Bianchi et al. *Journal of Inequalities and Applications* (2015) 2015:113
DOI 10.1186/s13660-015-0624-5

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Bounds for the global cyclicity index of a general network via weighted majorization

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Abstract

In this paper we define a new graph-theoretic cyclicity index $CW(G)$ as a natural generalization of the global cyclicity index $C(G)$ when arbitrary resistances are allocated to each edge of an electrical network. Upper and lower bounds for $CW(G)$ are then provided using a powerful technique, based on p-majorization, which extends our prior studies (Bianchi *et al.* in Discrete Appl. Math., 2014, doi:10.1016/j.dam.2014.10.037; Bianchi *et al.* in Math. Inequal. Appl. 16(2):329-347, 2013). These new results on weighted majorization are of interest in themselves and may be applied also in other scenarios.

Keywords: p-majorization; p-Schur-convex functions; graphs; weighted global cyclicity index

1 Introduction

A simple connected undirected *graph* (or *network*) $G = (V, E)$, where each edge is endowed with a unit resistance, is the basic model for molecules in mathematical chemistry: vertices represent atoms and edges represent bonds. Several graph measures based on distances, degrees, and graph entropies have been investigated in the recent literature to study a variety of physicochemical properties of these molecules [1, 2]. In the sequel, among the mathematical descriptors in the field, we focus on those indices based on electrical network theory like the Kirchhoff index [3–9] defined as

$$K(G) = \sum_{i<j} R_{ij},$$

where R_{ij} is the effective resistance between vertices i and j, and the global cyclicity index [10] defined as

$$C(G) = \sum_{(i,j)\in E} \frac{1}{R_{ij}} - m,$$

where $m = |E|$.

When introducing a new topological index it is desirable to investigate its discriminative power (see [11] and [12]). In this regard, the global cyclicity index appears to be able to capture cyclicity meaningfully (see [10] for a thorough discussion on this matter and for a plethora of examples). Besides its appearance in the realm of mathematical chemistry, cyclicity is a key concept in other purely mathematical contexts, like measures of connectivity or complexity of graphs [13].

Through the majorization technique discussed in [14–16] and [17] significant bounds have been obtained by the authors for the Kirchhoff index as well as for some of its generalizations like the additive/multiplicative degree-Kirchhoff indices. Recently, using this powerful tool, Yang provided good bounds for the global cyclicity index [18].

One natural generalization of the molecular model described above is to endow the edges with arbitrary resistances. This is customary when studying random walks [19] but it has not received much attention in the area of molecular descriptors. We can single out the article [20] where they minimize the Kirchhoff index of a graph when there is a fixed total conductance to be allocated among the edges of the graph. Also, in references [21] and [22], they study the Kirchhoff index for networks where the effective resistances R_{ij} are made to depend on a single parameter λ and a set of weights on the vertices.

In this direction, we allow the edges of our graph to have arbitrary resistances r_{ij} and we consider the *weighted global cyclicity index*

$$CW(G) = \sum_{(i,j)\in E} \left(\frac{1}{R_{ij}} - \frac{1}{r_{ij}} \right),$$

as a natural extension of the global cyclicity index, which can be recovered when $r_{ij} = 1$ for all $(i,j) \in E$.

In order to tackle this new descriptor, we extend prior results of majorization into the realm of weighted majorization [23], and then we express the descriptor as an appropriate p-Schur-convex function to which the new technique can be applied. We believe that the new results on weighted majorization are of interest in themselves and may be of interest in other scenarios.

2 Notations and preliminaries

We recall in this section some notions on p-majorization (for further details see [23] and [24]).

