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Some elliptic system and reduction method

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Abstract

We get a theorem which shows the existence of at least three solutions for some elliptic system with Dirichlet boundary condition. We obtain this result by using the finite dimensional reduction method for the dimension of the system which reduces the infinite dimensional problem to the finite dimensional one. We also use critical point theory on the reduced finite dimensional subspace.

MSC: 35J50; 35J55

Keywords: elliptic system; reduction method; (PS) condition; critical point theory; variational method

1 Introduction

In this paper we are concerned with multiple solutions for a class of systems of elliptic equations with Dirichlet boundary condition

$$\begin{aligned} &-\Delta u_1 = F_{u_1}(x, u_1, \dots, u_n) \quad \text{in } \Omega, \\ &-\Delta u_2 = F_{u_2}(x, u_1, \dots, u_n) \quad \text{in } \Omega, \\ &\vdots \\ &-\Delta u_n = F_{u_n}(x, u_1, \dots, u_n) \quad \text{in } \Omega, \\ &u_i(x) = 0, \quad i = 1, \dots, n, \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded subset of R^n with smooth boundary $\partial\Omega$, $n \geq 3$, $u_i(x) \in W_0^{1,2}(\Omega)$, $F : R^n \times R^n \to R$ is a C^2 function such that $F(x, \theta) = 0$, $\theta = (0, \dots, 0)$ and $F_{u_i}(x, u_1, \dots, u_n) = \frac{\partial F(x,u_1,\dots,u_n)}{\partial u_i}$, $i = 1, \dots, n$. Let $U = (u_1, \dots, u_n)$ and $\| \cdot \|_{R^n}$ denote the Euclidean norm in R^n. Let us define

$$d_U F(x, U) = F_U(x, U) = \operatorname{grad}_U F(x, U) = \bigl(F_{u_1}(x, u_1, \dots, u_n), \dots, F_{u_n}(x, u_1, \dots, u_n)\bigr)$$

and

$$d_U^2 F(U) \cdot U = d\bigl(F_U(x, U)\bigr) \cdot U \quad \forall U \in E.$$

Let $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ be eigenvalues of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$, and ϕ_k be an eigenfunction belonging to the eigenvalue λ_k, $k \geq 1$.

We assume that F satisfies the following conditions:

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(F1) $F \in C^2(R^n \times R^n, R)$, $F(x,\theta) = 0$, $F_U(x,\theta) = \theta$, $x \in \Omega$, $\theta = (0,\dots,0)$.

(F2) There exist constants α and β (α, β are not eigenvalues of the elliptic eigenvalue problem) such that $\alpha < \beta$ and

$$\alpha I \le d_U^2 F(x,U) \le \beta I \quad \forall (x,U) \in R^n \times R^n,$$

and there exists $k \in N^*$ such that $\alpha I < \lambda_k I < d_U^2 F(x,U) < \lambda_{k+1} I < \beta I$ for every U, where I is the $n \times n$ identity matrix.

(F3) There exist eigenvalues $\lambda_{h+1}, \dots, \lambda_{h+m}$ such that

$$\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{h+m} < \beta < \lambda_{h+m+1},$$

where $h \ge 1$, $m \ge 1$.

(F4) There exist γ and C such that $\lambda_{h+m} < \gamma < \beta$ and

$$F(x,U) \ge \frac{1}{2}\gamma \|U\|_{R^n}^2 - C, \quad \forall (x,U) \in R^n \times R^n.$$

Some papers of Lee [1–4] concerning the semilinear elliptic system and some papers of the other several authors [5, 6] have treated the system of this like nonlinear elliptic equations. Some papers of Chang [7] and Choi and Jung [8] considered the existence and multiplicity of weak solutions for nonlinear boundary value problems with asymptotically linear term. The authors obtained some results for those problems by approaching the variational method, critical point theory and the topological method.

Let $W_0^{1,2}(\Omega, R)$ be the Sobolev space with the norm

$$\|u\|_{W_0^{1,2}(\Omega,R)}^2 = \int_\Omega |\nabla u|^2 \, dx \quad \text{for } u \in W_0^{1,2}(\Omega)$$

and the scalar product

$$(u,v)_{W_0^{1,2}(\Omega,R)} = (\nabla u, \nabla v)_{L^2(\Omega,R)}.$$

Let E be a cartesian product of the Sobolev spaces $W_0^{1,2}(\Omega,R)$, *i.e.*,

$$E = W_0^{1,2}(\Omega,R) \times \cdots \times W_0^{1,2}(\Omega,R).$$

We endow the Hilbert space E with the norm

$$\|U\|_E^2 = \sum_{i=1}^n \|u_i\|_{W_0^{1,2}(\Omega)}^2,$$

where $\|u_i\|_{W_0^{1,2}(\Omega,R)}^2 = \int_\Omega |\nabla u_i(x)|^2 \, dx$. From now on we shall denote by $W_0^{1,2}(\Omega)$ instead of $W_0^{1,2}(\Omega,R)$.

