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On a generalized Hölder inequality

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Abstract

Refining the Hölder inequality, a result of S Wu is extended to the case of multiple sequences.

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1 Introduction

There are a lot of generalizations and extensions of the classical Hölder inequality. Among them is the existence of a function

$$g(a, b)$$

for which

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} g(a, b) \leq \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}}.$$

Here $a = \{a_i\}$, $b = \{b_i\}$ are positive sequences and $1 \leq p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Hu and Wu published several results [1–3] on this topic. Among them is the following theorem.

Theorem A ([3], Theorem 1) *Let $a_i > 0$, $b_i > 0$ ($i = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), and let $p \geq q > 0$. Then*

$$\sum_{i=1}^n a_i b_i \leq n^{1-\min\{\frac{1}{p}+\frac{1}{q}, 1\}} \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \left[1 - \left\{ \frac{\sum_{i=1}^n a_i^p e_i}{\sum_{i=1}^n a_i^p} - \frac{\sum_{i=1}^n b_i^q e_i}{\sum_{i=1}^n b_i^q} \right\}^2 \right]^{\frac{1}{2p}}.$$

Our purpose of this paper is to extend Theorem A to the case of multiple sequences. The following are the main results of this paper.

Theorem 1.1 *Let $a_{ji} > 0$, $p_j > 0$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $\Lambda := \sum_{j=1}^m \frac{1}{p_j}$ and $\{p_{j1}, p_{j2}\} \subset \{p_1, p_2, \dots, p_m\}$. Then*

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \leq n^{1-\min\{\Lambda, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2 \max\{p_{j_1}, p_{j_2}\}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}.$$

Theorem 1.2 Let $a_{ji} > 0$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $p_j < 0$ ($j = 1, 2, \dots, m-1$), $0 < \sum_{j=1}^m \frac{1}{p_j} \leq 1$ and $\{p_{j_1}, p_{j_2}\} \subset \{p_1, p_2, \dots, p_m\}$. Then

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \geq \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2 \min\{p_{j_1}, p_{j_2}\}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}.$$

If $m = 2$, then Theorem 1.1 reduces to Theorem A. The case $\Lambda = 1$ and $\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} = \frac{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}}}$ of Theorem 1.1 is the well-known Hölder inequality.

Theorem 1.2 is a reversed version of Theorem 1.1. The case $m = 2$ with $\Lambda = 1$ and $\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} = \frac{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}}}$ of Theorem 1.2 reduces to the well-known reversed Hölder inequality.

The idea of proofs of Theorem 1.1 and Theorem 1.2 are essentially the same as that of Theorem A. The point is that a variant of the methods of Wu in [3] works for the case of multiple sequences. The process will be done in Section 3 and Section 4 after the preparing lemmas in Section 2. In Section 5, as an application of Theorem 1.1, we refine a reversed Hölder inequality different from Theorem 1.2. We refer to [4, 5] for the general theory of inequalities.

2 Lemmas

In order to prove Theorem 1.1 and Theorem 1.2, we need the following lemmas.

Lemma 2.1 ([2], Lemma 1(1)) Let $a_{ji} > 0$, $\lambda_j > 0$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$) and $\Lambda := \sum_{j=1}^m \lambda_j$. Then

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji}^{\lambda_j} \leq n^{1-\min\{\Lambda, 1\}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji} \right)^{\lambda_j}$$

with equality holding if and only if $a_{j1} = a_{j2} = \dots = a_{jn}$ ($j = 1, 2, \dots, m$) for $\Lambda < 1$, or $\frac{a_{1i}}{\sum_{i=1}^n a_{1i}} = \frac{a_{2i}}{\sum_{i=1}^n a_{2i}} = \dots = \frac{a_{mi}}{\sum_{i=1}^n a_{mi}}$ ($i = 1, 2, \dots, n$) for $\Lambda = 1$.

