2}-1}$. Then by Lemma 2.2, we have

$$\begin{aligned}
&\left|\left\{y\in B(x,t):\left|\mathcal{I}_{\Omega,\alpha}\vec{f}(y)\right|>\beta\right\}\right|\\
&\quad\le C\left|\left\{y\in B(x,t):C\left[\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)\right]^{\frac{1}{2}}\left[\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)\right]^{\frac{1}{2}}>\beta\right\}\right|\\
&\quad\le C\left|\left\{y\in B(x,t):\sqrt{C}\left[\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)\right]^{\frac{1}{2}}>\mu\right\}\right|\\
&\qquad+\left|\left\{y\in B(x,t):\sqrt{C}\left[\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)\right]^{\frac{1}{2}}>\beta/\mu\right\}\right|\\
&\quad\le C\left|\left\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha+\epsilon}\vec{f}(x)>C\mu^2\right\}\right|+\left|\left\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha-\epsilon}\vec{f}(x)>C\beta^2/\mu^2\right\}\right|\\
&\quad\le Ct^\lambda\left[\left(\frac{1}{\mu^2}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\right)^{\tilde{q}_1}+C\left(\frac{\mu^2}{\beta^2}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\right)^{\tilde{q}_2}\right]\\
&\quad\le Ct^\lambda\left(\frac{1}{\beta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\right)^q.
\end{aligned}$$

Hence, we obtain the following inequality:

$$\|\mathcal{I}_{\Omega,\alpha}\vec{f}\|_{WL^{q,\lambda}}=\sup_{\beta>0}\beta\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\left|\left\{y\in B(x,t):\left|\mathcal{I}_{\Omega,\alpha}\vec{f}(y)\right|>\beta\right\}\right|\right)^{\frac{1}{q}}\le C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}.$$

Thus we complete the proof of Theorem 1.1. □

3 Boundedness on modified Morrey spaces

In this section we study the boundedness of $\mathcal{M}_{\Omega,\alpha}$ and $\mathcal{I}_{\Omega,\alpha}$ on modified Morrey spaces. The following inequality for M_α in Modified Morrey spaces is valid.

Lemma 3.1 [14] *Let* $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n - \alpha p$.

(i) *If* $p > 1$, *then the condition* $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{p,\lambda}$ *to* $\widetilde{L}^{q,\lambda}$.

(ii) *If* $p = 1$, *then the condition* $\alpha/n \le 1 - 1/q \le \alpha/(n-\lambda)$ *is necessary and sufficient for the boundedness of the operator* M_α *from* $\widetilde{L}^{1,\lambda}$ *to* $W\widetilde{L}^{q,\lambda}$.

We are ready to prove Theorem 1.2.

Proof Similar to the proofs of sufficiency in Theorem 1.1, we will get the sufficiency parts for $\mathcal{M}_{\Omega,\alpha}$ and $\mathcal{I}_{\Omega,\alpha}$, respectively. Now, we give only the proof of necessity for $\mathcal{M}_{\Omega,\alpha}$, since the main steps and the ideas are almost the same as $\mathcal{I}_{\Omega,\alpha}$.

Let $[\epsilon]_{1,+} = \max\{1,\epsilon\}$. Then by (2.1), we obtain

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{\widetilde{L}^{q,\lambda}} &= \epsilon^{-\alpha} \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^\lambda}\int_{B(x,t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(\epsilon y)\right|^q dy\right)^{1/q} \\
&= \epsilon^{-\alpha-n/q} \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^\lambda}\int_{B(\epsilon x,\epsilon t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\right|^q dy\right)^{1/q} \\
&= \epsilon^{-\alpha-n/q} \sup_{t>0}\left(\frac{[\epsilon t]_1}{[t]_1}\right)^{\lambda/q} \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[\epsilon t]_1^\lambda}\int_{B(\epsilon x,\epsilon t)}\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\right|^q dy\right)^{1/q} \\
&= \epsilon^{-\alpha-n/q}[\epsilon]_{1,+}^{\frac{\lambda}{q}}\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{\widetilde{L}^{q,\lambda}}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{W\widetilde{L}^{q,\lambda}} &= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(x,t):|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon(y)|>r\}} dy\right)^{\frac{1}{q}} \\
&= \sup_{r>0} r \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(x,t):|\mathcal{M}_{\Omega,\alpha}\vec{f}(\epsilon y)|>r\epsilon^\alpha\}} dy\right)^{\frac{1}{q}} \\
&= \epsilon^{-n/q}\sup_{r>0} r \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^\lambda}\int_{\{y\in B(\epsilon x,\epsilon t):|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)|>r\epsilon^\alpha\}} dy\right)^{\frac{1}{q}} \\
&= \epsilon^{-\alpha-n/q}\sup_{t>0}\left(\frac{[\epsilon t]_1}{[t]_1}\right)^{\lambda/q} \\
&\quad\times \sup_{r>0} r\epsilon^\alpha \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[\epsilon t]_1^\lambda}\left|\left\{y\in B(\epsilon x,\epsilon t):\left|\mathcal{M}_{\Omega,\alpha}\vec{f}(y)\right|>r\epsilon^\alpha\right\}\right|\right)^{\frac{1}{q}} \\
&= \epsilon^{-\alpha-n/q}[\epsilon]_{1,+}^{\frac{\lambda}{q}}\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{W\widetilde{L}^{q,\lambda}}.
\end{aligned}$$