System (1.1) can be rewritten by

$$\begin{aligned} -\Delta U &= \operatorname{grad}_U F(x,U) \quad \text{in } \Omega, \qquad (1.2)\\ U &= \theta \quad \text{on } \partial\Omega, \end{aligned}$$

where $-\Delta U = (-\Delta u_1, \dots, -\Delta u_n)$ and $\theta = (0, \dots, 0)$. In this paper we are looking for weak solutions of system (1.1) in E, that is, $U = (u_1, \dots, u_n) \in E$ such that

$$\int_\Omega [-\Delta U \cdot V]\,dx - \int_\Omega F_U(x, U) \cdot V = 0 \quad \text{for all } V \in E.$$

Our main result is the following.

Theorem 1.1 *Assume that F satisfies conditions* (F1)-(F4). *Then system* (1.1) *has at least three nontrivial weak solutions.*

The proof of Theorem 1.1 is organized as follows: We approach the variational method and use the finite dimensional reduction method for the dimension of the system, which reduces the infinite dimensional problem to the finite dimensional one, and we get critical points of the functional on the infinite dimensional space E from that of the reduced functional on the finite dimensional subspace of E. We also use critical point theory on the reduced finite dimensional subspace. In Section 2, we approach the variational method and the reduction method. We show that the reduced functional satisfies the (PS) condition. In Section 3, we prove Theorem 1.1.

2 Reduction approach

We assume that $F \in C^2(R^n \times R^n, R)$, $F(x, \theta) = 0$, $F_U(x, \theta) = \theta$, $\theta = (0, \dots, 0)$ and there exist constants α and β (α, β are not eigenvalues of the elliptic eigenvalue problem) such that $\alpha < \beta$ and

$$\alpha I \le d_U^2 F(x, U) \le \beta I \quad \forall (x, U) \in R^n \times R^n,$$

and there exists $k \in N^*$ such that $\alpha I < \lambda_k I < d_U^2 F(x, U) < \lambda_{k+1} I < \beta I$ for every U, where $U = (u_1, \dots, u_n)$ and there exist eigenvalues $\lambda_{h+1}, \dots, \lambda_{h+m}$ such that

$$\lambda_h < \alpha < \lambda_{h+1} < \cdots < \lambda_{h+m} < \beta < \lambda_{h+m+1},$$

where $h \ge 1$, $m \ge 1$.

Lemma 2.1 *Let $F_{u_i}(x, U) \in L^2(\Omega)$, $U = (u_1, \dots, u_i, \dots, u_n)$, $i = 1, \dots, n$. Then all the solutions of*

$$-\Delta U = \operatorname{grad}_U F(x, U)$$

belong to E.

Proof Let $F_{u_i}(x, U) \in L^2(\Omega)$. We note that $\{\lambda_n : |\lambda_n| < |c|\}$ is finite. Then $F_{u_i}(x, u_1, \dots, u_n) \in L^2(\Omega)$, $i = 1, \dots, n$, can be expressed by

$$F_{u_i}(x, u_1, \dots, u_n) = \sum_{k=1}^{\infty} h_k \phi_k, \qquad \sum_{k=1}^{\infty} h_k^2 < \infty \quad \text{for each } i = 1, \dots, n.$$

Then

$$(-\Delta)^{-1}F_{u_i}(x,u_1,\dots,u_n) = \sum \frac{1}{\lambda_k} h_k \phi_k.$$

Hence we have the inequality

$$\big\|(-\Delta)^{-1}F_{u_i}(x,u_1,\dots,u_n)\big\|^2_{W_0^{1,2}(\Omega)} = \sum \lambda_k \frac{1}{\lambda_k^2} h_k^2 \le \sum h_k^2 < \infty,$$

which means that

$$\big\|(-\Delta)^{-1}F_{u_i}(x,u_1,\dots,u_n)\big\|_{W_0^{1,2}(\Omega)} \le \big\|F_{u_i}(x,u_1,\dots,u_n)\big\|_{L^2(\Omega)} < \infty,$$

so $\|u_i\|_{W_0^{1,2}(\Omega)} < \infty$. Thus

$$\|U\|_E = \left(\sum_{i=1}^n \|u_i\|_{W_0^{1,2}(\Omega)}\right)^{\frac{1}{2}} < \infty.$$ □

By the following Lemma 2.2, weak solutions of system (1.1) coincide with critical points of the associated functional I

$$\begin{aligned} &I \in C^{1,1}(E,R), \\ &I(U) = \int_\Omega \left[\frac{1}{2}|\nabla U|^2 - F(x,U)\right] dx, \end{aligned} \tag{2.1}$$

where $U = (u_1,\dots,u_n)$ and $\int_\Omega |\nabla U|^2\,dx = \sum_{i=1}^n \int_\Omega |\nabla u_i|^2\,dx$, $n \ge 1$.