Lemma 2.2 ([2], Lemma 1(2)) Let $a_{ji} > 0$ ($i = 1, 2, \dots, n; j = 1, 2, \dots, m$), $\lambda_j < 0$ ($j = 1, 2, \dots, m-1$), λ_m be a real number, and $\Lambda := \sum_{j=1}^m \lambda_j$. Then

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji}^{\lambda_j} \geq n^{1-\max\{\Lambda, 1\}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji} \right)^{\lambda_j}$$

with equality holding if and only if $a_{j1} = a_{j2} = \dots = a_{jn}$ ($j = 1, 2, \dots, m$) for $\Lambda > 1$, or $\frac{a_{1i}}{\sum_{i=1}^n a_{1i}} = \frac{a_{2i}}{\sum_{i=1}^n a_{2i}} = \dots = \frac{a_{mi}}{\sum_{i=1}^n a_{mi}}$ ($i = 1, 2, \dots, n$) for $\Lambda = 1$.

3 Proof of Theorem 1.1

We may suppose $p_{j_1} \leq p_{j_2}$.

Case (1): $\Lambda > 1$. We show

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \leq \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{i=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2p_{j_2}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \quad (3.1)$$

It is simple to see that

$$\begin{aligned}
 \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^2 &= \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \\
 &= \left\{ \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{1-\Lambda} \left\{ \sum_{i=1}^n \sum_{k=1}^n (1-e_i+e_k) \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{\Lambda} \\
 &\leq \sum_{i=1}^n \sum_{k=1}^n \left[\left\{ \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{1-\Lambda} \left\{ (1-e_i+e_k) \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{\Lambda} \right] \\
 &= \sum_{i=1}^n \sum_{k=1}^n \left\{ (1-e_i+e_k)^{\Lambda} \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\} \\
 &= \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m \left\{ a_{lk} (1-e_i+e_k)^{\frac{1}{p_l}} \right\} \prod_{j=1}^m a_{ji} \\
 &\leq \sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \left\{ a_{lk}^{p_l} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_l}} \prod_{j=1}^m a_{ji}
 \end{aligned}$$

by applying Lemma 2.2 and Lemma 2.1. On the other hand, it is straightforward to see that

$$\begin{aligned}
 &\left[\sum_{k=1}^n \left\{ a_{j_1 k}^{p_{j_1}} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_{j_1}}} \left[\sum_{k=1}^n \left\{ a_{j_2 k}^{p_{j_2}} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_{j_2}}} a_{j_1 i} a_{j_2 i} \\
 &= \left[\sum_{k=1}^n \left\{ a_{j_1 k}^{p_{j_1}} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left[\sum_{k=1}^n \left\{ a_{j_1 k}^{p_{j_1}} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_{j_2}}} \\
 &\quad \times \left[\sum_{k=1}^n \left\{ a_{j_2 k}^{p_{j_2}} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_{j_2}}} a_{j_1 i} a_{j_2 i} \\
 &= \left(\sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}}
 \end{aligned}$$

and

$$\left[\sum_{k=1}^n \left\{ a_{j k}^{p_j} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_j}} a_{ji} = \left(\sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}},$$

where $A_{jl} = A_{jl}(i, k) = a_{jk}^{p_j} a_{li}^{p_l} (1-e_i+e_k)$ ($j, l = 1, 2, 3, \dots, m$). Hence

$$\begin{aligned}
 &\sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \left\{ a_{lk}^{p_l} (1-e_i+e_k) \right\} \right]^{\frac{1}{p_l}} \prod_{j=1}^m a_{ji} \\
 &= \sum_{i=1}^n \left\{ \left(\sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}} \right\} \\
 &\leq \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}}
 \end{aligned}$$

by applying Lemma 2.1 again. On the other hand, it is also straightforward to see that

