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Hardy inequalities with Aharonov-Bohm type magnetic field on the Heisenberg group

Yingxiong Xiao*

*Correspondence:
yxxiao2011@163.com
School of Mathematics and Statistics, Hubei Engineering University, Xiaogan, Hubei 432000, People's Republic of China

Abstract
We introduce an Aharonov-Bohm type magnetic field on three-dimensional Heisenberg group and show this quadratic form satisfy an improved Hardy inequality with weights.

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1 Introduction

The classical Hardy inequality states that, for $N \geq 3$ and for all $u \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2\,dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx \tag{1.1}$$

and $(\frac{N-2}{2})^2$ is the best constant in (1.1). If $N = 2$, the classical Hardy inequality fails. However, for some magnetic forms in dimension two, the Hardy inequality becomes possible. In fact, if $\beta\mathbf{a}$ is the Aharonov-Bohm magnetic field

$$\beta\mathbf{a} = \beta\left(-\frac{x_2}{x_1^2+x_2^2}, \frac{x_1}{x_1^2+x_2^2}\right),$$

then for all $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ (*cf.* [1]),

$$\int_{\mathbb{R}^2} \left|(\nabla + i\beta\mathbf{a})u\right|^2 dx \geq \min_{k\in\mathbb{Z}} |k-\beta|^2 \int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\,dx. \tag{1.2}$$

Recently, a version of the Aharonov-Bohm magnetic field for a Grushin subelliptic operator has been introduced by Aermark and Laptev [2]. Furthermore, such quadratic form also satisfies an improved Hardy inequality. In the same paper, they asked the following question: does there exist a similar result for the Heisenberg quadratic form?

Recall that the three-dimension Heisenberg group $\mathbb{H}_1 = (\mathbb{R}^2 \times \mathbb{R}, \circ)$ is a step-two nilpotent group whose group structure is given by

$$(x,y,t) \circ (x',y',t') = \left(x+x', y+y', t+t' - \frac{1}{2}(xy'-yx')\right).$$

The vector fields

$$X = \frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - \frac{x}{2}\frac{\partial}{\partial t}$$

are left invariant and generate the Lie algebra of $\mathbb{H}_1$. The Kohn sub-Laplacian on $\mathbb{H}_1$ is

$$\Delta_{\mathbb{H}} = X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{x^2+y^2}{4}\frac{\partial^2}{\partial t^2} + \frac{1}{4}\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial t}$$

and the subgradient is the vector given by $\nabla_{\mathbb{H}} = (X, Y)$. For simplicity, we let $z = x + yi$. Then $|z| = \sqrt{x^2+y^2}$. Denote

$$\rho := \rho(z,t) = \left(|z|^4 + 16t^2\right)^{\frac{1}{4}}.$$

Similar as in [1, 2], we define an Aharonov-Bohm type magnetic field $\mathcal{A}$ on $\mathbb{H}_1$:

$$\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2) = \left(-\frac{Y\rho}{\rho}, \frac{X\rho}{\rho}\right). \tag{1.3}$$

To our surprise, for such magnetic field (1.3), we cannot deal with the Hardy inequality

$$\int_{\mathbb{H}_1} |\nabla_{\mathbb{H}} u|^2 \, dz\, dt \ge \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2} |\nabla_{\mathbb{H}}\rho|^2 \, dz\, dt, \quad u \in C_0^\infty(\mathbb{H}_1), \tag{1.4}$$

but the Hardy inequality with weight

$$\int_{\mathbb{H}_1} \frac{|\nabla_{\mathbb{H}} u|^2}{|\nabla_{\mathbb{H}}\rho|^2} \, dz\, dt \ge \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2} \, dz\, dt, \quad u \in C_0^\infty(\mathbb{H}_1).$$

For the reasons, see Remark 2.2. For more Hardy inequalities on Heisenberg groups, we refer to [3–10].

The main result is the following theorem.