Lemma 2.2 *Assume that F satisfies conditions* (F1)-(F4). *Then the functional $I(U)$ is continuous, Fréchet differentiable with Fréchet derivative*

$$DI(U)\cdot V = \int_\Omega \big[(-\Delta U)\cdot V - F_U(x,U)\cdot V\big]\,dx.$$

Moreover, $DI \in C$. That is, $I \in C^1$.

Proof First we shall prove that $I(U)$ is continuous. For $U, V \in E$,

$$\begin{aligned} \big|I(U+V) - I(U)\big| &= \bigg|\frac{1}{2}\int_\Omega (-\Delta U - \Delta V)\cdot(U+V)\,dx - \int_\Omega F(x,U+V)\,dx \\ &\quad - \frac{1}{2}\int_\Omega (-\Delta U)\cdot U\,dx + \int_\Omega F(x,U)\,dx\bigg| \\ &= \bigg|\frac{1}{2}\int_\Omega (-\Delta U\cdot V - \Delta V\cdot U - \Delta V\cdot V)\,dx \\ &\quad - \int_\Omega \big(F(x,U+V) - F(x,U)\big)\,dx\bigg|. \end{aligned}$$

We have

$$\left|\int_\Omega \big[F(x,U+V) - F(x,U)\big]\,dx\right| \le \left|\int_\Omega \big[F_U(x,U)\cdot V + O\big(\|V\|_E\big)\big]\,dx\right| = O\big(\|V\|_E\big). \tag{2.2}$$

Thus we have

$$\begin{aligned}&\bigl|I(U+V)-I(U)\bigr| = O\bigl(\|V\|_E\bigr),\\&\bigl|I(U+V)-I(U)-DI(U)\cdot V\bigr| = O\bigl(\|V\|_E^2\bigr).\end{aligned} \tag{2.3}$$

Next we shall prove that $I(U)$ is Fréchet differentiable. For $U, V \in E$,

$$\begin{aligned}&\bigl|I(U+V)-I(U)-DI(U)\cdot V\bigr|\\&\quad = \biggl|\frac{1}{2}\int_\Omega (-\Delta U-\Delta V)\cdot(U+V)\,dx - \int_\Omega F(x,U+V)\,dx\\&\qquad - \frac{1}{2}\int_\Omega(-\Delta U)\cdot U\,dx + \int_\Omega F(x,U)\,dx - \int_\Omega \bigl(-\Delta U - F_U(x,U)\bigr)\cdot V\,dx\biggr|\\&\quad = \biggl|\frac{1}{2}\int_\Omega [-\Delta U\cdot V - \Delta V\cdot U - \Delta V\cdot V]\,dx\\&\qquad - \int_\Omega \bigl[F(x,U+V)-F(x,U)\bigr]\,dx - \int_\Omega \bigl[\bigl(-\Delta U - F_U(x,U)\bigr)\cdot V\bigr]\,dx\biggr|.\end{aligned}$$

By (2.2),

$$\bigl\|I(U+V)-I(U)-DI(U)\cdot V\bigr\| = O\bigl(\|V\|_E^2\bigr).$$

Thus $I \in C^1$. □

Let $E = W_0^{1,2}(\Omega,R)\times\cdots\times W_0^{1,2}(\Omega,R)$ and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis in R^n. Then

$$L^2\bigl(\Omega,R^n\bigr) = \bigoplus_{m\in N} M(\lambda_m),$$

where N is a natural number and $M(\lambda_m) = \operatorname{span}\{\phi_m e_1, \ldots, \phi_m e_n\}$ is the eigenspace of $-\Delta$ with eigenvalue λ_m, $\dim M(\lambda_m) = n$, $m = 1, 2, \ldots$. Let

$$L_1 = \bigoplus_{-\infty<\lambda_m<\alpha} M(\lambda_m), \qquad L_0 = \bigoplus_{\alpha<\lambda_m<\beta} M(\lambda_m), \qquad L_2 = \bigoplus_{\beta<\lambda_m<\infty} M(\lambda_m).$$

Then

$$L^2\bigl(\Omega,R^n\bigr) = L_1 \oplus L_0 \oplus L_2. \tag{2.4}$$

For each $X \in L^2(\Omega,R^n)$, we have the composition

$$X = X_1 + X_0 + X_2,$$

where $X_1 \in L_1$, $X_0 \in L_0$, $X_2 \in L_2$. Let P_1 be the orthogonal projection from $L^2(\Omega,R^n)$ onto L_1, P_0 be that from $L^2(\Omega,R^n)$ onto L_0 and P_2 be that from $L^2(\Omega,R^n)$ onto L_2. Let

$$V = (-\Delta)^{-1}L_0, \qquad W_1 = (-\Delta)^{-1}L_1, \qquad W_2 = (-\Delta)^{-1}L_2.$$