We will denote by $[x_1^{\alpha_1}, x_2^{\alpha_2}, \ldots, x_p^{\alpha_p}]$ a vector in $\mathbb{R}^n$ with α_i components equal to x_i, where $\sum_{i=1}^p \alpha_i = n$. If $\alpha_i = 1$, we use for convenience x_i instead of x_i^1, while x_i^0 means that the component x_i is not present. Let $\mathbf{e^j}$, $j = 1, \ldots, n$, be the fundamental vectors of $\mathbb{R}^n$. Recalling that the Hadamard product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ is defined as follows:

$$\mathbf{x} \circ \mathbf{y} = [x_1y_1, x_2y_2, \ldots, x_ny_n]^T,$$

it is easy to verify the following properties, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, $\mathbf{s^j} = [1^j, 0^{n-j}]$, with $j = 1, 2, \ldots, n$ and $\mathbf{v^j} = [0^j, 1^{n-j}]$, with $j = 0, \ldots, n$:

(i) $\langle \mathbf{x} \circ \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \circ \mathbf{z} \rangle$,
(ii) $\langle \mathbf{s^h}, \mathbf{v^k} \rangle = h - \min\{h, k\}$,
(iii) $\mathbf{s^k} \circ \mathbf{s^j} = \mathbf{s^h}$, $h = \min\{k, j\}$,
(iv) $\mathbf{v^k} \circ \mathbf{s^j} = \mathbf{s^j} - \mathbf{s^h} = \mathbf{v^h} - \mathbf{v^j}$, $h = \min\{k, j\}$.

Definition 1 Fix $\mathbf{p} > \mathbf{0}$. Given two vectors $\mathbf{y}, \mathbf{z} \in D = \{\mathbf{x} \in \mathbb{R}^n : x_1 \geq x_2 \geq \cdots \geq x_n\}$, the p-majorization order $\mathbf{y} \trianglelefteq_p \mathbf{z}$ means:

$$\langle \mathbf{p} \circ \mathbf{y}, \mathbf{s^k} \rangle \leq \langle \mathbf{p} \circ \mathbf{z}, \mathbf{s^k} \rangle, \quad k = 1, \ldots, (n-1)$$

and

$$\langle \mathbf{p} \circ \mathbf{y}, \mathbf{s^n} \rangle = \langle \mathbf{p} \circ \mathbf{z}, \mathbf{s^n} \rangle.$$

Notice that for $\mathbf{p} = \mathbf{s^n}$ p-majorization reduces to the usual majorization. Thus in the sequel all our results entail as particular cases the results known for majorization (see [24, 25]).

Let $\mathbf{x}^{*\mathbf{p}}(S)$ and $\mathbf{x}_{*\mathbf{p}}(S)$ denote the maximal and the minimal elements of a subset $S \subseteq \mathbb{R}^n$ with respect to the p-majorization order.

Given a positive real number a, let

$$\Sigma_a = D \cap \{\mathbf{x} \in \mathbb{R}^n_+ : \langle \mathbf{x}, \mathbf{p} \rangle = a\}.$$

By direct calculations we can show that the maximal and the minimal elements of Σ_a with respect to the p-majorization order are, respectively,

$$\mathbf{x}^{*\mathbf{p}}(\Sigma_a) = \frac{a}{p_1}\mathbf{e^1} \quad \text{and} \quad \mathbf{x}_{*\mathbf{p}}(\Sigma_a) = \left[\left(\frac{a}{\sum_{i=1}^n p_i} \right)^n \right]$$

(see also [23]).

For the maximal element we have $\langle \mathbf{p} \circ \mathbf{x}^{*\mathbf{p}}(\Sigma_a), \mathbf{s^j} \rangle = a$, for all $j = 1, 2, \ldots, n$ and thus $\langle \mathbf{p} \circ \mathbf{x}, \mathbf{s^j} \rangle \leq \langle \mathbf{p} \circ \mathbf{x}^{*\mathbf{p}}(\Sigma_a), \mathbf{s^j} \rangle$ for all $\mathbf{x} \in \Sigma_a$, $j = 1, 2, \ldots, (n-1)$.

Now we prove that for each $\mathbf{x} \in \Sigma_a$ and every $k = 1, 2, \ldots, (n-1)$ we have

$$\frac{a}{\sum_{i=1}^n p_i} \sum_{i=1}^k p_i \leq \sum_{i=1}^k p_i x_i. \tag{1}$$

