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Approximation of Schurer type q-Bernstein-Kantorovich operators

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Abstract
In this paper, a kind of Schurer type q-Bernstein-Kantorovich operators is introduced. The Korovkin type approximation theorem of these operators is investigated. The rates of convergence of these operators are also studied by means of the modulus of continuity and the help of functions of the Lipschitz class. Then, the global approximation property is given for these operators.

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Keywords: Schurer type q-Bernstein-Kantorovich operators; Korovkin type approximation theorem; rate of convergence; q-integers; modulus of continuity

1 Introduction

In 1997, Phillips [1] introduced and studied q analogue of Bernstein polynomials. During the last decade, the applications of q-calculus in the approximation theory have become one of the main areas of research, q-calculus has been extensively used for constructing various generalizations of many classical approximation processes. It is well known that many q-extensions of the classical objects arising in the approximation theory have been introduced and studied (*e.g.*, see [2–7]). Very recently, the book *Convergence Estimates in Approximation Theory* written by Gupta and Agarwal (see [8]) introduced some approximation properties of certain complex q-operators in compact disks. Also, the Stancu variants of some q-operators have been recently discussed (*e.g.*, see [9–14]).

The goal of this paper is to introduce a kind of Schurer type q-Bernstein-Kantorovich operators and to study the approximation properties of these operators with the help of the Korovkin type approximation theorem. We also estimate the rate of convergence of these operators by using the modulus of continuity and the help of functions of the Lipschitz class. Then, we give the global approximation property for these operators.

Throughout the paper, we use some basic definitions and notations of q-calculus which can be found in Aral *et al.* [3].

In the paper, C is a positive constant. In different places, the value of C may be different. For $f \in C[a,b]$, we denote $\|f\| = \max\{|f(x)| : x \in [a,b]\}$.

2 Construction of the operators

Let $p \in \mathbf{N} \cup \{0\}$ be fixed. In 1962, Schurer [15] introduced and studied the linear positive operators $B_{n,p} : C[0,1+p] \to C[0,1]$ defined for any $n \in \mathbf{N}$ and any $f \in C[0,1+p]$ as follows:

$$B_{n,p}(f;x) = \sum_{k=0}^{n+p} \binom{n+p}{k} x^k (1-x)^{n+p-k} f(k/n), \quad x \in [0,1].$$

In 2011, Muraru [16] introduced and studied the following q-Bernstein-Schurer operators for any fixed $p \in \mathbf{N} \cup \{0\}$:

$$B_{n,p}(f;q;x) = \sum_{k=0}^{n+p} P_{n+p,k}(q;x) f\left([k]_q/[n]_q\right),$$

where $P_{n+p,k}(q;x) := \left[\begin{smallmatrix} n+p \\ k \end{smallmatrix}\right]_q x^k (1-x)_q^{n+p-k}$ and $f \in C[0,1+p]$, $x \in [0,1]$, $n \in \mathbf{N}$, $0 < q < 1$.

The moments of these operators $B_{n,p}(f;q;x)$ were obtained as follows (see [16]).

Remark 1 For $B_{n,p}(t^j;q;x)$, $j = 0,1,2$, we have

$$B_{n,p}(1;q;x) = 1, \qquad B_{n,p}(t;q;x) = \frac{[n+p]_q x}{[n]_q},$$

$$B_{n,p}\left(t^2;q;x\right) = \frac{[n+p]_q}{[n]_q^2}\left([n+p]_q x^2 + x(1-x)\right).$$

In 2013, Mahmudov and Sabancigil [17] defined q-Bernstein-Kantorovich operators as follows:

$$B_{n,q}^*(f,x) = \sum_{k=0}^{n} P_{n,k}(q;x) \int_0^1 f\left(\frac{[k]_q + q^k t}{[n+1]_q}\right) d_q t,$$

where $P_{n,k}(q;x) := \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q x^k (1-x)_q^{n-k}$ and $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbf{N}$, $0 < q < 1$.

Inspired by the operators above, we introduce a kind of Schurer type q-Bernstein-Kantorovich operators as follows.