Then $E = V \oplus W_1 \oplus W_2$, and for $U \in E$, U has the decomposition $U = Y + Z_1 + Z_2 \in E$, where

$$Y = (-\Delta)^{-1}X_0 \in V, \qquad Z_1 = (-\Delta)^{-1}X_1 \in W_1, \qquad Z_2 = (-\Delta)^{-1}X_2 \in W_2. \tag{2.5}$$

Let $W = W_1 \oplus W_2$. Then W is the orthogonal complement of V in E. Let $P : E \to V$ be the orthogonal projection of E onto V and $I - P : E \to W$ denote that of E onto W. Then every element $U \in E$ is expressed by $U = Y + Z$, $Y = PU$, $Z = (I - P)U$. Then (1.2) is equivalent to the two systems in the two unknowns Y and Z:

$$-\Delta Y = P\big(\mathrm{grad}_U F(x, Y + Z)\big) \quad \text{in } \Omega, \tag{2.6}$$

$$-\Delta Z = (I - P)\big(\mathrm{grad}_U F(x, Y + Z)\big) \quad \text{in } \Omega, \tag{2.7}$$

$$Y = (0, \dots, 0), \qquad Z = (0, \dots, 0) \quad \text{on } \partial\Omega.$$

Let $Y \in V$ be fixed and consider the function $h : W_1 \times W_2 \to R$ defined by

$$h(Z_1, Z_2) = I(Y + Z_1 + Z_2).$$

The function h has continuous partial Fréchet derivatives D_1h and D_2h with respect to its first and second variables given by

$$D_ih(Z_1, Z_2) \cdot X_i = DI(Y + Z_1 + Z_2) \cdot X_i \tag{2.8}$$

for $X_i \in W_i$, $i = 1, 2$. By Lemma 2.2, I is a functional of class C^1.

By the following Lemma 2.3, we can get critical points of the functional $I(U)$ on the infinite dimensional space E from that of the functional on the finite dimensional subspace V.

Lemma 2.3 (Reduction lemma) *Assume that F satisfies conditions* (F1)-(F4). *Then*

(i) *there exists a unique solution $Z \in W$ of the equation*

$$-\Delta Z = (I - P)\big(\mathrm{grad}_U F(x, Y + Z)\big) \quad \text{in } \Omega,$$

$$Z = (0, \dots, 0) \quad \text{on } \partial\Omega.$$

If we put $Z = \Theta(Y)$, then Θ is continuous on V and satisfies a uniform Lipschitz condition in V with respect to the L^2 norm (*also norm* $\|\cdot\|$). *Moreover,*

$$DI\big(Y + \Theta(Y)\big) \cdot X = 0 \quad \text{for all } X \in W.$$

(ii) *There exists $m_1 < 0$ such that if Z_1 and X_1 are in W_1 and $Z_2 \in W_2$, then*

$$\big(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2)\big)(Z_1 - X_1) \le m_1 \|Z_1 - X_1\|^2.$$

(iii) *There exists $m_2 > 0$ such that if Z_2 and X_2 are in W_2 and $Z_1 \in W_1$, then*

$$\big(D_2h(Z_1, Z_2) - D_2h(Z_1, X_2)\big) \cdot (Z_2 - X_2) \ge m_2 \|Z_2 - X_2\|^2.$$

(iv) *If* $\tilde{I}: V \to R$ *is defined by* $\tilde{I}(Y) = I(Y + \Theta(Y))$, *then* $\tilde{I}$ *has a continuous Fréchet derivative* $D\tilde{I}$ *with respect to* Y, *and*

$$D\tilde{I}(Y) \cdot B = DI\big(Y + \Theta(Y)\big) \cdot B \quad \textit{for all } Y, B \in V. \tag{2.9}$$

(v) $Y_0 \in V$ *is a critical point of* $\tilde{I}$ *if and only if* $Y_0 + \Theta(Y_0)$ *is a critical point of* I.

Proof (i) Let $\delta = \frac{\alpha+\beta}{2}$. Equation (2.7) is equivalent to

$$Z = (-\Delta - \delta)^{-1}(I - P)\big(\operatorname{grad}_U F(x, Y + Z) - \delta(Y + Z)\big). \tag{2.10}$$

