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Some inequalities for the minimum eigenvalue of the Hadamard product of an M-matrix and an inverse M-matrix

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Abstract

Several convergent sequences of the lower bounds of the minimum eigenvalue for the Hadamard product of an M-matrix and an inverse M-matrix are given. Numerical examples show that these sequences could reach the true value of the minimum eigenvalue in some cases. These bounds in this paper improve some existing results.

MSC: 15A06; 15A15; 15A48

Keywords: sequences; M-matrix; Hadamard product; minimum eigenvalue; lower bounds

1 Introduction

For a positive integer n, N denotes the set $\{1, 2, \ldots, n\}$, and $\mathbb{R}^{n\times n}(\mathbb{C}^{n\times n})$ denotes the set of all $n \times n$ real (complex) matrices throughout.

It is well known that a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is called a nonsingular M-matrix if $a_{ij} \le 0$, $i, j \in N$, $i \ne j$, A is nonsingular and $A^{-1} \ge 0$ (see [1, 2]). Denote by $\mathcal{M}_n$ the set of all $n \times n$ nonsingular M-matrices.

If A is a nonsingular M-matrix, then there exists a positive eigenvalue of A equal to $\tau(A) \equiv [\rho(A^{-1})]^{-1}$, where $\rho(A^{-1})$ is the Perron eigenvalue of the nonnegative matrix A^{-1}. It is easy to prove that $\tau(A) = \min\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ denotes the spectrum of A (see [3]).

A matrix A is called reducible if there exists a nonempty proper subset $I \subset N$ such that $a_{ij} = 0$, $\forall i \in I$, $\forall j \notin I$. If A is not reducible, then we call A irreducible (see [4]).

For two real matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size, the Hadamard product of A and B is defined as the matrix $A \circ B = [a_{ij}b_{ij}]$. If A and B are two nonsingular M-matrices, then it was proved in [3] that $A \circ B^{-1}$ is also a nonsingular M-matrix.

Let $A = [a_{ij}]$ be an $n \times n$ matrix with all diagonal entries being nonzero throughout. For $i, j, k \in N$, $j \ne i$, denote

$$R_i = \sum_{j\ne i} |a_{ij}|, \qquad d_i = \frac{R_i}{|a_{ii}|}; \qquad s_{ji} = \frac{|a_{ji}| + \sum_{k\ne j,i} |a_{jk}| d_k}{|a_{jj}|}, \qquad s_i = \max_{j\ne i}\{s_{ij}\};$$

$$r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\ne j,i} |a_{jk}|}, \qquad r_i = \max_{j\ne i}\{r_{ji}\};$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|}, \qquad m_i = \max_{j\neq i}\{m_{ij}\};$$
$$u_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|m_{ki}}{|a_{jj}|}, \qquad u_i = \max_{j\neq i}\{u_{ij}\}.$$

In 2013, Zhou *et al.* [5] gave the following result: If $A = [a_{ij}] \in \mathcal{M}_n$ is a strictly row diagonally dominant matrix, $B = [b_{ij}] \in \mathcal{M}_n$ and $A^{-1} = [\alpha_{ij}]$, then

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i\in N}\left\{\frac{b_{ii} - m_i \sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}. \tag{1}$$

In 2013, Cheng *et al.* [6] obtained the following result: If $A = [a_{ij}] \in \mathcal{M}_n$ and $A^{-1} = [\alpha_{ij}]$ is a doubly stochastic matrix, then

$$\tau\left(A \circ A^{-1}\right) \geq \min_{i\in N}\left\{\frac{a_{ii} - u_i R_i}{1 + \sum_{j\neq i} u_{ji}}\right\}. \tag{2}$$

In this paper, we present several convergent sequences of the lower bounds of $\tau(B\circ A^{-1})$ and $\tau(A\circ A^{-1})$, which improve (1) and (2). Numerical examples show that these sequences could reach the true value of $\tau(A \circ A^{-1})$ in some cases.

2 Some lemmas and notations

In this section, we first give the following notations; these will be useful in the following proofs.

Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$. For $i,j,k \in N$, $j \neq i$, $t = 1,2,\ldots$, denote

$$q_{ji} = \min\{s_{ji}, m_{ji}\}, \qquad h_i = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum_{k\neq j,i}|a_{jk}|q_{ki}}\right\},$$
$$v_{ji}^{(0)} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|q_{ki}h_i}{|a_{jj}|}, \qquad p_{ji}^{(t)} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|v_{ki}^{(t-1)}}{|a_{jj}|},$$
$$p_i^{(t)} = \max_{j\neq i}\{p_{ij}^{(t)}\}, \qquad h_i^{(t)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(t)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}}\right\},$$
$$v_{ji}^{(t)} = \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(t)}h_i^{(t)}}{|a_{jj}|}.$$

Lemma 1 *If $A = [a_{ij}] \in \mathcal{M}_n$ is strictly row diagonally dominant, then, for all $i,j \in N$, $j \neq i$, $t = 1,2,\ldots$,*

(a) $1 > q_{ji} \geq v_{ji}^{(0)} \geq p_{ji}^{(1)} \geq v_{ji}^{(1)} \geq p_{ji}^{(2)} \geq v_{ji}^{(2)} \geq \cdots \geq p_{ji}^{(t)} \geq v_{ji}^{(t)} \geq \cdots \geq 0$;
(b) $1 \geq h_i \geq 0$, $1 \geq h_i^{(t)} \geq 0$.

Proof Since A is a strictly row diagonally dominant matrix, that is, $|a_{jj}| > \sum_{k\neq j}|a_{jk}| = \sum_{k\neq j,i}|a_{jk}| + |a_{ji}|$, we have $0 \leq r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i}|a_{jk}|} < 1$. By the definition of r_i, we obtain $0 \leq r_i < 1$. Since $r_i = \max_{j\neq i}\{r_{ji}\}$, so $r_i \geq r_{ji} = \frac{|a_{ji}|}{|a_{jj}| - \sum_{k\neq j,i}|a_{jk}|}$, *i.e.*, $r_i \geq \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|}$, from the definition of m_{ji}, we have $1 > r_i \geq m_{ji} \geq 0$.

Since A is a strictly row diagonally dominant matrix, $1 > d_j \geq s_{ji} \geq 0$. Then, by the definition of q_{ji}, it is easy to see that $0 \leq q_{ji} < 1$. Hence, if $q_{ji} = s_{ji}$, then

$$\frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum_{k\neq j,i}|a_{jk}|q_{ki}} = \frac{|a_{ji}|}{|a_{jj}|s_{ji} - \sum_{k\neq j,i}|a_{jk}|s_{ki}} = \frac{|a_{jj}|s_{ji} - \sum_{k\neq j,i}|a_{jk}|d_k}{|a_{jj}|s_{ji} - \sum_{k\neq j,i}|a_{jk}|s_{ki}} \leq 1;$$

else, *i.e.*, if $q_{ji} = m_{ji}$, then

$$\frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum_{k\neq j,i}|a_{jk}|q_{ki}} = \frac{|a_{ji}|}{|a_{jj}|m_{ji} - \sum_{k\neq j,i}|a_{jk}|m_{ki}} = \frac{|a_{jj}|m_{ji} - \sum_{k\neq j,i}|a_{jk}|r_i}{|a_{jj}|m_{ji} - \sum_{k\neq j,i}|a_{jk}|m_{ki}} \leq 1,$$

furthermore, from the definition of h_i, we have $0 \leq h_i \leq 1$.

Since

$$h_i = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum_{k\neq j,i}|a_{jk}|q_{ki}}\right\},$$

we have

$$h_i \geq \frac{|a_{ji}|}{|a_{jj}|q_{ji} - \sum_{k\neq j,i}|a_{jk}|q_{ki}}, \quad \textit{i.e.}, \; q_{ji}h_i \geq \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|q_{ki}h_i}{|a_{jj}|} = v_{ji}^{(0)}.$$

By $0 \leq h_i \leq 1$, we have $q_{ji} \geq v_{ji}^{(0)} \geq 0$. From the definition of $v_{ji}^{(0)}, p_{ji}^{(1)}$, we have $v_{ji}^{(0)} \geq p_{ji}^{(1)} \geq 0$.

Hence,

$$\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(1)}} = \frac{|a_{jj}|p_{ji}^{(1)} - \sum_{k\neq j,i}|a_{jk}|v_{ki}^{(0)}}{|a_{jj}|p_{ji}^{(1)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(1)}} \leq 1,$$

furthermore, by the definition of $h_i^{(1)}$, we have $0 \leq h_i^{(1)} \leq 1$, $i \in N$.