Indeed, if $\frac{a}{\sum_{i=1}^n p_i} \sum_{i=1}^k p_i > \sum_{i=1}^k p_i x_i$, then

$$a - \sum_{i=k+1}^n p_i x_i < \frac{a}{\sum_{i=1}^n p_i} \sum_{i=1}^k p_i$$

and rearranging the terms

$$\sum_{i=k+1}^n p_i x_i > \frac{a}{\sum_{i=1}^n p_i} \sum_{i=k+1}^n p_i.$$

Thus

$$x_{k+1} \sum_{i=k+1}^n p_i > \frac{a}{\sum_{i=1}^n p_i} \sum_{i=k+1}^n p_i$$

and consequently $x_1 \geq x_2 \geq \cdots \geq x_{k+1} > \frac{a}{\sum_{i=1}^{n} p_i}$. But this implies (1), a contradiction. Thus condition (1) holds, and it simply implies

$$\langle \mathbf{p} \circ \mathbf{x}, \mathbf{s^j} \rangle \geq \langle \mathbf{p} \circ \mathbf{x}_{*\mathbf{p}}(\Sigma_a), \mathbf{s^j} \rangle$$

for all $\mathbf{x} \in \Sigma_a$, $j = 1, 2, \ldots, (n-1)$.

We finally recall the notion of p-Schur-convex functions (see [23]). Let π be a permutation of $\{1, \ldots n\}$ and $\mathbf{x}^\pi$ be the vector obtained exchanging the components of $\mathbf{x}$ according to π.

Given a fixed vector of positive components $\mathbf{p}$, a function $\phi(\cdot, \mathbf{p}) : \mathbb{R}^n \longrightarrow \mathbb{R}$, is said to be p-Schur-convex if it preserves the p-majorization order, that is, if

$$\phi(\mathbf{x}, \mathbf{p}) = \phi(\mathbf{x}^\pi, \mathbf{p}^\pi) \quad \text{for all } \pi, \tag{2}$$

$$\phi(\mathbf{x}, \mathbf{p}) \leq \phi(\mathbf{y}, \mathbf{p}) \quad \text{whenever } \mathbf{x}, \mathbf{y} \in D \text{ and } \mathbf{x} \trianglelefteq_p \mathbf{y}. \tag{3}$$

The following result gives an important characterization of differentiable p-Schur-convex functions.

Theorem 2 (see [23]) *Let $\mathbf{p} > \mathbf{0}$ be fixed and $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function satisfying* (2). *Then the function ϕ is p-Schur-convex if and only if, for all* $\mathbf{x}$,

$$(x_i - x_j)\left(\frac{1}{p_i} \frac{\partial \phi(\mathbf{x}; \mathbf{p})}{\partial x_i} - \frac{1}{p_j} \frac{\partial \phi(\mathbf{x}; \mathbf{p})}{\partial x_j} \right) \geq 0 \quad \textit{for all } i, j = 1, \ldots, n. \tag{4}$$

Note that for $\mathbf{p} = \mathbf{s^n}$ we recover the classical notion of Schur-convex function (see [24]).

3 Some results on *p*-majorization

Now let us consider the subset of Σ_a given by

$$S_a = \Sigma_a \cap \left\{ \mathbf{x} \in \mathbb{R}^n : M_i \geq x_i \geq m_i, i = 1, \ldots, n \right\}, \tag{5}$$

where $\mathbf{m} = [m_1, m_2, \ldots, m_n]^T$ and $\mathbf{M} = [M_1, M_2, \ldots, M_n]^T$ are two assigned vectors arranged in nonincreasing order with $0 \leq m_i \leq M_i$, for all $i = 1, \ldots, n$, and a is a positive real number such that

$$\langle \mathbf{m}, \mathbf{p} \rangle \leq a \leq \langle \mathbf{M}, \mathbf{p} \rangle.$$

The existence of maximal and minimal elements of S_a with respect to the p-majorization are ensured by the compactness of the set S_a and by the closure of the upper and lower level sets:

$$U(\mathbf{x}) = \{ \mathbf{z} \in S_a : \mathbf{x} \trianglelefteq_p \mathbf{z} \}, \qquad L(\mathbf{x}) = \{ \mathbf{z} \in S_a : \mathbf{z} \trianglelefteq_p \mathbf{x} \}.$$

For the sequel we are interested in computing the maximal element, with respect to the p-majorization order, of the set S_a.