$$\begin{aligned}
 & \sum_{i=1}^n \sum_{k=1}^n A_{j_2 j_1} \sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_2} \\
 &= \left(\sum_{i=1}^n \sum_{k=1}^n a_{j_2 i}^{p_{j_2}} a_{j_1 k}^{p_{j_1}} - \sum_{i=1}^n \sum_{k=1}^n a_{j_2 i}^{p_{j_2}} a_{j_1 k}^{p_{j_1}} e_i + \sum_{i=1}^n \sum_{k=1}^n a_{j_2 i}^{p_{j_2}} a_{j_1 k}^{p_{j_1}} e_k \right) \\
 & \quad \times \left(\sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} - \sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} e_i + \sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} e_k \right) \\
 &= \left(\sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} \right)^2 - \left(\sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} e_i - \sum_{i=1}^n \sum_{k=1}^n a_{j_1 i}^{p_{j_1}} a_{j_2 k}^{p_{j_2}} e_k \right)^2 \\
 &= \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^2 \left(\sum_{k=1}^n a_{j_2 k}^{p_{j_2}} \right)^2 \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 k}^{p_{j_2}} e_k}{\sum_{k=1}^n a_{j_2 k}^{p_{j_2}}} \right)^2 \right\}
 \end{aligned}$$

and, for $j = 1, 2, 3, \dots, m$, that

$$\sum_{i=1}^n \sum_{k=1}^n A_{jj} = \sum_{i=1}^n \sum_{k=1}^n a_{ji}^{p_j} a_{jk}^{p_j} - \sum_{i=1}^n \sum_{k=1}^n a_{ji}^{p_j} a_{jk}^{p_j} e_i + \sum_{i=1}^n \sum_{k=1}^n a_{ji}^{p_j} a_{jk}^{p_j} e_k = \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^2.$$

Hence,

$$\begin{aligned}
 & \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}} \\
 &= \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^{\frac{2}{p_{j_1}} - \frac{2}{p_{j_2}}} \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^{\frac{2}{p_{j_2}}} \left(\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} \right)^{\frac{2}{p_{j_2}}} \\
 & \quad \times \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{p_{j_2}}} \\
 &= \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^{\frac{2}{p_{j_1}}} \left(\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} \right)^{\frac{2}{p_{j_2}}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{p_{j_2}}}
 \end{aligned}$$

and

$$\prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}} = \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{2}{p_j}}.$$

Therefore (3.1) follows.

Case (2): $0 < \Lambda \leq 1$. We show

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \leq n^{1-\Lambda} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2p_{j_2}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \quad (3.2)$$

Since $\sum_{j=1}^m \frac{1}{p_j \Lambda} = 1$, we have

$$\begin{aligned} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^2 &= \sum_{i=1}^n \sum_{k=1}^n \left\{ (1 - e_i + e_k)^{\sum_{l=1}^m \frac{1}{p_l \Lambda}} \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\} \\ &= \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m \left\{ a_{lk} (1 - e_i + e_k)^{\frac{1}{p_l \Lambda}} \right\} \prod_{j=1}^m a_{ji} \\ &\leq \sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \left\{ a_{lk}^{p_l \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_l \Lambda}} \prod_{j=1}^m a_{ji} \end{aligned}$$

by applying Lemma 2.1. On the other hand, it is straightforward to see that

$$\begin{aligned} &\left[\sum_{k=1}^n \left\{ a_{j_1 k}^{p_{j_1} \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_{j_1} \Lambda}} \left[\sum_{k=1}^n \left\{ a_{j_2 k}^{p_{j_2} \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_{j_2} \Lambda}} a_{j_1 i} a_{j_2 i} \\ &= \left(\sum_{k=1}^n B_{j_1 j_1} \right)^{\frac{1}{p_{j_1} \Lambda} - \frac{1}{p_{j_2} \Lambda}} \left(\sum_{k=1}^n B_{j_2 j_1} \right)^{\frac{1}{p_{j_2} \Lambda}} \left(\sum_{k=1}^n B_{j_1 j_2} \right)^{\frac{1}{p_{j_2} \Lambda}} \end{aligned}$$

and

$$\left[\sum_{k=1}^n \left\{ a_{jk}^{p_j \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_j \Lambda}} a_{ji} = \left(\sum_{k=1}^n B_{jj} \right)^{\frac{1}{p_j \Lambda}},$$

where $B_{jl} = B_{jl}(i, k) = a_{jk}^{p_j \Lambda} a_{li}^{p_l \Lambda} (1 - e_i + e_k)$ ($j, l = 1, 2, 3, \dots, m$). Hence