Let $f \in C[0,1+p]$ and $p \in \mathbf{N} \cup \{0\}$ be fixed. For $x \in [0,1]$, $n \in \mathbf{N}$, $0 < q < 1$, we define the Schurer type q-Bernstein-Kantorovich operators by

$$S_{n,p}(f;q;x) = \sum_{k=0}^{n+p} P_{n+p,k}(q;x) \int_0^1 f\left(\frac{[k]_q + q^k t}{[n+1]_q}\right) d_q t, \tag{1}$$

where

$$P_{n+p,k}(q;x) = \begin{bmatrix} n+p \\ k \end{bmatrix}_q x^k (1-x)_q^{n+p-k}.$$

In 2013, Özarslan and Vedi [18] introduced the q-Bernstein-Schurer-Kantorovich operators K_n^p. Comparing the results of our present paper with [18], we find that the literature [18] only estimated the rate of convergence in the pointwise sense for these operators K_n^p. In the present paper, we not only estimate the rate of convergence in the pointwise sense, but also give the global approximation for these operators $S_{n,p}$ defined by (1), and about the estimate of the rate of convergence in the pointwise sense for these operators $S_{n,p}$, we get some new results, which are different from those in [18]. As regards [19], the q-Bernstein-Schurer-Stancu-Kantorovich operators $K_{n,q}^{(\alpha,\beta)}$ were introduced. When $\alpha = \beta = 0$, these operators $K_{n,q}^{(\alpha,\beta)}$ defined by [19] are reduced to $K_{n,q}^{(0,0)}$, which are q-Bernstein-Schurer-Kantorovich type operators, but these operators $K_{n,q}^{(0,0)}$ are quite different from operators $S_{n,p}$ defined by (1), and our research work is different from that in [19], where statistical approximation properties were studied for $K_{n,q}^{(\alpha,\beta)}$.

Now, we give some lemmas, which are necessary to prove our results.

Lemma 1 *Let $p \in \mathbf{N} \cup \{0\}$ be fixed. For any $m \in \mathbf{N} \cup \{0\}$, $n \in \mathbf{N}$, $x \in [0,1]$, $0 < q < 1$, we have*

$$S_{n,p}(t^m;q;x) = \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j}{[n+1]_q^m [m+1-j]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,p}(t^{j+i};q;x).$$

Proof When $0 < q < 1$, we have $q^k - 1 = [k]_q(q-1)$, so

$$\begin{aligned}
&S_{n,p}(t^m;q;x) \\
&\quad= \sum_{k=0}^{n+p} P_{n+p,k}(q;x) \sum_{j=0}^{m} \int_0^1 \binom{m}{j} \frac{[k]_q^j q^{k(m-j)} t^{m-j}}{[n+1]_q^m} \, d_q t \\
&\quad= \sum_{k=0}^{n+p} P_{n+p,k}(q;x) \sum_{j=0}^{m} \binom{m}{j} \frac{[k]_q^j q^{k(m-j)}}{[n+1]_q^m [m+1-j]_q} \\
&\quad= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j}{[n+1]_q^m [m+1-j]_q} \sum_{k=0}^{n+p} (q^k - 1 + 1)^{m-j} \frac{[k]_q^j}{[n]_q^j} P_{n+p,k}(q;x) \\
&\quad= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j}{[n+1]_q^m [m+1-j]_q} \sum_{k=0}^{n+p} \sum_{l=0}^{m-j} \binom{m-j}{i} (q^k - 1)^i \frac{[k]_q^j}{[n]_q^j} P_{n+p,k}(q;x) \\
&\quad= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j}{[n+1]_q^m [m+1-j]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i \sum_{k=0}^{n+p} \frac{[k]_q^{j+i}}{[n]_q^{j+i}} P_{n+p,k}(q;x) \\
&\quad= \sum_{j=0}^{m} \binom{m}{j} \frac{[n]_q^j}{[n+1]_q^m [m+1-j]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^n - 1)^i B_{n,p}(t^{j+i};q;x).
\end{aligned}$$

□

Lemma 2 *For $S_{n,p}(t^i;q;x)$, $i = 0,1,2$, we have*

(i) $S_{n,p}(1;q;x) = 1$;