Theorem 1.1 *We have, for $u \in C_0^\infty(\mathbb{H}_1)$,*

$$\int_{\mathbb{H}_1} \frac{|(\nabla_{\mathbb{H}} + i\beta\mathcal{A})u|^2}{|\nabla_{\mathbb{H}}\rho|^2} \, dz\, dt \ge \left(1 + \min_{k\in\mathbb{Z}} |k-\beta|^2\right) \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2} \, dz\, dt. \tag{1.5}$$

2 Proof of Theorem 1.1

Before the proof of Theorem 1.1, we need a polar coordinate associated with ρ on $\mathbb{H}_1$. We describe it as follows. For each real number $\lambda > 0$, there is a dilation naturally associated with the group structure which is usually denoted $\delta_\lambda(x,y,t) = (\lambda x, \lambda y, \lambda^2 t)$. The Jacobian determinant of δ_λ is λ^Q, where $Q = 4$ is the homogeneous dimension of $\mathbb{H}_1$. For simplicity, we use the notation $\lambda(x,y,t) = \delta_\lambda(x,y,t)$. Given any $\xi = (x,y,t) \in \mathbb{H}_1$, set $x^* = \frac{x}{\rho}$, $y^* = \frac{y}{\rho}$, $t^* = \frac{t}{\rho^2}$, and $\xi^* = (x^*, y^*, t^*)$ if $\rho(\xi) \neq 0$. The polar coordinate on $\mathbb{H}_1$ associated with ρ is the following (*cf.* [11], Proposition (1.15)):

$$\int_{\mathbb{H}_1} f(\xi)\, dz\, dt = \int_0^\infty \int_\Sigma f\left(\lambda\xi^*\right)\lambda^3 \, d\sigma\, d\lambda, \quad f \in L^1(\mathbb{H}_1),$$

where $\Sigma = \{(x,y,t) \in \mathbb{H}_1 : \rho(x,y,t) = 1\}$ is the unit sphere associated with ρ. Moreover, there is a parametrization of this polar coordinate (*cf.* [12], Theorem 5.12):

$$\begin{cases} x = \rho\sqrt{\cos\alpha}\cos\theta; \\ y = \rho\sqrt{\cos\alpha}\sin\theta; \\ t = \frac{1}{4}\rho^2\sin\alpha, \end{cases} \tag{2.1}$$

where $\alpha \in [-\pi/2, \pi/2)$, $\theta \in [0, 2\pi)$, and $0 \le \rho < \infty$. Using this parametrization, we can rewrite the polar coordinate as follows:

$$\int_{\mathbb{H}_1} f(\xi)\,dz\,dt = \frac{1}{4}\int_0^\infty \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} f(\xi)\rho^3\,d\rho\,d\alpha\,d\theta, \quad f \in L^1(\mathbb{H}_1). \tag{2.2}$$

Proof of Theorem 1.1 Using the identity

$$\left(a^2+b^2\right)\left(|z_1|^2+|z_2|^2\right) = |az_1+bz_2|^2 + |az_2-bz_1|^2, \quad a,b \in \mathbb{R}, z_1, z_2 \in \mathbb{C},$$

we have

$$\begin{aligned} &|\nabla_{\mathbb{H}}\rho|^2\left|(\nabla_{\mathbb{H}}+i\beta\mathcal{A})u\right|^2 \\ &\quad = \left(|X\rho|^2+|Y\rho|^2\right)\left(\left|Xu - i\beta\frac{Y\rho}{\rho}\right|^2 + \left|Yu + i\beta\frac{X\rho}{\rho}\right|^2\right) \\ &\quad = |X\rho\cdot Xu + Y\rho\cdot Yu|^2 + \left|X\rho\cdot Yu - Y\rho\cdot Xu + i\beta\frac{|\nabla_{\mathbb{H}}\rho|^2 u}{\rho}\right|^2. \end{aligned} \tag{2.3}$$