$$\begin{aligned} &\sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \left\{ a_{lk}^{p_l \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_l \Lambda}} \prod_{j=1}^m a_{ji} \\ &= \sum_{i=1}^n \left\{ \left(\sum_{k=1}^n B_{j_1 j_1} \right)^{\frac{1}{p_{j_1} \Lambda} - \frac{1}{p_{j_2} \Lambda}} \left(\sum_{k=1}^n B_{j_2 j_1} \right)^{\frac{1}{p_{j_2} \Lambda}} \left(\sum_{k=1}^n B_{j_1 j_2} \right)^{\frac{1}{p_{j_2} \Lambda}} \prod_{j \neq j_1, j_2} \left(\sum_{k=1}^n B_{jj} \right)^{\frac{1}{p_j \Lambda}} \right\} \\ &\leq \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_1 j_1} \right)^{\frac{1}{p_{j_1} \Lambda} - \frac{1}{p_{j_2} \Lambda}} \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_2 j_1} \right)^{\frac{1}{p_{j_2} \Lambda}} \\ &\quad \times \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_1 j_2} \right)^{\frac{1}{p_{j_2} \Lambda}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n B_{jj} \right)^{\frac{1}{p_j \Lambda}} \end{aligned}$$

by applying Lemma 2.1. Applying Lemma 2.1 again, we have

$$\begin{aligned} \left(\sum_{i=1}^n \sum_{k=1}^n B_{ts} \right)^{\frac{1}{p_h \Lambda}} &= \left[\sum_{i=1}^n \sum_{k=1}^n \left\{ a_{ti}^{p_t \Lambda} a_{sk}^{p_s \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{1}{p_h \Lambda}} \\ &\leq \left[\sum_{i=1}^n \sum_{k=1}^n \left\{ a_{ti}^{p_t \Lambda} a_{sk}^{p_s \Lambda} (1 - e_i + e_k) \right\} \right]^{\frac{\Lambda}{p_h \Lambda}} \left\{ \sum_{i=1}^n \sum_{k=1}^n (1 - e_i + e_k) \right\}^{\frac{1-\Lambda}{p_h \Lambda}} \\ &= \left(\sum_{i=1}^n \sum_{k=1}^n A_{ts} \right)^{\frac{1}{p_h}} \left\{ \sum_{i=1}^n \sum_{k=1}^n (1 - e_i + e_k) \right\}^{\frac{1-\Lambda}{p_h \Lambda}} \end{aligned}$$

and

$$\left(\sum_{i=1}^n \sum_{k=1}^n B_{ti} \right)^{\frac{1}{p_t \Lambda} - \frac{1}{p_s \Lambda}} \leq \left(\sum_{i=1}^n \sum_{k=1}^n A_{ti} \right)^{\frac{1}{p_t} - \frac{1}{p_s}} \left\{ \sum_{i=1}^n \sum_{k=1}^n (1 - e_i + e_k) \right\}^{\frac{1-\Lambda}{p_t \Lambda} - \frac{1-\Lambda}{p_s \Lambda}}.$$

Hence

$$\begin{aligned} & \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_1 i_1} \right)^{\frac{1}{p_{j_1} \Lambda} - \frac{1}{p_{j_2} \Lambda}} \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_2 i_1} \right)^{\frac{1}{p_{j_2} \Lambda}} \\ & \quad \times \left(\sum_{i=1}^n \sum_{k=1}^n B_{j_1 i_2} \right)^{\frac{1}{p_{j_2} \Lambda}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n B_{jj} \right)^{\frac{1}{p_j \Lambda}} \\ & \leq n^{2(1-\Lambda)} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 i_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 i_1} \right)^{\frac{1}{p_{j_2}}} \\ & \quad \times \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 i_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}}. \end{aligned}$$

By the same process as in the proof of Case (1),

$$\begin{aligned} & \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 i_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 i_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 i_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}} \\ & = \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2p_{j_2}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{2}{p_j}}. \end{aligned}$$

Therefore (3.2) follows. The proof is complete.