(ii) $S_{n,p}(t;q;x) = \frac{2q[n+p]_q}{[2]_q[n+1]_q} x + \frac{1}{[2]_q[n+1]_q}$;

(iii) $$\begin{aligned} S_{n,p}(t^2;q;x) &= \frac{(4q^3 + q^2 + q)q[n+p-1]_q[n+p]_q x^2}{[2]_q[3]_q[n+1]_q^2} \\ &\quad+ \frac{(4q^3 + 5q^2 + 3q)[n+p]_q x}{[2]_q[3]_q[n+1]_q^2} + \frac{1}{[3]_q[n+1]_q^2}. \end{aligned}$$

Proof (i) For $i = 0$, since $\sum_{k=0}^{n} P_{n,k}(q;x) = 1$, $\int_0^1 d_q t = (1-q) \sum_{j=0}^{\infty} q^j = 1$, by (1) we can get $S_{n,p}(1;q;x) = 1$.

In view of Lemma 1 and Remark 1, by direct computation, we obtain explicit formulas for $S_{n,p}(t^i;q;x)$, $i = 1,2$ as follows.

$$\begin{aligned}
S_{n,p}(t;q;x) &= \frac{1}{[2]_q[n+1]_q}\big(B_{n,p}(1;q;x) + \big(q^n-1\big)B_{n,p}(t;q;x)\big) + \frac{[n]_q}{[n+1]_q}B_{n,p}(t;q;x) \\
&= \left(\frac{q^n-1}{[2]_q[n+1]_q} + \frac{[n]_q}{[n+1]_q}\right)\frac{[n+p]_q x}{[n]_q} + \frac{1}{[2]_q[n+1]_q} \\
&= \frac{2q[n+p]_q}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q},
\end{aligned}$$

$$\begin{aligned}
&S_{n,p}\big(t^2;q;x\big) \\
&= \frac{1}{[3]_q[n+1]_q^2}\big(B_{n,p}(1;q;x) + 2\big(q^n-1\big)B_{n,p}(t;q;x) \\
&\quad + \big(q^n-1\big)^2 B_{n,p}\big(t^2;q;x\big)\big) + \frac{2[n]_q}{[2]_q[n+1]_q^2}\big(B_{n,p}(t;q;x) \\
&\quad + \big(q^n-1\big)B_{n,p}\big(t^2;q;x\big)\big) + \frac{[n]_q^2}{[n+1]_q^2}B_{n,p}\big(t^2;q;x\big) \\
&= \left(\frac{(q^n-1)^2}{[3]_q[n+1]_q^2} + \frac{2[n]_q(q^n-1)}{[2]_q[n+1]_q^2} + \frac{[n]_q^2}{[n+1]_q^2}\right)\frac{q[n+p-1]_q[n+p]_q}{[n]_q^2}x^2 \\
&\quad + \left(\frac{2(q^n-1)}{[3]_q[n+1]_q^2} + \frac{(q^n-1)^2}{[3]_q[n]_q[n+1]_q^2} + \frac{2[n]_q}{[2]_q[n+1]_q^2} + \frac{2[n]_q(q^n-1)}{[2]_q[n]_q[n+1]_q^2}\right. \\
&\quad \left. + \frac{[n]_q^2}{[n]_q[n+1]_q^2}\right)\frac{[n+p]_q}{[n]_q}x + \frac{1}{[3]_q[n+1]_q^2} \\
&= \frac{(4q^3+q^2+q)q[n+p-1]_q[n+p]_q x^2}{[2]_q[3]_q[n+1]_q^2} + \frac{(4q^3+5q^2+3q)[n+p]_q x}{[2]_q[3]_q[n+1]_q^2} \\
&\quad + \frac{1}{[3]_q[n+1]_q^2}.
\end{aligned}$$

□

Lemma 3 *Let $p \in \mathbf{N} \cup \{0\}$ be fixed. For $x \in [0,1]$, $n \in \mathbf{N}$, $0 < q < 1$, we have*

$$B_{n,p}\big((t-x)^2;q;x\big) = x^2\left(\frac{[n+p]_q}{[n]_q} - 1\right)^2 + x(1-x)\frac{[n+p]_q}{[n]_q^2}.$$