By (2.1),

$$\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \tag{2.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial\alpha} &= \frac{\partial x}{\partial\alpha}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\alpha}\frac{\partial}{\partial y} + \frac{\partial t}{\partial\alpha}\frac{\partial}{\partial t} \\ &= -\frac{\rho\sin\alpha\cos\theta}{2\sqrt{\cos\alpha}}\frac{\partial}{\partial x} - \frac{\rho\sin\alpha\sin\theta}{2\sqrt{\cos\alpha}}\frac{\partial}{\partial y} + \frac{1}{4}\rho^2\cos\alpha\frac{\partial}{\partial t} \\ &= -\frac{2xt}{|z|^2}\frac{\partial}{\partial x} - \frac{2yt}{|z|^2}\frac{\partial}{\partial y} + \frac{|z|^2}{4}\frac{\partial}{\partial t}. \end{aligned} \tag{2.5}$$

Therefore, by (2.4) and (2.5)

$$\begin{aligned} X\rho\cdot Yu - Y\rho\cdot Xu &= \frac{|z|^2x+4yt}{\rho^3}\left(\frac{\partial u}{\partial y} - \frac{x}{2}\frac{\partial u}{\partial t}\right) - \frac{|z|^2y-4xt}{\rho^3}\left(\frac{\partial u}{\partial x} + \frac{y}{2}\frac{\partial u}{\partial t}\right) \\ &= \frac{|z|^2}{\rho^3}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right) + \frac{4xt}{\rho^3}\frac{\partial u}{\partial x} + \frac{4yt}{\rho^3}\frac{\partial u}{\partial y} - \frac{|z|^4}{2\rho^3}\frac{\partial u}{\partial t} \\ &= \frac{|z|^2}{\rho^3}\frac{\partial u}{\partial\theta} + \frac{|z|^2}{\rho^3}\left(\frac{4xt}{|z|^2}\frac{\partial}{\partial x} + \frac{4yt}{|z|^2}\frac{\partial}{\partial y} - \frac{|z|^2}{2}\frac{\partial}{\partial t}\right) \\ &= \frac{|z|^2}{\rho^3}\left(\frac{\partial u}{\partial\theta} - 2\frac{\partial u}{\partial\alpha}\right) = \frac{|\nabla_{\mathbb{H}}\rho|^2}{\rho}\left(\frac{\partial u}{\partial\theta} - 2\frac{\partial u}{\partial\alpha}\right). \end{aligned} \tag{2.6}$$

To get the last inequality above, we use the fact $|\nabla_{\mathbb{H}}\rho| = \frac{|z|}{\rho}$. Combining (2.3) and (2.6) yields

$$\begin{aligned}\int_{\mathbb{H}_1} \frac{|(\nabla_{\mathbb{H}} + i\beta\mathcal{A})u|^2}{|\nabla_{\mathbb{H}}\rho|^2}\,dz\,dt &= \int_{\mathbb{H}_1} \frac{|X\rho\cdot Xu + Y\rho\cdot Yu|^2}{|\nabla_{\mathbb{H}}\rho|^4}\,dz\,dt \\ &\quad + \int_{\mathbb{H}_1} \frac{|\frac{\partial u}{\partial\theta} - 2\frac{\partial u}{\partial\alpha} + i\beta u|^2}{\rho^2}\,dz\,dt \\ &= (I) + (II),\end{aligned} \tag{2.7}$$

where

$$(I) := \int_{\mathbb{H}_1} \frac{|X\rho\cdot Xu + Y\rho\cdot Yu|^2}{|\nabla_{\mathbb{H}}\rho|^4}\,dz\,dt$$

and

$$\begin{aligned}(II) &:= \int_{\mathbb{H}_1} \frac{|\frac{\partial u}{\partial\theta} - 2\frac{\partial u}{\partial\alpha} + i\beta u|^2}{\rho^2}\,dz\,dt \\ &= \frac{1}{4}\int_0^\infty \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} \left|\frac{\partial u}{\partial\theta} - 2\frac{\partial u}{\partial\alpha} + i\beta u\right|^2 \rho\,d\rho\,d\alpha\,d\theta.\end{aligned}$$