4 Proof of Theorem 1.2

We may suppose $p_{j_2} \leq p_{j_1}$ and let $\sum_{j=1}^m \frac{1}{p_j} = \Lambda$. We show

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \geq \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{2p_{j_2}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \quad (4.1)$$

It is simple to see that

$$\begin{aligned} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^2 &= \left\{ \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{1-\Lambda} \left\{ \sum_{i=1}^n \sum_{k=1}^n (1 - e_i + e_k) \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{\Lambda} \\ &\geq \sum_{i=1}^n \sum_{k=1}^n \left[\left\{ \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{1-\Lambda} \left\{ (1 - e_i + e_k) \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\}^{\Lambda} \right] \\ &= \sum_{i=1}^n \sum_{k=1}^n \left\{ (1 - e_i + e_k)^{\Lambda} \prod_{l=1}^m a_{lk} \prod_{j=1}^m a_{ji} \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \sum_{k=1}^n \prod_{l=1}^m \{a_{lk}(1-e_i+e_k)^{\frac{1}{p_l}}\} \prod_{j=1}^m a_{ji} \\
&\geq \sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \{a_{lk}^{p_l}(1-e_i+e_k)\} \right]^{\frac{1}{p_l}} \prod_{j=1}^m a_{ji}
\end{aligned}$$

by applying Lemma 2.1 and Lemma 2.2. On the other hand, it is straightforward to see that

$$\begin{aligned}
&\left[\sum_{k=1}^n \{a_{j_1 k}^{p_{j_1}}(1-e_i+e_k)\} \right]^{\frac{1}{p_{j_1}}} \left[\sum_{k=1}^n \{a_{j_2 k}^{p_{j_2}}(1-e_i+e_k)\} \right]^{\frac{1}{p_{j_2}}} a_{j_1 i} a_{j_2 i} \\
&= \left(\sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}}
\end{aligned}$$

and

$$\left[\sum_{k=1}^n \{a_{j k}^{p_j}(1-e_i+e_k)\} \right]^{\frac{1}{p_j}} a_{ji} = \left(\sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}},$$

where $A_{jl} = A_{jl}(i, k) = a_{jk}^{p_j} a_{li}^{p_l} (1-e_i+e_k)$ ($j, l = 1, 2, 3, \dots, m$). Since only one of the numbers $\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}, \frac{1}{p_{j_2}}, \frac{1}{p_j}$ ($j \neq j_1, j_2$) is positive, by applying Lemma 2.2 again,

$$\begin{aligned}
&\sum_{i=1}^n \prod_{l=1}^m \left[\sum_{k=1}^n \{a_{lk}^{p_l}(1-e_i+e_k)\} \right]^{\frac{1}{p_l}} \prod_{j=1}^m a_{ji} \\
&= \sum_{i=1}^n \left\{ \left(\sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \left(\sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}} \right\} \\
&\geq \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 j_1} \right)^{\frac{1}{p_{j_2}}} \\
&\quad \times \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{jj} \right)^{\frac{1}{p_j}}.
\end{aligned}$$

On the other hand, it is also straightforward to see that

$$\begin{aligned}
&\sum_{i=1}^n \sum_{k=1}^n A_{j_2 j_1} \sum_{i=1}^n \sum_{k=1}^n A_{j_1 j_2} \\
&= \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^2 \left(\sum_{k=1}^n a_{j_2 k}^{p_{j_2}} \right)^2 \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 k}^{p_{j_2}} e_k}{\sum_{k=1}^n a_{j_2 k}^{p_{j_2}}} \right)^2 \right\}
\end{aligned}$$

and, for $j = 1, 2, 3, \dots, m$, that

$$\sum_{i=1}^n \sum_{k=1}^n A_{jj} = \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^2.$$