Proof For $x \in [0,1]$, $n \in \mathbf{N}$, $0 < q < 1$, by Remark 1, we have

$$\begin{aligned}
B_{n,p}\big((t-x)^2;q;x\big) &= B_{n,p}\big(t^2;q;x\big) - 2xB_{n,p}(t;q;x) + x^2 \\
&= \frac{[n+p]_q}{[n]_q^2}\big([n+p]_q x^2 + x(1-x)\big) - \frac{2[n+p]_q x^2}{[n]_q} + x^2 \\
&= x^2\left(\frac{[n+p]_q}{[n]_q} - 1\right)^2 + x(1-x)\frac{[n+p]_q}{[n]_q^2}.
\end{aligned}$$

□

Lemma 4 *Let $p \in \mathbf{N} \cup \{0\}$ be fixed. For any $n \in \mathbf{N}$, $x \in [0,1]$ and $0 < q < 1$, we have*

(i) $$S_{n,p}(t-x;q;x) = \frac{(2[p]_q - 1)q^{n+1} - 1}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q};$$

(ii) $S_{n,p}\big((t-x)^2;q;x\big) \le \frac{4}{[n]_q}\left(x(1-x)+\frac{([p]_q+1)^2}{[n]_q}\right).$

Proof (i) In view of $[n+p]_q = [n]_q + q^n[p]_q$, $[n+1]_q = [n]_q + q^n$, by Lemma 2, we can easily obtain

$$S_{n,p}(t-x;q;x) = \left(\frac{2q[n+p]_q}{[2]_q[n+1]_q}-1\right)x+\frac{1}{[2]_q[n+1]_q} = \frac{(2[p]_q-1)q^{n+1}-1}{[2]_q[n+1]_q}x+\frac{1}{[2]_q[n+1]_q}.$$

(ii) For $p \in \mathbf{N}\cup\{0\}$ and any $n \in \mathbf{N}$, using Remark 1, Lemma 3 and $[n+p]_q = [n]_q + q^n[p]_q$, we have

$$\begin{aligned}
&S_{n,p}\big((t-x)^2;q;x\big)\\
&\quad= \sum_{k=0}^{n+p} P_{n+p,k}(q;x)\int_0^1 \left(\frac{[k]_q+q^k t}{[n+1]_q}-x\right)^2 d_q t\\
&\quad= \sum_{k=0}^{n+p} P_{n+p,k}(q;x)\int_0^1 \left(\frac{q^k t}{[n+1]_q}-\frac{q^n[k]_q}{[n]_q[n+1]_q}+\frac{[k]_q}{[n]_q}-x\right)^2 d_q t\\
&\quad\le 2\sum_{k=0}^{n+p} P_{n+p,k}(q;x)\int_0^1 \left(\frac{q^k t}{[n+1]_q}-\frac{q^n[k]_q}{[n]_q[n+1]_q}\right)^2 d_q t\\
&\qquad+2\sum_{k=0}^{n+p} P_{n+p,k}(q;x)\int_0^1 \left(\frac{[k]_q}{[n]_q}-x\right)^2 d_q t\\
&\quad\le \frac{2}{[n+1]^2}B_{n,p}\big(t^2;q;x\big)+2\sum_{k=0}^{n+p} P_{n+p,k}(q;x)\frac{\int_0^1 t^2\,d_q t}{[n+1]^2}+2B_{n,p}\big((t-x)^2;q;x\big)\\
&\quad\le \frac{4}{[n]_q}\left(x(1-x)+\frac{([p]_q+1)^2}{[n]_q}\right).
\end{aligned}$$

□

Lemma 5 *For* $f \in C[0,1+p]$, $x \in [0,1]$ *and* $n \in \mathrm{N}$, *we have*

$$\big|S_{n,p}(f;q;x)\big| \le \|f\|.$$

Proof In view of the definition given by (1) and Lemma 2, we have

$$\big|S_{n,p}(f;q;x)\big| \le S_{n,p}(1;q;x)\|f\| = \|f\|.$$

□

Let $W^2 = \{g \in C[0,1+p] : g', g'' \in C[0,1+p]\}$. For $\delta > 0$, $f \in C[0,1+p]$, the Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf\big\{\|f-g\| + \delta\|g''\| : g \in W^2\big\}. \tag{2}$$