Since

$$h_i^{(1)} = \max_{j\neq i}\left\{\frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(1)}}\right\},$$

we have

$$h_i^{(1)} \geq \frac{|a_{ji}|}{|a_{jj}|p_{ji}^{(1)} - \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(1)}}, \quad \textit{i.e.}, \; p_{ji}^{(1)}h_i^{(1)} \geq \frac{|a_{ji}| + \sum_{k\neq j,i}|a_{jk}|p_{ki}^{(1)}h_i^{(1)}}{|a_{jj}|} = v_{ji}^{(1)}.$$

By $0 \leq h_i^{(1)} \leq 1$, we have $p_{ji}^{(1)} \geq v_{ji}^{(1)} \geq 0$. From the definition of $v_{ji}^{(1)}$, $p_{ji}^{(2)}$, we obtain $v_{ji}^{(1)} \geq p_{ji}^{(2)} \geq 0$.

In the same way as above, we can also prove that

$$p_{ji}^{(2)} \geq v_{ji}^{(2)} \geq \cdots \geq p_{ji}^{(t)} \geq v_{ji}^{(t)} \geq \cdots \geq 0, \quad 1 \geq h_i^{(t)} \geq 0, t = 2, 3, \ldots.$$

The proof is completed. □

Using the same technique as the proof of Lemma 2.2, Lemma 2.3, Lemma 3.1 in [6], we can obtain Lemma 2, Lemma 3, Lemma 4, respectively.

Lemma 2 *If $A = [a_{ij}] \in \mathcal{M}_n$ is a strictly row diagonally dominant matrix, then $A^{-1} = [\alpha_{ij}]$ exists, and*

$$\alpha_{ji} \leq \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| v_{ki}^{(t)}}{a_{jj}} \alpha_{ii} = p_{ji}^{(t+1)} \alpha_{ii}, \quad j, i \in N, j \neq i, t = 0, 1, 2, \ldots.$$

Lemma 3 *If $A = [a_{ij}] \in \mathcal{M}_n$ is a strictly row diagonally dominant matrix, then $A^{-1} = [\alpha_{ij}]$ exists, and*

$$\frac{1}{a_{ii}} \leq \alpha_{ii} \leq \frac{1}{a_{ii} - \sum_{j\neq i} |a_{ij}| p_{ji}^{(t)}}, \quad i, j \in N, t = 1, 2, \ldots.$$

Lemma 4 *If $A \in \mathcal{M}_n$ and $A^{-1} = [\alpha_{ij}]$ is a doubly stochastic matrix, then*

$$\alpha_{ii} \geq \frac{1}{1 + \sum_{j\neq i} p_{ji}^{(t)}}, \quad i, j \in N, t = 1, 2, \ldots.$$

Lemma 5 [7] *If A^{-1} is a doubly stochastic matrix, then $A^T e = e$, $Ae = e$, where $e = (1, 1, \ldots, 1)^T$.*

Lemma 6 [8] *Let $A = [a_{ij}] \in \mathbb{C}^{n\times n}$ and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq x_i \sum_{j\neq i} \frac{1}{x_j} |a_{ji}| \right\}.$$

3 Main results

In this section, we give several sequences of the lower bounds for $\tau(B \circ A^{-1})$ and $\tau(A \circ A^{-1})$.

Theorem 1 *Let $A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}_n$. Then, for $t = 1, 2, \ldots,$*

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i\in N} \left\{ \frac{b_{ii} - p_i^{(t)} \sum_{j\neq i} |b_{ji}|}{a_{ii}} \right\} = \Omega_t. \tag{3}$$

Proof It is evident that the result holds with equality for $n = 1$.

We next assume that $n \geq 2$.

Since $A \in \mathcal{M}_n$, there exists a positive diagonal matrix D such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix, and

$$\tau\left(B \circ A^{-1}\right) = \tau\left(D^{-1}\left(B \circ A^{-1}\right)D\right) = \tau\left(B \circ \left(D^{-1}AD\right)^{-1}\right).$$

Therefore, for convenience and without loss of generality, we assume that A is a strictly row diagonally dominant matrix.

If A is irreducible, then $0 < p_i^{(t)} < 1$, for any $i \in N$. Let $A^{-1} = [\alpha_{ij}]$. Since $\tau(B \circ A^{-1})$ is an eigenvalue of $B \circ A^{-1}$, by Lemma 2 and Lemma 6, there exists an i such that

$$\begin{aligned} \left|\tau\left(B \circ A^{-1}\right) - b_{ii}\alpha_{ii}\right| &\leq p_i^{(t)} \sum_{j\neq i} \frac{1}{p_j^{(t)}} |b_{ji}\alpha_{ji}| \leq p_i^{(t)} \sum_{j\neq i} \frac{1}{p_j^{(t)}} |b_{ji}| p_{ji}^{(t)} |\alpha_{ii}| \\ &\leq p_i^{(t)} \sum_{j\neq i} \frac{1}{p_j^{(t)}} |b_{ji}| p_j^{(t)} |\alpha_{ii}| = p_i^{(t)} |\alpha_{ii}| \sum_{j\neq i} |b_{ji}|. \end{aligned} \tag{4}$$