Hence,

$$\begin{aligned} & \left(\sum_{i=1}^n \sum_{k=1}^n A_{h_1 k_1} \right)^{\frac{1}{p_{j_1}} - \frac{1}{p_{j_2}}} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j_2 k_1} \right)^{\frac{1}{p_{j_2}}} \\ & \quad \times \left(\sum_{i=1}^n \sum_{k=1}^n A_{h_1 j_2} \right)^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n \sum_{k=1}^n A_{j j} \right)^{\frac{1}{p_j}} \\ & = \left(\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} \right)^{\frac{2}{p_{j_1}}} \left(\sum_{i=1}^n a_{j_2 i}^{p_{j_2}} \right)^{\frac{2}{p_{j_2}}} \\ & \quad \times \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}} e_i}{\sum_{k=1}^n a_{j_2 i}^{p_{j_2}}} \right)^2 \right\}^{\frac{1}{p_{j_2}}} \prod_{j \neq j_1, j_2} \left(\sum_{i=1}^n a_{j i}^{p_j} \right)^{\frac{2}{p_j}}. \end{aligned}$$

Therefore (4.1) follows. The proof is complete.

5 Further observations

As a generalization of the reversed Hölder inequality, JF Tian obtained the following.

Theorem B ([6], Theorem 2.2) *Let $a_i > 0$, $b_i > 0$ ($i = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $q < 0$, and $\frac{1}{p} + \frac{1}{q} \geq 0$. Then*

$$\begin{aligned} \sum_{i=1}^n a_i b_i & \geq n^{1 - \max\{\frac{1}{p} + \frac{1}{q}, 1\}} \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^q \right)^{\frac{1}{q}} \\ & \quad \times \left\{ 1 - \left(\frac{\sum_{i=1}^n b_i^q e_i}{\sum_{i=1}^n b_i^q} - \frac{\sum_{i=1}^n a_i b_i e_i}{\sum_{i=1}^n a_i b_i} \right)^2 \right\}^{\frac{\max\{-1, \frac{1}{q}\}}{2}}. \end{aligned}$$

We extended Theorem B to multiple sequences as follows. The idea of the proof is essentially the same as that of Theorem B and is different from that of Theorem 1.2. Note that there is no implication between Theorem 1.2 and Theorem 5.1.

Theorem 5.1 *Let $a_{ji} > 0$ ($i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $p_j < 0$ ($j = 1, 2, \dots, m-1$), $\Lambda := \sum_{j=1}^m \frac{1}{p_j} > 0$, and $\{p_{j_1}\} \subset \{p_1, p_2, \dots, p_{m-1}\}$. Then*

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} & \geq n^{1 - \max\{\Lambda, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j_1}\}}} \\ & \quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

Proof Case (1): $-1 \leq p_{j_1} < 0$. Since $\sum_{j=1}^m \frac{1}{p_j} > 0$, we have $p_m > 0$. Let $q_j = -\frac{p_j}{p_m} > 0$ ($j = 1, 2, \dots, m-1$), and $q_m = \frac{1}{p_m} > 0$, then $q_m \geq q_{j_1} > 0$.

When $0 < \sum_{j=1}^m \frac{1}{q_j} < 1$, we obtain

$$\begin{aligned} & \sum_{i=1}^n \left(a_{1i}^{-p_m} \times \cdots \times a_{m-1i}^{-p_m} \times \prod_{j=1}^m a_{ji}^{p_m} \right) \\ & \leq n^{1-\min\{\sum_{j=1}^m \frac{1}{q_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2\max\{q_m, q_{j_1}\}}} \\ & \quad \times \left\{ \prod_{j=1}^{m-1} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{q_j}} \right\} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^{\frac{1}{q_m}} \end{aligned}$$

by using Theorem 1.1. That is,

$$\begin{aligned} \sum_{i=1}^n a_{mi}^{p_m} & \leq n^{1-\sum_{j=1}^m \frac{1}{q_j}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{p_m}{2}} \\ & \quad \times \left\{ \prod_{j=1}^{m-1} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{-\frac{p_m}{p_j}} \right\} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^{p_m}. \end{aligned}$$

Therefore

$$\begin{aligned} \left(\sum_{i=1}^n a_{mi}^{p_m} \right)^{\frac{1}{p_m}} & \leq n^{\sum_{j=1}^m \frac{1}{p_j} - 1} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2}} \\ & \quad \times \left\{ \prod_{j=1}^{m-1} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{-\frac{1}{p_j}} \right\} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} & \geq n^{1-\sum_{j=1}^m \frac{1}{p_j}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{-\frac{1}{2}} \\ & \quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