(i) Let $\mathcal{M}_{\Omega,\alpha}$ be bounded from $\widetilde{L}^{p_1,\lambda_1}\times\cdots\times\widetilde{L}^{p_m,\lambda_m}$ to $\widetilde{L}^{q,\lambda}$. Then we have

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{\widetilde{L}^{q,\lambda}} &= \epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{\widetilde{L}^{q,\lambda}} \\
&\le C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m\left\|f_j(\epsilon\cdot)\right\|_{\widetilde{L}^{p_j,\lambda_j}} \\
&= C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^{\lambda_j}}\int_{B(x,t)}\left|f_j(\epsilon y)\right|^{p_j} dy\right)^{1/p_j} \\
&= C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m \epsilon^{-n/p_j}\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[t]_1^{\lambda_j}}\int_{B(\epsilon x,\epsilon t)}\left|f_j(y)\right|^{p_j} dy\right)^{1/p_j} \\
&\le C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^m \epsilon^{-n/p_j}\sup_{t>0}\left(\frac{[\epsilon t]_1}{[t]_1}\right)^{\lambda_j/p_j} \\
&\quad\times \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{[\epsilon t]^{\lambda_j}}\int_{B(\epsilon x,\epsilon t)}\left|f_j(y)\right|^{p_j} dy\right)^{1/p_j}
\end{aligned}$$

$$\begin{aligned} &\le C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}[\epsilon]_{1,+}^{\frac{\lambda}{p}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \\ &\le C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}, \end{aligned}$$

where C is independent of ϵ.

If $1/p < 1/q + \alpha/n$, then for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1} \times \cdots \times \widetilde{L}^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to 0$.

Also, if $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1} \times \cdots \times \widetilde{L}^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to \infty$.

Therefore we get $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$.

(ii) Let $\mathcal{M}_{\Omega,\alpha}$ be bounded from $\widetilde{L}^{p_1,\lambda_1} \times \cdots \times \widetilde{L}^{p_m,\lambda_m}$ to $W\widetilde{L}^{q,\lambda}$. Then we have

$$\begin{aligned} \|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{W\widetilde{L}^{q,\lambda}} &= \epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\|\mathcal{M}_{\Omega,\alpha}\vec{f}_\epsilon\|_{W\widetilde{L}^{q,\lambda}} \\ &\le C\epsilon^{\alpha+n/q}[\epsilon]_{1,+}^{-\frac{\lambda}{q}}\prod_{j=1}^{m}\big\|f_j(\epsilon\cdot)\big\|_{\widetilde{L}^{p_j,\lambda_j}} \\ &\le C\epsilon^{\alpha+n/q-n/p}[\epsilon]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\prod_{j=1}^{m}\|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}, \end{aligned}$$

where C is independent of ϵ.

If $1/p < 1/q + \alpha/n$, then for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1} \times \cdots \times \widetilde{L}^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{W\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to 0$.

Also, if $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\vec{f} \in \widetilde{L}^{p_1,\lambda_1} \times \cdots \times \widetilde{L}^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha}\vec{f}\|_{W\widetilde{L}^{q,\lambda}} = 0$ as $\epsilon \to \infty$.

Consequently, we get $\alpha/n \le 1/p - 1/q \le \alpha/(n-\lambda)$.

This completes the proof of Theorem 1.2. □

4 Some applications

As an application, we first obtain a result parallel to Theorem A for the operator $\mathcal{M}_{\Omega,\alpha}$ and $\mathcal{I}_{\Omega,\alpha}$.

Corollary 4.1 *Let α, Ω, s, p_j, λ_j, p, and λ be as in Theorem* 1.1, $1/q = 1/p - \alpha/n$, $\mu/q = \lambda/p$.

(i) *If $p > s'$ and $\frac{\lambda}{q} = \sum_{j=1}^{m}\frac{\lambda_j}{q_j}$, then there exists a constant $C < \infty$ such that*

$$\|M_{\Omega,\alpha}\vec{f}\|_{L^{q,\mu}(\mathbb{R}^n)} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)}.$$

(ii) *If $p = s'$ and $\lambda\sum_{j=1}^{m}\frac{1}{p_jq_j} = \sum_{j=1}^{m}\frac{\lambda_j}{p_jq_j}$, then there exists a constant $C < \infty$ such that*

$$\|M_{\Omega,\alpha}\vec{f}\|_{WL^{q,\mu}(\mathbb{R}^n)} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}(\mathbb{R}^n)}.$$

Moreover, similar estimates hold for $\mathcal{I}_{\Omega,\alpha}$.