System (2.10) can be rewritten as

$$\begin{aligned} &z_i = (-\Delta - \delta)^{-1}(I - P)\big(F_{u_i}(x, y_1 + z_1, \dots, y_n + z_n) - \delta(y_1 + z_1, \dots, y_n + z_n)\big), \\ &\quad i = 1, 2, \dots, n, \\ &y_1 = \cdots = y_n = 0, \qquad z_1 = \cdots = z_n = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.11}$$

where $u_i = y_i + z_i$, $i = 1, 2, \dots, n$, $U = (u_1, \dots, u_n)$, $Y = (y_1, \dots, y_n)$, $Z = (z_1, \dots, z_n)$. The operator $(-\Delta - \delta)^{-1}(I - P)$ is a self-adjoint, compact and linear map from $(I - P)L^2(\Omega, R)$ into itself, and by condition (F3) its L_2 norm is $\min\{\lambda_{h+m+1} - \delta, \delta - \lambda_h\}^{-1}$. Let $U_1, U_2 \in E$. Since

$$\begin{aligned} &\big(F_{u_i}(x, U_2) - \delta U_2\big) - \big(F_{u_i}(x, U_1) - \delta U_1\big) \\ &\quad \le \max\big\{|\alpha - \delta|, |\beta - \delta|\big\} \|U_2 - U_1\|_{R^n} = \frac{\alpha + \beta}{2} \|U_2 - U_1\|_{R^n}, \end{aligned}$$

it follows that the right-hand side of (2.11) defines, for fixed $Y \in V$, a Lipschitz mapping of $(I - P)L^2(\Omega, R)$ into itself with Lipschitz constant $r = (\min\{\lambda_{h+m+1} - \delta, \delta - \lambda_h\})^{-1} \times \frac{\alpha+\beta}{2} < 1$ because $\lambda_{h+m+1} - \delta > \frac{\alpha+\beta}{2}$ and $\delta - \lambda_h > \frac{\alpha+\beta}{2}$. Therefore, by the contraction mapping principle, for each given $Y \in V$, $i = 1, 2, \dots, n$, there exists a unique $z_i \in (I - P)L^2(\Omega, R)$ which satisfies (2.11). Thus, for fixed $Y \in V$, there exists a unique $Z \in (I - P)L^2(\Omega, R^n)$ which satisfies (2.10). If $\Theta(Y)$ denotes the unique $Z \in (I - P)L^2(\Omega, R^n)$ which solves (2.10), then Θ is continuous and satisfies a uniform Lipschitz condition in Y with respect to the L^2 norm (also norm $\|\cdot\|_E$). In fact, if $Z_1 = \Theta(Y_1)$ and $Z_2 = \Theta(Y_2)$, then

$$\begin{aligned} \|Z_1 - Z_2\|_{L^2(\Omega, R^n)} &= \big\|(-\Delta - \delta)^{-1}(I - P)\big(\operatorname{grad}_U F(x, Y_1 + Z_1) - \delta(Y_1 + Z_1)\big) \\ &\quad - \big(\operatorname{grad}_U F(x, Y_2 + Z_2) - \delta(Y_2 + Z_2)\big)\big\|_{L^2(\Omega, R^n)} \\ &\le r\big\|(Y_1 + Z_1) - (Y_2 + Z_2)\big\|_{L^2(\Omega, R^n)} \\ &\le r\big(\|Y_1 - Y_2\|_{L^2(\Omega, R^n)} + \|Z_1 - Z_2\|_{L^2(\Omega, R^n)}\big) \\ &\le r\|Y_1 - Y_2\|_E + r\|Z_1 - Z_2\|_E. \end{aligned}$$

Hence

$$\|Z_1 - Z_2\|_{L^2(\Omega, R^n)} \le C\|Y_1 - Y_2\|_{L^2(\Omega, R^n)}, \qquad C = \frac{r}{1 - r}.$$

Let $U = Y + Z$, $Y \in V$ and $Z = \Theta(Y)$. If $X \in (I-P)L^2(\Omega, R^n) \cap E$,

$$DI\big(Y + \Theta(Y)\big) \cdot X = \int_\Omega \big[-\Delta\big(Y + \Theta(Y)\big) \cdot X - P\big(\big(\mathrm{grad}_U F(x, Y+Z) - \delta(Y+Z)\big) \cdot X\big) \\ - (I-P)\big(\big(\mathrm{grad}_U F(x, Y+Z) - \delta(Y+Z)\big) \cdot X\big)\big]\, dx.$$

It follows from (2.7) that

$$\int_\Omega \big[-\Delta Z(x) \cdot X(x) - \mathrm{grad}_U F\big(x, Y(x) + Z(x)\big) \cdot X(x)\big]\, dx = 0.$$

Since

$$\int_\Omega -\Delta Y(x) \cdot X(x)\, dx = 0,$$

we have

$$DI\big(Y + \Theta(Y)\big) \cdot X = 0. \tag{2.12}$$

(ii) If Z_1 and X_1 are in W_1 and $Z_2 \in W_2$, then

$$\big(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2)\big)(Z_1 - X_1) = \int_\Omega \big[\big|\nabla(Z_1 - X_1)\big|^2 - \big(\mathrm{grad}_U F(x, Y + Z_1 + Z_2) \\ - \mathrm{grad}_U F(x, Y + X_1 + Z_2)\big) \cdot (Z_1 - X_1)\big]\, dx.$$