From $0 < \sum_{j=1}^m \frac{1}{q_j} < 1$, we have $\sum_{j=1}^m \frac{1}{p_j} > 1$. Consequently

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} & \geq n^{1-\max\{\sum_{j=1}^m \frac{1}{p_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}} e_i}{\sum_{i=1}^n a_{j_1 i}^{p_{j_1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2\min\{-1, p_{j_1}\}}} \\ & \quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

When $\sum_{j=1}^m \frac{1}{q_j} \geq 1$, then $\sum_{j=1}^m \frac{1}{p_j} \leq 1$, so that similarly we obtain

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} &\geq \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j1}\}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}} \\ &= n^{1 - \max\{\sum_{j=1}^m \frac{1}{p_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j1}\}}} \\ &\quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

Case (2): $p_{j1} < -1$. Since $\sum_{j=1}^m \frac{1}{p_j} > 0$, we have $p_m > 0$. Let $q_j = -\frac{p_j}{p_m} > 0$ ($j = 1, 2, \dots, m-1$), and $q_m = \frac{1}{p_m} > 0$, then $q_{j1} > q_m > 0$.

When $0 < \sum_{j=1}^m \frac{1}{q_j} < 1$, we obtain

$$\begin{aligned} \sum_{i=1}^n \left(a_{1i}^{-p_m} \times a_{2i}^{-p_m} \times \cdots \times a_{m-1i}^{-p_m} \times \prod_{j=1}^m a_{ji}^{p_m} \right) \\ \leq n^{1 - \min\{\sum_{j=1}^m \frac{1}{q_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \max\{q_m, q_{j1}\}}} \\ \times \left\{ \prod_{j=1}^{m-1} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{q_j}} \right\} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^{\frac{1}{q_m}} \end{aligned}$$

by using Theorem 1.1 again. That is,

$$\begin{aligned} \sum_{i=1}^n a_{mi}^{p_m} &\leq n^{1 - \sum_{j=1}^m \frac{1}{q_j}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{-\frac{p_m}{2p_{j1}}} \\ &\quad \times \left\{ \prod_{j=1}^{m-1} \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{-\frac{p_m}{p_j}} \right\} \left(\sum_{i=1}^n \prod_{j=1}^m a_{ji} \right)^{p_m}. \end{aligned}$$

Hence

$$\sum_{i=1}^n \prod_{j=1}^m a_{ji} \geq n^{1 - \sum_{j=1}^m \frac{1}{p_j}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2p_{j1}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}.$$

From $0 < \sum_{j=1}^m \frac{1}{q_j} < 1$, we have $\sum_{j=1}^m \frac{1}{p_j} > 1$. Consequently

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} &\geq n^{1 - \max\{\sum_{j=1}^m \frac{1}{p_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j1}\}}} \\ &\quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

When $\sum_{j=1}^m \frac{1}{q_j} \geq 1$, then $\sum_{j=1}^m \frac{1}{p_j} \leq 1$, so that similarly we obtain

$$\begin{aligned} \sum_{i=1}^n \prod_{j=1}^m a_{ji} &\geq \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j1}\}}} \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}} \\ &= n^{1 - \max\{\sum_{j=1}^m \frac{1}{p_j}, 1\}} \left\{ 1 - \left(\frac{\sum_{i=1}^n a_{ji}^{p_{j1}} e_i}{\sum_{i=1}^n a_{ji}^{p_{j1}}} - \frac{\sum_{i=1}^n \prod_{j=1}^m a_{ji} e_i}{\sum_{i=1}^n \prod_{j=1}^m a_{ji}} \right)^2 \right\}^{\frac{1}{2 \min\{-1, p_{j1}\}}} \\ &\quad \times \prod_{j=1}^m \left(\sum_{i=1}^n a_{ji}^{p_j} \right)^{\frac{1}{p_j}}. \end{aligned}$$

The proof is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

EGK has been involved in drafting the manuscript and JEB helped to draft the manuscript. All authors read and approved the final manuscript.

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