Theorem 3 *Let $k \geq 0$ be the smallest integer such that*

$$\langle \mathbf{p} \circ \mathbf{M}, \mathbf{s^k} \rangle + \langle \mathbf{p} \circ \mathbf{m}, \mathbf{v^k} \rangle \leq a < \langle \mathbf{p} \circ \mathbf{M}, \mathbf{s^{k+1}} \rangle + \langle \mathbf{p} \circ \mathbf{m}, \mathbf{v^{k+1}} \rangle \tag{6}$$

and $\theta = \frac{a-\langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\rangle-\langle\mathbf{p}\circ\mathbf{m},\mathbf{v^{k+1}}\rangle}{p_{k+1}}$. *Then*

$$\mathbf{x}^{*\mathbf{p}}(S_a) = \mathbf{M}\circ\mathbf{s^k} + \theta\mathbf{e^{k+1}} + \mathbf{m}\circ\mathbf{v^{k+1}}. \tag{7}$$

Proof First of all we verify that $\mathbf{x}^{*\mathbf{p}}(S_a)\in S_a$.

It easy to see that $\langle \mathbf{p}\circ\mathbf{x}^{*\mathbf{p}}(S_a),\mathbf{s^n}\rangle = a$ and that $m_i \le [\mathbf{x}^{*\mathbf{p}}(S_a)]_i \le M_i$ for $i \ne k+1$. To prove that $m_{k+1} \le \mathbf{x}^{*\mathbf{p}}_{k+1}(S_a) \le M_{k+1}$, notice that from (6)

$$p_{k+1}m_{k+1} \le a - \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\rangle - \langle\mathbf{p}\circ\mathbf{m},\mathbf{v^{k+1}}\rangle = \theta p_{k+1} < p_{k+1}M_{k+1}.$$

Now we show that $\mathbf{x} \trianglelefteq_p \mathbf{x}^{*\mathbf{p}}(S_a)$ for all $\mathbf{x}\in S_a$. By property (i) it follows that

$$\langle \mathbf{p}\circ\mathbf{x}^{*\mathbf{p}}(S_a),\mathbf{s^j}\rangle = \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\circ\mathbf{s^j}\rangle + \theta p_{k+1}\langle\mathbf{e^{k+1}},\mathbf{s^j}\rangle + \langle\mathbf{p}\circ\mathbf{m},\mathbf{v^{k+1}}\circ\mathbf{s^j}\rangle, \quad j=1,\dots,(n-1),$$

and by (iii) and (iv)

$$\langle \mathbf{p}\circ\mathbf{x}^{*\mathbf{p}}(S_a),\mathbf{s^j}\rangle = \begin{cases} \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^j}\rangle, & 1\le j\le k, \\ \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\rangle + \theta p_{k+1} + \langle\mathbf{p}\circ\mathbf{m},\mathbf{s^j}-\mathbf{s^{k+1}}\rangle, & (k+1)\le j\le(n-1). \end{cases}$$

Thus, given a vector $\mathbf{x}\in S_a$, for $1\le j\le k$ we obtain

$$\langle \mathbf{p}\circ\mathbf{x},\mathbf{s^j}\rangle \le \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^j}\rangle = \langle \mathbf{p}\circ\mathbf{x}^{*\mathbf{p}}(S_a),\mathbf{s^j}\rangle,$$

while for $(k+1)\le j\le(n-1)$, by (iii),

$$\begin{aligned}\langle \mathbf{p}\circ\mathbf{x},\mathbf{s^j}\rangle &= \langle \mathbf{p}\circ\mathbf{x},\mathbf{s^n}\rangle - \langle\mathbf{p}\circ\mathbf{x},\mathbf{v^j}\rangle \le a - \langle\mathbf{p}\circ\mathbf{m},\mathbf{v^j}\rangle \\ &= \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\rangle + \theta p_{k+1} + \langle\mathbf{p}\circ\mathbf{m},\mathbf{v^{k+1}}\rangle - \langle\mathbf{p}\circ\mathbf{m},\mathbf{v^j}\rangle \\ &= \langle \mathbf{p}\circ\mathbf{M},\mathbf{s^k}\rangle + \theta p_{k+1} + \langle\mathbf{p}\circ\mathbf{m},\mathbf{s^j}-\mathbf{s^{k+1}}\rangle \\ &= \langle \mathbf{p}\circ\mathbf{x}^{*\mathbf{p}}(S_a),\mathbf{s^j}\rangle,\end{aligned}$$

and the result follows. □

From this general result, the maximal element of particular subsets of S_a can be deduced.