Since $\int_\Omega |\nabla(Z_1 - X_1)|^2 = \|Z_1 - X_1\|_E^2 \leq \lambda_h \|Z_1 - X_1\|^2_{L^2(\Omega, R^n)}$ and

$$\int_\Omega \big(\mathrm{grad}_U F(x, Y + Z_1 + Z_2) - \mathrm{grad}_U F(x, Y + X_1 + Z_2)\big) \cdot (Z_1 - X_1) \\ \geq \alpha \|Z_1 - X_1\|_{L^2(\Omega, R^n)} \geq \frac{\alpha}{\lambda_h} \|Z_1 - X_1\|_E,$$

$$\big(D_1h(Z_1, Z_2) - D_1h(X_1, Z_2)\big)(Z_1 - X_1) \leq \left(1 - \frac{\alpha}{\lambda_h}\right) \|Z_1 - X_1\|_E^2,$$

where $1 - \frac{\alpha}{\lambda_h} < 0$.

(iii) Similarly, using the fact that $\int_\Omega |\nabla(Z_2 - X_2)|^2\, dx = \|Z_2 - X_2\|_E^2 \geq \lambda_{h+m+1} \|Z_2 - X_2\|^2_{L^2(\Omega, R^n)}$ and

$$\int_\Omega \big(\mathrm{grad}_U F(x, Y + Z_1 + Z_2) - \mathrm{grad}_U F(x, Y + Z_1 + X_2)\big) \cdot (Z_2 - X_2) \\ \leq \beta \|Z_2 - X_2\|_{L^2(\Omega, R^n)} \leq \frac{\beta}{\lambda_{h+m+1}} \|Z_2 - X_2\|_E^2,$$

we see that if Z_2 and X_2 are in W_2 and $Z_1 \in W_1$, then

$$\big(D_2h(Z_1, Z_2) - D_2h(Z_1, X_2)\big)(Z_2 - X_2) \geq \left(1 - \frac{\beta}{\lambda_{h+m+1}}\right) \|Z_2 - X_2\|_E^2,$$

where $(1 - \frac{\beta}{\lambda_{h+m+1}}) > 0$.

(iv) Since the functional I has a continuous Fréchet derivative DI, $\tilde{I}$ has a continuous Fréchet derivative $D\tilde{I}$ with respect to Y.

(v) Suppose that there exists $Y_0 \in V$ such that $D\tilde{I}(Y_0) = 0$. From $D\tilde{I}(Y) \cdot B = DI(Y + \Theta(Y)) \cdot B$ for all $Y, B \in V$, $DI(Y_0 + \Theta(Y_0))(B) = D\tilde{I}(Y_0)(B) = 0$ for all $B \in V$. Since $DI(Y + \Theta(Y)) \cdot B = 0$ for all $B \in W$ and E is the direct sum of V and W, it follows that $DI(Y_0 + \Theta(Y_0)) = 0$. Thus $Y_0 + \Theta(Y_0)$ is a solution of (1.1). Conversely, if U is a solution of (1.1) and $Y = PU$, then $D\tilde{I}(Y) = 0$. □

Remark We note that if $Y \in V$, then $\Theta(Y) = 0$.

3 Proof of Theorem 1.1

Lemma 3.1 ((PS) condition) *Assume that F satisfies conditions* (F1)-(F4). *Then $-\tilde{I}(v)$ is bounded from below and $\tilde{I}(v)$ satisfies the Palais-Smale condition.*

Proof We have

$$\begin{aligned}
\tilde{I}(Y) &= I\bigl(Y + \Theta_1(Y) + \Theta_2(Y)\bigr) \\
&= \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y + \Theta_1(Y) + \Theta_2(Y)\bigr) \cdot \bigl(Y + \Theta_1(Y) + \Theta_2(Y)\bigr)\bigr)\,dx \\
&\quad - \int_\Omega F\bigl(x, Y + \Theta_1(Y) + \Theta_2(Y)\bigr)\,dx \\
&= \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y + \Theta_1(Y)\bigr) \cdot \bigl(Y + \Theta_1(Y)\bigr)\bigr)\,dx \\
&\quad - \int_\Omega F\bigl(x, Y + \Theta_1(Y)\bigr)\,dx + \frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx \\
&\quad - \int_\Omega \bigl[F\bigl(x, Y + \Theta_1(Y) + \Theta_2(Y)\bigr) - F\bigl(x, Y + \Theta_1(Y)\bigr)\bigr]\,dx.
\end{aligned}$$

We claim that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx - \int_\Omega \bigl[F\bigl(x, Y + \Theta_1(Y) + \Theta_2(Y)\bigr) - F\bigl(x, Y + \Theta_1(Y)\bigr)\bigr]\,dx \\
&\quad \le 0.
\end{aligned}$$