Proof The proof follows from similar steps in Corollary 3.1, [17], here we omit the proof. □

As another application, we obtain the Olsen inequality which is a multi-version of the results considered by Olsen in [18] in the study of the Schrödinger equation with perturbed potentials W on $\mathbb{R}^n$. As a consequence of Theorem 1.1 and the Hölder inequality, we have the following.

Corollary 4.2 *Let α, Ω, s, p_j, λ_j, p, and λ be as in Theorem* 1.1, $1/p - 1/q = \alpha/(n-\lambda)$ *and let $W \in L^{(n-\lambda)/\alpha,\lambda}$. We get the following.*

(i) *If $p > s'$ and $\frac{\lambda}{q} = \sum_{j=1}^m \frac{\lambda_j}{q_j}$, then there exists a constant $C < \infty$ such that*

$$\|W \cdot M_{\Omega,\alpha}\vec{f}\|_{L^{p,\lambda}(\mathbb{R}^n)} \le C\|W\|_{L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)}\|f_1\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \times \cdots \times \|f_m\|_{L^{p_m,\lambda_m}(\mathbb{R}^n)}.$$

(ii) *If $p = s'$ and $\lambda \sum_{j=1}^m \frac{1}{p_j q_j} = \sum_{j=1}^m \frac{\lambda_j}{p_j q_j}$, then there exists a constant $C < \infty$ such that*

$$\|W \cdot M_{\Omega,\alpha}\vec{f}\|_{WL^{p,\lambda}(\mathbb{R}^n)} \le C\|W\|_{WL^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)}\|f_1\|_{L^{p_1,\lambda_1}(\mathbb{R}^n)} \times \cdots \times \|f_m\|_{L^{p_m,\lambda_m}(\mathbb{R}^n)}.$$

Moreover, similar estimates hold for $\mathcal{I}_{\Omega,\alpha}$.

Remark 4.1 We point out that similar results in Corollary in 4.1 and 4.2 hold on modified Morrey spaces; we do not list them here.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed in all parts to an equal extent, and they read and approved the final manuscript.

Author details
[1]School of Computer Science and Technique, Henan Polytechnic University, Jiaozuo, 454000, P.R. China. [2]School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, 454000, P.R. China.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (No. 11401175) and Doctor Foundation of Henan Polytechnic University (No. B2012-055).

Received: 18 November 2014 Accepted: 10 March 2015 Published online: 19 March 2015

References
1. Kenig, CE, Stein, EM: Multilinear estimates and fractional integration. Math. Res. Lett. **6**, 1-15 (1999)
2. Chen, X, Xue, Q: Weighted estimates for a class of multilinear fractional type operators. J. Math. Anal. Appl. **362**, 355-373 (2010)
3. Moen, K: Weighted inequalities for multilinear fractional integral operators. Collect. Math. **60**, 213-238 (2009)
4. Shi, Y, Tao, X: Multilinear Riesz potential operators on Herz-type spaces and generalized Morrey spaces. Hokkaido Math. J. **38**, 635-662 (2009)
5. Si, Z, Lu, S: Weighted estimates for iterated commutators of multilinear fractional operators. Acta Math. Sin. Engl. Ser. **28**, 1769-1778 (2012)
6. Si, Z: λ-Central BMO estimates for multilinear commutators of fractional integrals. Acta Math. Sin. Engl. Ser. **26**, 2093-2108 (2010)
7. Xue, Q: Weighted estimates for the iterated commutators of multilinear maximal and fractional type operators. Stud. Math. **217**, 97-122 (2013)
8. Ding, Y, Lu, S: The $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_k}$ boundedness for some rough operators. J. Math. Anal. Appl. **203**, 166-186 (1996)
9. Lu, S, Ding, Y, Yan, D: Singular Integrals and Related Topics. World Scientific, Singapore (2006)
10. Stein, EM: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton (1993)
11. Morrey, CB: On the solutions of quasi-linear elliptic partial differential equations. Trans. Am. Math. Soc. **43**, 126-166 (1938)
12. Adams, DR: A note on Riesz potentials. Duke Math. J. **42**, 765-778 (1975)
13. Peetre, J: On the theory of $L^{p,\lambda}$. J. Funct. Anal. **4**, 71-87 (1969)

14. Guliyev, V, Hasnov, J, Zeren, Y: Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces. J. Math. Inequal. **5**, 491-506 (2011)
15. Si, Z, Shi, Y: Iterated commutators of multilinear fractional operators with rough kernels. J. Inequal. Appl. **2012**, 80 (2012)
16. Ding, Y, Lin, C-C: Rough bilinear fractional integrals. Math. Nachr. **246**, 47-52 (2002)
17. Shi, Y, Si, Z: Necessary and sufficient conditions for boundedness of multilinear fractional integrals with rough kernels on Morrey type spaces. Preprint
18. Olsen, PA: Fractional integration, Morrey spaces and a Schrödinger equation. Commun. Partial Differ. Equ. **20**, 2005-2055 (1995)