Corollary 4 *Let* $0\le m<M$ *and* $m \le \frac{a}{\sum_{i=1}^n p_i} \le M$. *Given the set*

$$S_a^1 = \Sigma_a \cap \{\mathbf{x}\in\mathbb{R}^n : M\ge x_1\ge x_2\ge\cdots\ge x_n\ge m\}$$

we have

$$\mathbf{x}^{*\mathbf{p}}(S_a^1) = M\mathbf{s^k} + \theta\mathbf{e^{k+1}} + m\mathbf{v^{k+1}},$$

where k *is the first integer such that*

$$M\sum_{i=1}^k p_i + m\sum_{i=k+1}^n p_i \le a < M\sum_{i=1}^{k+1} p_i + m\sum_{i=k+2}^n p_i$$

and $\theta = \frac{a-M\sum_{i=1}^k p_i - m\sum_{i=k+2}^n p_i}{p_{k+1}}$.

Remark 5

(1) When $\mathbf{p} = \mathbf{s^n}$ we get the maximal element given in [24] (see also Corollary 6 in [25]).

(2) The minimal element of the set S_a^1 is again $\mathbf{x}_{*\mathbf{p}}(\Sigma_a^1) = [(\frac{a}{\sum_{i=1}^n p_i})^n]$.

4 Effective resistance in general electric networks

Let $G = (V, E)$ denote a simple connected network with n vertices and m edges. To each edge $(i,j) \in E$ we associate the resistance r_{ij} and the effective resistance R_{ij}, which can be computed using Ohm's law.

If we deal with electric network with $r_{ij} = 1$ the following relations are well known:

(1) $\sum_{(i,j)\in E} R_{ij} = n - 1$ (Foster's first formula);

(2) $\frac{2}{n} \leq R_{ij} \leq 1$.

The inequality on the left hand side of (2) follows taking $d_i = d_j = n-1$ in the general bound proved in [26]

$$R_{ij} \geq \frac{d_i + d_i - 2}{d_i d_j - 1}, \tag{8}$$

where $(i,j) \in E$ and d_i denotes the degree of vertex i. The inequality on the right hand side follows noting that the effective resistance R_{ij} between two adjacent vertices i and j is equal to one if there is only one path connecting them, otherwise it is strictly less than one.

For a general electric network, assuming $k \leq r_{ij} \leq K$, the previous relations generalize as follows:

(1′) $\sum_{(i,j)\in E} \frac{R_{ij}}{r_{ij}} = n - 1$ (generalized Foster's first formula);

(2′) $\frac{2k}{n} \leq R_{ij} \leq K$.

Relation (2′) can be obtained via electric arguments as we will show below. Indeed, we can prove a more general result that extends the lower bound (8).

Proposition 6 *If* $(i,j) \in E$ *then*

$$R_{ij} \geq \frac{k(d_i + d_j - 2)}{d_i d_j - 1}. \tag{9}$$

Proof The monotonicity principle states that if the resistance of an individual resistor anywhere in the graph is increased (decreased) then the effective resistance between any two vertices in the graph can only increase (decrease) (see Doyle and Snell [19], p.67). Thus R_{ij} is greater than or equal to the effective resistance between i and j when the resistance of all the edges is reduced to k, and so we may assume that $r_{st} = k$ for all $(s,t) \in E$.

Let us consider $(i,j) \in E$. If either $d_i = 1$ or $d_j = 1$ then $R_{ij} = r_{ij} = k$ and (9) holds. So we take $d_i \geq 2$ and $d_j \geq 2$. Consider now all the endpoints of all the other $d_i - 1$ edges stemming out of i and all the $d_j - 1$ edges stemming out of j. Short all these. Then we get two edges in parallel between i and j: one with resistance k and the other with resistance $\frac{k}{d_i-1} + \frac{k}{d_j-1}$. Solving this into a single resistor finishes the proof. □

Corollary 7 *If* $d_i \leq d$ *for all* $i \in V$ *then*

$$R_{ij} \geq \frac{2k}{d+1} \tag{10}$$

for all $(i,j) \in E$.