In fact, we note that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx \\
&\quad = -\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx + \int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx \\
&\quad = -\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx + \int_\Omega F_U\bigl(x, Y + \Theta_1(Y) + \Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dx
\end{aligned}$$

by (1.1). We also note that

$$\begin{aligned}
&F\bigl(x, Y + \Theta_1(Y) + \Theta_2(Y)\bigr) - F\bigl(x, Y + \Theta_1(Y)\bigr) \\
&\quad = \int_0^1 F_U\bigl(x, Y + \Theta_1(Y) + t\Theta_2(Y)\bigr) \cdot \Theta_2(Y)\,dt.
\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega\bigl(-\Delta\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx-\int_\Omega\bigl[F\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)-F\bigl(x,Y+\Theta_1(Y)\bigr)\bigr]\,dx\\&\quad=-\frac{1}{2}\int_\Omega\bigl(-\Delta\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx+\int_\Omega F_U\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx\\&\qquad-\int_0^1\int_\Omega F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx\,dt.\end{aligned}$$

We note that

$$\begin{aligned}&\frac{d}{dt}\bigl(F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\\&\quad=\bigl(d_U^2F\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\\&\qquad+F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y),\end{aligned}$$

which leads to

$$\begin{aligned}&\int_0^1\frac{d}{dt}\bigl(F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\,dt\\&\quad=\int_0^1\bigl(d_U^2F\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt\\&\qquad+\int_0^1F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt.\end{aligned}$$

That is,

$$\begin{aligned}&F_U\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\\&\quad=\int_0^1\bigl(d_U^2F\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt\\&\qquad+\int_0^1F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt.\end{aligned}$$

Thus we have

$$\begin{aligned}&F_U\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)\cdot\Theta_2(Y)-\int_0^1F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt\\&\quad=\int_0^1\bigl(d_U^2F\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dt.\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{1}{2}\int_\Omega\bigl(-\Delta\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx-\int_\Omega\bigl[F\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)-F\bigl(x,Y+\Theta_1(Y)\bigr)\bigr]\,dx\\&\quad=-\frac{1}{2}\int_\Omega\bigl(-\Delta\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx+\int_\Omega F_U\bigl(x,Y+\Theta_1(Y)+\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx\\&\qquad-\int_0^1\int_\Omega F_U\bigl(x,Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx\,dt\end{aligned}$$

$$= -\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx$$
$$+ \int_0^1\int_\Omega \bigl(d_U^2F\bigl(x, Y+\Theta_1(Y)+t\Theta_2(Y)\bigr)\cdot t\Theta_2(Y)\bigr)\cdot\Theta_2(Y)\,dx\,dt \le 0$$

by condition (F2). Thus by condition (F3) we have

$$\tilde{I}(Y) \le \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\,dx - \int_\Omega F\bigl(x, Y+\Theta_1(Y)\bigr)\,dx$$
$$\le \frac{1}{2}(\lambda_{h+m}-\gamma)\bigl\|Y+\Theta_1(Y)\bigr\|_E^2 + C \to -\infty \quad \text{as } \bigl\|Y+\Theta_1(Y)\bigr\|_E \to \infty. \tag{3.1}$$

Thus $-I(\tilde{Y})$ is bounded from below and satisfies the (PS) condition. □

Lemma 3.2 *Assume that F satisfies conditions* (F1)-(F4). *Then $Y = \theta$, $\theta = (0,\dots,0)$ is neither minimum nor degenerate.*

Proof By (3.1) we have

$$\tilde{I}(Y) \le \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\,dx - \int_\Omega F\bigl(x, Y+\Theta_1(Y)\bigr)\,dx. \tag{3.2}$$

Since

$$F\bigl(x, Y+\Theta_1(Y)\bigr) = \int_0^1 \frac{dF}{dt}\bigl(x, t\bigl(Y+\Theta_1(Y)\bigr)\bigr)\,dt$$
$$= \int_0^1 F_U\bigl(x, t\bigl(Y+\Theta_1(Y)\bigr)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\,dt,$$

we have that

$$\left|\int_\Omega F\bigl(x, Y+\Theta_1(Y)\bigr) - \frac{1}{2}\int_\Omega d_U^2F(x,\theta)\cdot\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\,dx\right|$$
$$= \left|\int_0^1\int_\Omega \bigl[F_U\bigl(x, t\bigl(Y+\Theta_1(Y)\bigr)\bigr) - \bigl(d_U^2F(x,\theta)\cdot t\bigl(Y+\Theta_1(Y)\bigr)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr]\,dx\,dt\right|$$
$$\le \frac{1}{2}\sup_{0<t<1}\bigl\|d_U^2F\bigl(x, t\bigl(Y+\Theta_1(Y)\bigr)\bigr) - d_U^2F(x,\theta)\bigr\|_{\mathcal{L}(V,V)}\bigl\|Y+\Theta_1(Y)\bigr\|_E^2.$$