By Lemma 3, inequality (4), and $\tau(B\circ A^{-1})\le b_{ii}\alpha_{ii}$ for all $i\in N$, we have

$$\tau\left(B\circ A^{-1}\right)\ge b_{ii}\alpha_{ii}-p_i^{(t)}|\alpha_{ii}|\sum_{j\neq i}|b_{ji}|\ge\frac{b_{ii}-p_i^{(t)}\sum_{j\neq i}|b_{ji}|}{a_{ii}}\ge\min_{i\in N}\left\{\frac{b_{ii}-p_i^{(t)}\sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}.$$

If A is reducible, it is well known that a matrix in $Z_n=\{A=[a_{ij}]\in\mathbb{R}^{n\times n}:a_{ij}\le 0,i\neq j\}$ is a nonsingular M-matrix if and only if all its leading principal minors are positive (see condition (E17) of Theorem 6.2.3 of [1]). If we denote by $C=[c_{ij}]$ the $n\times n$ permutation matrix with $c_{12}=c_{23}=\cdots=c_{n-1,n}=c_{n1}=1$, the remaining c_{ij} zero, then $A-\varepsilon C$ is an irreducible nonsingular M-matrix for any chosen positive real number ε, sufficiently small such that all the leading principal minors of $A-\varepsilon C$ are positive. Now we substitute $A-\varepsilon C$ for A, in the previous case, and then, letting $\varepsilon\to 0$, the result follows by continuity. □

Theorem 2 *The sequence* $\{\Omega_t\}$, $t=1,2,\ldots$ *obtained from Theorem* 1 *is monotone increasing with an upper bound* $\tau(B\circ A^{-1})$ *and, consequently, is convergent.*

Proof By Lemma 1, we have $p_{ji}^{(t)}\ge p_{ji}^{(t+1)}\ge 0$, $t=1,2,\ldots$, so by the definition of $p_i^{(t)}$, it is easy to see that the sequence $\{p_i^{(t)}\}$ is monotone decreasing. Then Ω_t is a monotonically increasing sequence. Hence, the sequence is convergent. □

Remark 1 We give a simple comparison between (1) and (3). According to Lemma 1, we know that $q_{ji}=\min\{s_{ji},m_{ji}\}\ge p_{ji}^{(t)}$. Furthermore, by the definition of m_i, $p_i^{(t)}$, we have $m_i\ge p_i^{(t)}$. Therefore for $t=1,2,\ldots$,

$$\tau\left(B\circ A^{-1}\right)\ge\min_{i\in N}\left\{\frac{b_{ii}-p_i^{(t)}\sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}\ge\min_{i\in N}\left\{\frac{b_{ii}-m_i\sum_{j\neq i}|b_{ji}|}{a_{ii}}\right\}.$$

So the bound in (3) is bigger than the bound in (1).

Let $A=[a_{ij}]\in\mathcal{M}_n$. By Lemma 5, we know that if A^{-1} is a doubly stochastic matrix, then $A^Te=e$, $Ae=e$, that is, $a_{ii}=1+\sum_{j\neq i}|a_{ij}|=1+\sum_{j\neq i}|a_{ji}|$. So A is strictly diagonally dominant matrix by row and by column. By using Lemma 4 and Theorem 1, we can get the following corollaries.

Corollary 1 *Let* $A=[a_{ij}],B=[b_{ij}]\in\mathcal{M}_n$ *and* A^{-1} *be a doubly stochastic matrix. Then, for* $t=1,2,\ldots$,

$$\tau\left(B\circ A^{-1}\right)\ge\min_{i\in N}\left\{\frac{b_{ii}-p_i^{(t)}\sum_{j\neq i}|b_{ji}|}{1+\sum_{j\neq i}p_{ji}^{(t)}}\right\}.$$

Corollary 2 *Let* $A=[a_{ij}]\in\mathcal{M}_n$ *and* A^{-1} *be a doubly stochastic matrix. Then, for* $t=1,2,\ldots$,

$$\tau\left(A\circ A^{-1}\right)\ge\min_{i\in N}\left\{\frac{a_{ii}-p_i^{(t)}R_i}{1+\sum_{j\neq i}p_{ji}^{(t)}}\right\}=\Gamma_t.\tag{5}$$

Remark 2 (i) The sequence $\{\Gamma_t\}$, $t = 1, 2, \ldots$ obtained from Corollary 2 is monotone increasing with an upper bound $\tau(A \circ A^{-1})$ and, consequently, is convergent.