Proof By differentiating the functions $F(x) = \frac{k(x+d_j-2)}{xd_j-1}$ and $G(x) = \frac{k(d+x-2)}{dx-1}$, they are easily seen to be decreasing and thus

$$R_{ij} \geq F(d_i) \geq F(d) = G(d_j) \geq G(d) = \frac{2k}{d+1}. \qquad \square$$

Note that the bound (10) holds in particular if the graph is d-regular. Finally, since $d_i \leq n-1$ for all $i \in V$, it follows that

$$R_{ij} \geq \frac{2k}{n}, \quad \text{for all } (i,j) \in E.$$

5 Bounds for the global cyclicity index

In [10], by means of the concept of effective resistances, the global cyclicity index has been proposed:

$$C(G) = \sum_{(i,j)\in E} \frac{1}{R_{ij}} - m. \tag{11}$$

Yang in [18] continued the study of this new cyclicity measure for connected graphs. Following Bianchi *et al.* [16, 25] and computing the extremal values of the Schur-convex function $f(R_{ij}) = \sum_{(i,j)\in E} \frac{1}{R_{ij}}$ on the set $S = \{R_{ij} \in \mathbb{R}^m : \sum_{(i,j)\in E} R_{ij} = n-1, \frac{2}{n} \leq R_{ij} \leq 1\}$, he obtained the following bounds for $C(G)$:

$$\frac{m(m-n+1)}{n-1} \leq C(G) \leq \frac{n(m-n+1)}{2}, \tag{12}$$

where $(m-n+1)$ is the well-known cyclomatic number of a graph (see Theorems 3.13, 3.15 and Corollary 3.14 in [18]).

The aim of this section is to extend bounds (12) to the case of general networks where a resistance r_{ij}, $k \leq r_{ij} \leq K$, is associated to any edge. We show next how the weighted majorization technique proposed in Section 2 can be a fruitful tool to bound the weighted global cyclicity index, throughout the p-Schur-convex functions.

First of all, let us define the *weighted global cyclicity index* as a natural extension of the global cyclicity index (11) which can be recovered when $r_{ij} = 1$ for all $(i,j) \in E$:

$$CW(G) = \sum_{(i,j)\in E} \left(\frac{1}{R_{ij}} - \frac{1}{r_{ij}} \right).$$

If we define the variables $x_{ij} = \frac{R_{ij}}{\sqrt{r_{ij}}}$ and the weights $p_{ij} = \frac{1}{\sqrt{r_{ij}}}$, the weighted global cyclicity index can be written as a function of x_{ij} and p_{ij} as follows:

$$CW(G) = f(x_{ij}, p_{ij}) = \sum_{(i,j)\in E} \left(\frac{p_{ij}}{x_{ij}} - p_{ij}^2 \right).$$

The choice of the variables and of the weights ensures that the function f is p-Schur-convex. Indeed, applying Theorem 2 we get

$$(x_{ij} - x_{i'j'}) \left(\frac{1}{p_{ij}} \frac{\partial f}{\partial x_{ij}} - \frac{1}{p_{i'j'}} \frac{\partial f}{\partial x_{i'j'}} \right) = \frac{(x_{ij} - x_{i'j'})^2 (x_{ij} + x_{i'j'})}{x_{ij}^2 x_{i'j'}^2} \geq 0$$

for all $(i,j), (i',j') \in E$.

Remark 8 Note that other possible choices of the variables and of the weights are not fruitful:

(1) if we use as variables $x_{ij} = R_{ij}$ and as weights $p_{ij} = \frac{1}{r_{ij}}$ the function $CW(G)$ is not p-Schur-convex;

(2) if we use as variables $x_{ij} = \frac{R_{ij}}{r_{ij}}$ and as weights $p_{ij} = 1$ the function $CW(G)$ is not Schur-convex.

We can now state our main result.