If we represent u by the Fourier series

$$u(\rho,\alpha,\theta) = \sum_{n=-\infty}^{+\infty} u_n(\rho,\alpha)e^{in\theta}/\sqrt{2\pi},$$

then

$$(II) = \sum_{n=-\infty}^{+\infty} \frac{1}{4}\int_0^\infty \int_{-\pi/2}^{\pi/2} \left|2\frac{\partial u_n(\rho,\alpha)}{\partial\alpha} - i(\beta+n)u_n(\rho,\alpha)\right|^2 \rho\,d\rho\,d\alpha. \tag{2.8}$$

Similarly, representing $u_n(\rho,\alpha)$ by the Fourier series

$$u_n(\rho,\alpha) = \sum_{k=-\infty}^{+\infty} u_{n,k}(\rho)e^{i2k\alpha}/\sqrt{\pi},$$

we have

$$\begin{aligned}(II) &= \sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} \frac{|4k-n-\beta|^2}{4}\int_0^\infty \left|u_{n,k}(\rho)\right|^2\rho\,d\rho \\ &\geq \min_{k\in\mathbb{Z}}|k-\beta|^2\cdot\frac{1}{4}\sum_{n=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty}\int_0^\infty \left|u_{n,k}(\rho)\right|^2\rho\,d\rho \\ &= \min_{k\in\mathbb{Z}}|k-\beta|^2\cdot\frac{1}{4}\int_0^\infty\int_{-\pi/2}^{\pi/2}\int_0^{2\pi}\left|u(\rho,\alpha,\theta)\right|^2\rho\,d\rho\,d\alpha\,d\theta \\ &= \min_{k\in\mathbb{Z}}|k-\beta|^2\int_{\mathbb{H}_1}\frac{|u|^2}{\rho^2}\,dz\,dt.\end{aligned} \tag{2.9}$$

To finish the proof, it is enough to show

$$(I) = \int_{\mathbb{H}_1} \frac{|X\rho \cdot Xu + Y\rho \cdot Yu|^2}{|\nabla_{\mathbb{H}}\rho|^4}\,dx\,dy\,dt \geq \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2}\,dx\,dy\,dt.$$

This will be done in Lemma 2.1. The proof of Theorem 1.1 is therefore completed. □

Before we turn to the proof of Lemma 2.1, we need the horizontal polar coordinates on $\mathbb{H}_1$ which have been introduced by Korányi and Reimann [13] (see also [14], pp.110-112). Set

$$\gamma_\xi(\rho) = \left(sze^{4i\frac{t}{|z|^2}\log\rho}, \frac{1}{4}\rho^2 t\right), \quad \xi = (z,t) \in \Sigma.$$

The horizontal polar coordinate on $\mathbb{H}_1$ is

$$\int_{\mathbb{H}_1} f(z,t)\,dz\,dt = \int_0^\infty \int_\Sigma f(\gamma_\xi(\rho))\rho^3\,ds\,d\sigma, \quad f \in L^1(\mathbb{H}_1).$$

Furthermore, we can also give a parametrization of this polar coordinate through setting ([14], pp.111-112)

$$\Phi : \begin{cases} x = \rho\sqrt{\cos\alpha}\cos(\theta + 4\tan\alpha\log\rho); \\ y = \rho\sqrt{\cos\alpha}\sin(\theta + 4\tan\alpha\log\rho); \\ t = \rho^2\sin\alpha. \end{cases}$$