Thus we have

$$-\int_\Omega F\bigl(x, Y+\Theta_1(Y)\bigr)$$
$$\le -\frac{1}{2}\int_\Omega \bigl(d_U^2F(x,\theta)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr) + o\bigl(\bigl\|Y+\Theta_1(Y)\bigr\|^2\bigr)_E.$$

Since $\Theta_1 \in C^1(V, W_1)$, it follows that if $\|Y\| \to 0$, then $\|\Theta_1(Y)\|_E = O(\|Y\|_E)$ because $\Theta_1(\theta) = \theta$. Thus

$$\bigl\|Y+\Theta_1(Y)\bigr\|_E = O\bigl(\|Y\|_E\bigr).$$

Since $F_U(x,\theta) = \theta$, there exists a bounded self-adjoint operator $A \in \mathcal{L}(E,E)$ which commutes with P_o and P_1 such that

$$\lambda_{h+1} I \leq A \leq d_U^2 F(x,\theta).$$

Thus we have

$$\begin{aligned}
\tilde{I}(Y) &\leq \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\,dx \\
&\quad - \frac{1}{2}\int_\Omega \bigl(d_U^2 F(x,\theta)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\,dx + o\bigl(\|Y\|_E^2\bigr) \\
&\leq \frac{1}{2}\int_\Omega \bigl(-\Delta\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr)\bigr)\,dx \\
&\quad - \frac{1}{2}\int_\Omega A\bigl(Y+\Theta_1(Y)\bigr)\cdot\bigl(Y+\Theta_1(Y)\bigr) + o\bigl(\|Y\|_E^2\bigr) \\
&= \frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_1(Y)\cdot\Theta_1(Y)\bigr)\,dx - \frac{1}{2}\int_\Omega A\bigl(\Theta_1(Y)\bigr)\cdot\Theta_1(Y) + \frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)\,dx \\
&\quad - \frac{1}{2}\int_\Omega A(Y)\cdot Y + o\bigl(\|Y\|_E^2\bigr)
\end{aligned}$$

as $\|Y\|_E \to 0$. Since $\lambda_{h+1} I \leq A$, it follows that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_1(Y)\cdot\Theta_1(Y)\bigr)\,dx - \frac{1}{2}\int_\Omega A\bigl(\Theta_1(Y)\bigr)\cdot\Theta_1(Y) \\
&\quad \leq \frac{1}{2}\int_\Omega \bigl(-\Delta\Theta_1(Y)\cdot\Theta_1(Y)\bigr)\,dx - \frac{1}{2}\int_\Omega \lambda_{h+1}\Theta_1(Y)\cdot\Theta_1(Y) \leq 0.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\tilde{I}(Y) &\leq \frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)\,dx - \frac{1}{2}\int_\Omega A(Y)\cdot Y + o\bigl(\|Y\|_E^2\bigr) \\
&\leq \frac{1}{2}\int_\Omega \bigl[(-\Delta Y)\cdot Y - \lambda_{h+1} Y^2\bigr]\,dx + o\bigl(\|Y\|_E^2\bigr)
\end{aligned}$$

as $\|Y\|_E \to 0$. Similarly we can choose a bounded self-adjoint operator $B \in \mathcal{L}(E,E)$ which commutes with P_o and P_2 such that

$$d_U^2 F(x,\theta) \leq B \leq \lambda_{h+m+1} I.$$

This leads to

$$\begin{aligned}
\tilde{I}(Y) &\geq \frac{1}{2}\int_\Omega (-\Delta Y\cdot Y)\,dx - \frac{1}{2}\int_\Omega B(Y)\cdot Y + o\bigl(\|Y\|_E^2\bigr) \\
&\geq \frac{1}{2}\int_\Omega \bigl[(-\Delta Y)\cdot Y - \lambda_{h+m+1} Y^2\bigr]\,dx + o\bigl(\|Y\|_E^2\bigr)
\end{aligned}$$

as $\|Y\|_E \to 0$. Thus $Y = \theta$, $\theta = (0,\ldots,0)$ is neither minimum nor degenerate. □

Lemma 3.3 *Assume that F satisfies conditions* (F1)-(F4). *Then*

$$\tilde{I}(Y) \to -\infty \quad \textit{as } \|Y\|_E \to \infty.$$

Proof The proof can be found in the proof of Lemma 3.1. □

Proof of Theorem 1.1 By Lemma 2.2, $\tilde{I}(Y)$ is continuous and Fréchet differentiable in V. By Lemma 3.1, $-\tilde{I}(Y)$ satisfies the (PS) condition. By Lemma 3.2, $Y = \theta$ is neither minimum nor degenerate. By Lemma 3.3, $\tilde{I}(Y) \to -\infty$ as $\|Y\|_E \to \infty$. We note that $\max_{Y\in V} \tilde{I}(Y) > 0$ is another critical value of $\tilde{I}$. Thus there exists the third critical point of $\tilde{I}(Y)$. Thus (1.1) has at least three nontrivial solutions. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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