Theorem 9 *Let $G = (V,E)$ a connected network with n vertices and m edges. Let r_{ij}, $k \le r_{ij} \le K$, be the resistances associated to any edge $(i,j) \in E$ and let*

$$C = \sum_{(i,j)\in E} \frac{1}{r_{ij}}, \qquad C' = \sum_{(i,j)\in E} \frac{1}{\sqrt{r_{ij}}}, \qquad C'' = \max\left\{\frac{nK}{2k}, \frac{\sqrt{k}}{\sqrt{K}} + \frac{n\sqrt{K}}{2k\sqrt{k}} - \frac{1}{K}\right\}.$$

Then

$$\frac{(C')^2}{n-1} - C \le CW(G) \le C'' + \left(\frac{n\sqrt{K}}{2k} + \frac{\sqrt{k}}{K}\right)C' - \frac{n}{2k} - \frac{n^2-n}{2\sqrt{k}\sqrt{K}} - C. \tag{13}$$

Proof Let $e_1, e_2, \ldots, e_m$ be the edges of G. For simplicity denote the resistances and the effective resistances between the end vertices of the edge e_i, as r_i, and R_i, respectively and let $p_i = \frac{1}{\sqrt{r_i}}$. Moreover, let us assume, without loss of generality, that the variables $x_i = \frac{R_i}{\sqrt{r_i}}$ are arranged in nonincreasing order: $x_1 \ge x_2 \ge \cdots \ge x_m$.

Recalling that by Foster's first formula

$$\sum_{e_i\in E} p_i \cdot x_i = (n-1),$$

and that

$$\frac{2k}{n\sqrt{K}} \le x_i \le \frac{K}{\sqrt{k}}, \quad \text{for all } 1 \le i \le m,$$

let us now consider the set

$$S = \left\{\mathbf{x} \in \mathbb{R}^m : \sum_{i=1}^m x_i p_i = (n-1), \frac{K}{\sqrt{k}} \ge x_1 \ge x_2 \ge \cdots \ge x_m \ge \frac{2k}{n\sqrt{K}}\right\}.$$

The function $CW(G) = f(x_i, p_i)$ is p-Schur-convex and thus its lower and upper bounds on S are attained at the minimum and maximum element of S with respect to the p-majorization order, respectively.

From Remark 5 we know that the minimal element of S is $\mathbf{x}_{*\mathbf{p}}(S) = [(\frac{n-1}{\sum_{i=1}^m p_i})]^m$. Thus the lower bound is

$$f(\mathbf{x}_{*\mathbf{p}}, \mathbf{p}) = \sum_{i=1}^m p_i \cdot \frac{\sum_{i=1}^m p_i}{n-1} - C = \frac{(C')^2}{n-1} - C.$$

The maximal element $\mathbf{x}^{*\mathbf{p}}$ of the set S can be computed with Corollary 4, yielding

$$\mathbf{x}^{*\mathbf{p}} = \Bigg[\underbrace{\frac{K}{\sqrt{k}},\dots,\frac{K}{\sqrt{k}}}_{l\text{-times}},\theta,\underbrace{\frac{2k}{n\sqrt{K}},\dots,\frac{2k}{n\sqrt{K}}}_{(m-l-1)\text{-times}}\Bigg],$$

where l is the first integer such that

$$\frac{K}{\sqrt{k}}\sum_{i=1}^{l}p_i + \frac{2k}{n\sqrt{K}}\sum_{i=l+1}^{m}p_i \leq (n-1) < \frac{K}{\sqrt{k}}\sum_{i=1}^{l+1}p_i + \frac{2k}{n\sqrt{K}}\sum_{i=l+2}^{m}p_i \tag{14}$$

and $\theta = p_{l+1}\cdot(n-1-\frac{K}{\sqrt{k}}\sum_{i=1}^{l}p_i - \frac{2k}{n\sqrt{K}}\sum_{i=l+2}^{m}p_i)$. Let $D = \sum_{i=1}^{l}p_i$ and

$$H = \frac{n-1-\frac{2k}{n\sqrt{K}}C'}{\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}}}.$$

From (14) easy computations show that

$$0 \leq (H-D) < p_{l+1}.$$

Moreover,

$$f\big(\mathbf{x}^{*\mathbf{p}},\mathbf{p}\big) = \left(\frac{n\sqrt{K}}{2k}-\frac{\sqrt{k}}{K}\right)(H-D) + \frac{1}{(\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}})(H-D)+\frac{2k}{n\sqrt{K}}\cdot p_{l+1}} + T,$$

where

$$T = \left(\frac{n\sqrt{K}}{2k}+\frac{\sqrt{k}}{K}\right)C' - \frac{n\sqrt{K}}{2k}\cdot p_{l+1} - \frac{n^2-n}{2\sqrt{k}\sqrt{K}} - C.$$