The Jacobian determinant of Φ is ρ^3 so that

$$\begin{aligned}\int_{\mathbb{H}_1} f(z,t)\,dz\,dt &= \int_0^\infty \int_\Sigma f(\gamma_\xi(\rho))\rho^3\,d\rho\,d\xi \\ &= \frac{1}{4}\int_0^\infty \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} f(\gamma_\xi(\rho))\rho^3\,d\rho\,d\alpha\,d\theta. \end{aligned} \tag{2.10}$$

Using this parametrization, we have (see [14], p.112)

$$\begin{aligned}\frac{d}{d\rho}f(\gamma_\xi(\rho)) &= \frac{1}{\rho|z|^2}\left(\left(x|z|^2 - 4yt\right)Xf + \left(y|z|^2 + 4xt\right)Yf\right) \\ &= \frac{1}{4}\frac{\langle\nabla_{\mathbb{H}}\rho^4, \nabla_{\mathbb{H}}f\rangle}{\rho|z|^2} = \frac{\rho^2}{|z|^2}\langle\nabla_{\mathbb{H}}\rho, \nabla_{\mathbb{H}}f\rangle \\ &= \frac{\langle\nabla_{\mathbb{H}}\rho, \nabla_{\mathbb{H}}f\rangle}{|\nabla_{\mathbb{H}}\rho|^2}. \end{aligned} \tag{2.11}$$

Lemma 2.1 *We have, for $u \in C_0^\infty(\mathbb{H}_1)$,*

$$\int_{\mathbb{H}_1} \frac{|X\rho \cdot Xu + Y\rho \cdot Yu|^2}{|\nabla_{\mathbb{H}}\rho|^4}\,dz\,dt \geq \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2}\,dz\,dt.$$

Proof Notice that

$$\begin{aligned}&\int_0^\infty \left|\frac{d}{d\rho}f(\gamma_\xi(\rho))\right|^2\rho^3\,d\rho - \int_0^\infty \left|f(\gamma_\xi(\rho))\right|^2\rho\,d\rho \\ &\quad = \int_0^\infty \left|\frac{d(\rho f(\gamma_\xi(\rho)))}{d\rho}\right|^2\rho\,d\rho \geq 0. \end{aligned} \tag{2.12}$$

Integrating over $-\pi/2 \le \alpha \le \pi/2$ and $0 \le \theta \le 2\pi$ and using (2.10) yields

$$\int_{\mathbb{H}_1} \left| \frac{d}{d\rho} f\big(\gamma_\xi(\rho)\big) \right|^2 dz\,dt \ge \int_{\mathbb{H}_1} \frac{|f(\gamma_\xi(\rho))|^2}{\rho^2}\,dz\,dt.$$

Combining the inequality above and (2.11) gives

$$\int_{\mathbb{H}_1} \frac{|X\rho \cdot Xu + Y\rho \cdot Yu|^2}{|\nabla_{\mathbb{H}}\rho|^4}\,dz\,dt \ge \int_{\mathbb{H}_1} \frac{|u|^2}{\rho^2}\,dz\,dt.$$

This completes the proof of Lemma 2.1. □

Remark 2.2 If one considers the Hardy inequality (1.4) with the Aharonov-Bohm type magnetic field $\mathcal{A}$ on $\mathbb{H}_1$, then, following the proof above, one needs to show

$$\int_{-\pi/2}^{\pi/2} \left| \frac{\partial u}{\partial \alpha} - i\beta u \right|^2 \cos\alpha\,d\alpha \ge \min_{k\in\mathbb{Z}} |2k-\beta|^2 \int_{-\pi/2}^{\pi/2} |u|^2 \cos\alpha\,d\alpha. \tag{2.13}$$

However, to the best of our knowledge, it is not known whether inequality (2.13) is valid. The reason is that, in this case, $\{e^{2ki\pi}/\sqrt{\pi}\}$ is not an orthonormal basis because there exists a weight $\cos\alpha$ in (2.13).

Competing interests
The author declares to have no competing interests.

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