(ii) Next, we give a simple comparison between (2) and (5). By Lemma 1, we know that $q_{ji} = \min\{s_{ji}, m_{ji}\} \geq v_{ji}^{(0)}$, so $u_{ji} = \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|m_{ki}}{|a_{jj}|} \geq \frac{|a_{ji}|+\sum_{k\neq j,i}|a_{jk}|v_{ki}^{(0)}}{|a_{jj}|} = p_{ji}^{(1)} \geq p_{ji}^{(t)}$. Furthermore, by the definition of u_i, $p_i^{(t)}$, we have $u_i \geq p_i^{(t)}$. Obviously,

$$\tau\left(A \circ A^{-1}\right) \geq \Gamma_t \geq \min_{i \in N}\left\{\frac{a_{ii} - u_i R_i}{1 + \sum_{j\neq i} u_{ji}}\right\}.$$

So the bound in (5) is bigger than the bound in (2).

Using the same technique as the proof of Theorem 1, the another lower upper of $\tau(B \circ A^{-1})$ is given.

Theorem 3 *Let $A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}_n$. Then, for $t = 1, 2, \ldots,$*

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i \in N}\left\{\frac{b_{ii} - s_i \sum_{j\neq i} \frac{|b_{ji}|p_{ji}^{(t)}}{s_j}}{a_{ii}}\right\} = \Delta_t.$$

Using the same method as the proof of Theorem 2, the following theorem is obtained.

Theorem 4 *The sequence $\{\Delta_t\}$, $t = 1, 2, \ldots$ obtained from Theorem 3 is monotone increasing with an upper bound $\tau(B \circ A^{-1})$ and, consequently, is convergent.*

Similarly, by Lemma 4 and Theorem 2, we can get the following corollaries.

Corollary 3 *Let $A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}_n$ and A^{-1} be a doubly stochastic matrix. Then, for $t = 1, 2, \ldots,$*

$$\tau\left(B \circ A^{-1}\right) \geq \min_{i \in N}\left\{\frac{b_{ii} - s_i \sum_{j\neq i} \frac{|b_{ji}|p_{ji}^{(t)}}{s_j}}{1 + \sum_{j\neq i} p_{ji}^{(t)}}\right\}.$$

Corollary 4 *Let $A = [a_{ij}] \in \mathcal{M}_n$ and A^{-1} be a doubly stochastic matrix. Then, for $t = 1, 2, \ldots,$*

$$\tau\left(A \circ A^{-1}\right) \geq \min_{i \in N}\left\{\frac{a_{ii} - s_i \sum_{j\neq i} \frac{|a_{ji}|p_{ji}^{(t)}}{s_j}}{1 + \sum_{j\neq i} p_{ji}^{(t)}}\right\} = \mathrm{T}_t.$$

Remark 3 The sequence $\{\mathrm{T}_t\}$, $t = 1, 2, \ldots,$ obtained from Corollary 4 is monotone increasing with an upper bound $\tau(A \circ A^{-1})$ and, consequently, is convergent.

Let $\Upsilon_t = \max\{\Gamma_t, \mathrm{T}_t\}$. By Corollary 2 and Corollary 4, the following theorem is easily found.

Theorem 5 *Let $A = [a_{ij}] \in \mathcal{M}_n$ and A^{-1} be a doubly stochastic matrix. Then, for $t = 1, 2, \ldots,$*

$$\tau\left(A \circ A^{-1}\right) \geq \Upsilon_t.$$

4 Numerical examples

In this section, several numerical examples are given to verify the theoretical results.