Let $y = H-D$ and consider the function

$$h(y) = \left(\frac{n\sqrt{K}}{2k}-\frac{\sqrt{k}}{K}\right)y + \frac{1}{(\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}})y+\frac{2k}{n\sqrt{K}}\cdot p_{l+1}} + T.$$

The first derivative is

$$h'(y) = -\frac{\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}}}{[(\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}})y+\frac{2k}{n\sqrt{K}}\cdot p_{l+1}]^2} + \frac{n\sqrt{K}}{2k} - \frac{\sqrt{k}}{K}$$

and the only nonnegative stationary point is

$$\widehat{y} = \frac{\sqrt{\frac{2\sqrt{kK}}{n}} - \frac{2k}{n\sqrt{K}}\cdot p_{l+1}}{\frac{K}{\sqrt{k}}-\frac{2k}{n\sqrt{K}}}.$$

Assuming, without loss of generality that $k \leq 1 \leq K$, we can also be assured that $\widehat{y} < p_{l+1}$. The stationary point $\widehat{y}$ turns out to be a minimum. Thus the maximum value of the function h is attained at the extremum of the interval $[0,p_{l+1}]$. We have

$$h(0) = \frac{1}{\frac{2k}{n\sqrt{K}} \cdot p_{l+1}} + T,$$

$$h(p_{l+1}) = \frac{1}{\frac{K}{\sqrt{k}} \cdot p_{l+1}} + \left(\frac{n\sqrt{K}}{2k} - \frac{\sqrt{k}}{K}\right) \cdot p_{l+1} + T.$$

We can get rid of p_{l+1} by using the bounds on the resistances. We obtain

$$h(0) \le \frac{nK}{2k} + T,$$

$$h(p_{l+1}) \le \frac{\sqrt{k}}{\sqrt{K}} + \frac{n\sqrt{K}}{2k\sqrt{k}} - \frac{1}{K} + T.$$

The assertion easily follows from the bound

$$T \le \left(\frac{n\sqrt{K}}{2k} + \frac{\sqrt{k}}{K}\right)C' - \frac{n}{2k} - \frac{n^2 - n}{2\sqrt{k}\sqrt{K}} - C. \qquad \square$$

Noting that $\frac{m}{\sqrt{K}} \le C' \le \frac{m}{\sqrt{k}}$ and $\frac{m}{K} \le C \le \frac{m}{k}$, we get the following corollary.

Corollary 10 *Let $G = (V,E)$ a connected network with n vertices and m edges. Let r_{ij}, $k \le r_{ij} \le K$, be the resistances associated to any edge. Then*

$$\frac{m^2}{K(n-1)} - \frac{m}{k} \le CW(G) \le C'' + \frac{nm\sqrt{K}}{2k\sqrt{k}} - \frac{n}{2k} - \frac{n^2 - n}{2\sqrt{k}\sqrt{K}}, \tag{15}$$

where $C'' = \max\{\frac{nK}{2k}, \frac{\sqrt{k}}{\sqrt{K}} + \frac{n\sqrt{K}}{2k\sqrt{k}} - \frac{1}{K}\}$.

If in inequality (15) we set $k = K = 1$, that is, $r_{ij} = 1$ for all $(i,j) \in E$, we get

$$\frac{m(m-n+1)}{n-1} \le CW(G) \le \frac{n(m-n+1)}{2}$$

i.e. the bounds provided by Yang in [18], Theorem 3.13 and Corollary 3.14 for the global cyclicity index.

6 Summary and conclusion

In this article we defined the new *weighted global cyclicity index* that applies to graphs whose edges are endowed with arbitrary resistances and generalizes the global cyclicity index, introduced by Klein and Ivanciuc, which applies only to graphs whose edges are endowed with unit resistances. Through the use of p-majorization we provided upper and lower bounds for the new index that coincide with those given by Yang in the particular case of unit resistances. The maximal results obtained through p-majorization are of interest in themselves and may be applicable in other contexts.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Received: 6 November 2014 Accepted: 7 March 2015 Published online: 28 March 2015

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