Example 1 Let

$$A = \begin{pmatrix} 20 & -1 & -2 & -3 & -4 & -1 & -1 & -3 & -2 & -2 \\ -1 & 18 & -3 & -1 & -1 & -4 & -2 & -1 & -3 & -1 \\ -2 & -1 & 10 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -3 & -1 & 0 & 16 & -4 & -2 & -1 & -1 & -1 & -2 \\ -1 & -3 & 0 & -2 & 15 & -1 & -1 & -1 & -2 & -3 \\ -3 & -2 & -1 & -1 & -1 & 12 & -2 & 0 & -1 & 0 \\ -1 & -3 & -1 & -1 & 0 & -1 & 9 & 0 & -1 & 0 \\ -3 & -1 & -1 & -4 & -1 & 0 & 0 & 12 & 0 & -1 \\ -2 & -4 & -1 & -1 & -1 & 0 & -1 & -3 & 14 & 0 \\ -3 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -2 & 11 \end{pmatrix}.$$

By $Ae = e$, $A^T e = e$, we know that A is strictly diagonally dominant by row and column. Based on $A \in Z_n$, it is easy to see that A is nonsingular M-matrix and A^{-1} is doubly stochastic. Numerical results are given in Table 1 for the total number of iterations $T = 10$. In fact, $\tau(A \circ A^{-1}) = 0.9678$.

Remark 4 Numerical results in Table 1 show that:

(a) Lower bounds obtained from Theorem 5 are greater than the bound in Theorem 3.1 of [6].
(b) Sequence obtained from Theorem 5 is monotone increasing.
(c) The sequence obtained from Theorem 5 approximates effectively to the true value of $\tau(A \circ A^{-1})$, so we can estimate $\tau(A \circ A^{-1})$ by Theorem 5.

Example 2 A nonsingular M-matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ whose inverse is doubly stochastic is randomly generated by Matlab 7.1 (with 0-1 average distribution).

The numerical results obtained from Theorem 5 for $T = 500$ are listed in Table 2, where T are defined in Example 1.

Remark 5 Numerical results in Table 2 show that it is effective by Theorem 5 to estimate $\tau(A \circ A^{-1})$ for large order matrices.

Table 1 The lower upper of $\tau(A \circ A^{-1})$

Method	t	Υ_t
Theorem 3.1 of [6]		0.4471
Theorem 5	$t = 1$	0.7359
	$t = 2$	0.8441
	$t = 3$	0.8976
	$t = 4$	0.9233
	$t = 5$	0.9328
	$t = 6$	0.9350
	$t = 7$	0.9359
	$t = 8$	0.9363
	$t = 9$	0.9364
	$t = 10$	0.9365

Table 2 The lower upper of $\tau(A \circ A^{-1})$

t	$n = 200$	$n = 500$
$t = 1$	0.0311	0.0121
$t = 30$	0.3689	0.1568
$t = 60$	0.6198	0.2928
$t = 90$	0.7551	0.4149
$t = 120$	0.8430	0.5135
$t = 150$	0.8707	0.5911
$t = 180$	0.8873	0.6566
$t = 210$	0.8897	0.7041
$t = 240$	0.8928	0.7416
$t = 270$	0.8938	0.7699
$t = 300$	0.8943	0.7909
$t = 330$	0.8946	0.8065
$t = 360$	0.8947	0.8180
$t = 390$	0.8948	0.8264
$t = 420$	0.8948	0.8326
$t = 450$	0.8948	0.8371
$t = 480$	0.8948	0.8403
$t = 500$	0.8948	0.8420

Table 3 The lower upper of $\tau(A \circ A^{-1})$

t	$n = 10$	$n = 15$
$t = 1$	0.6667	0.1905
$t = 2$	0.7385	0.4364
$t = 3$	0.7500	0.6379
$t = 4$	0.7507	0.7191
$t = 5$		0.7422
$t = 6$		0.7481
$t = 7$		0.7495
$t = 8$		0.7499
$t = 9$		0.7500

Example 3 Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, where $a_{11} = a_{22} = \cdots = a_{n,n} = 2$, $a_{12} = a_{23} = \cdots = a_{n-1,n} = a_{n,1} = -1$, and $a_{ij} = 0$ elsewhere.

It is easy to see that A is a nonsingular M-matrix and A^{-1} is doubly stochastic. The results obtained from Theorem 5 for $n = 10, 100$ and $T = 10$ are listed in Table 3, where T is defined in Example 1. In fact, $\tau(A \circ A^{-1}) = 0.7507$ for $n = 10$ and $\tau(A \circ A^{-1}) = 0.7500$ for $n = 100$.

Remark 6 Numerical results in Table 3 show that the lower bound obtained from Theorem 5 could reach the true value of $\tau(A \circ A^{-1})$ in some cases.

5 Further work

In Theorem 5, we present a convergent sequence $\{\Upsilon_t\}$, $t = 1, 2, \ldots$, to approximate $\tau(A \circ A^{-1})$. Then an interesting problem is how accurately these bounds can be computed. At present, it is very difficult for the authors to give the error analysis. We will continue to study this problem in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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