Let $\delta > 0$, $f \in C[0,1+p]$, the second order modulus of smoothness for f is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x,x+2h\in[0,1+p]}\big|f(x+2h)-2f(x+h)+f(x)\big|,$$

the usual modulus of continuity for f is defined as

$$\omega(f,\delta) = \sup_{0<h\leq\delta}\ \sup_{x,x+h\in[0,1+p]} \left|f(x+h)-f(x)\right|.$$

For $f\in C[0,1+p]$, following [20], p.177, Theorem 2.4, there exists a constant $C>0$ such that

$$K_2(f,\delta)\leq C\omega_2(f,\sqrt{\delta}). \tag{3}$$

3 Main results

Firstly we give the following convergence theorem for the sequence $\{S_{n,p}(f;q)\}$.

Theorem 1 *Let $q_n\in(0,1)$. Then the sequence $\{S_{n,p}(f;q_n)\}$ converges to f uniformly on $[0,1]$ for any $f\in C[0,1+p]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

Proof Let $q_n\in(0,1)$ and $\lim_{n\to\infty} q_n=1$, we have $[n]_{q_n}\to\infty$ as $n\to\infty$ (see [21]). Thus, by Lemma 2, we have $\lim_{n\to\infty}\|\widetilde{S}_{n,p}(e_j;q_n;\cdot)-e_j\|_{C[0,1]}=0$ for $e_j(x)=x^j$, $j=0,1,2$. According to the well-known Bohman-Korovkin theorem [22], p.40, Theorem 1.9, we get that the sequence $\{\widetilde{S}_{n,p}(f;q_n)\}$ converges to f uniformly on $[0,1]$ for any $f\in C[0,1+p]$.

We prove the converse result by contradiction. If $\{q_n\}$ does not tend to 1 as $n\to\infty$, then it must contain a subsequence $\{q_{n_k}\}\subset(0,1)$ such that $\lim_{k\to\infty} q_{n_k}=q_0\in[0,1)$. Thus

$$\lim_{k\to\infty}\frac{1}{[n_k]_{q_{n_k}}} = \lim_{k\to\infty}\frac{1-q_{n_k}}{1-(q_{n_k})^{n_k}} = 1-q_0.$$

Taking $n=n_k$, $q=q_{n_k}$ in $S_{n,p}(t;q;x)$, by Lemma 2 we get

$$\begin{aligned} S_{n_k,p}(t;q_{n_k};x) &= \frac{2q_{n_k}[n_k+p]_{q_{n_k}}}{[2]_{q_{n_k}}[n_k+1]_{q_{n_k}}}x + \frac{1}{[2]_{q_{n_k}}[n_k+1]_{q_{n_k}}} \\ &\to \frac{1-q_0+2q_0x}{1+q_0}\neq x,\quad \text{as } k\to\infty. \end{aligned}$$

This leads to a contradiction, hence $\lim_{n\to\infty} q_n=1$. The theorem is proved. □

Next we estimate the rate of convergence.

Theorem 2 *Let $f\in C[0,1+p]$, $x\in[0,1]$, $q\in(0,1)$, we have $|S_{n,p}(f;q;x)-f(x)|\leq 2\omega(f,\delta_n(x))$, where*

$$\delta_n(x) = \left[\frac{4}{[n]_q}\left(x(1-x)+\frac{([p]_q+1)^2}{[n]_q}\right)\right]^{1/2}. \tag{4}$$

Proof By Lemma 2 we have

$$\left|S_{n,p}(f;q;x)-f(x)\right| = \left|S_{n,p}\big(f(t)-f(x);q;x\big)\right| \leq S_{n,p}\big(\left|f(t)-f(x)\right|;q;x\big).$$

Since for $t\in[0,1+p]$, $x\in[0,1]$ and any $\delta>0$ we have

$$\left|f(t)-f(x)\right|\leq\big(1+\delta^{-2}(t-x)^2\big)\omega(f,\delta),$$

we get

$$\left|S_{n,p}(f;q;x) - f(x)\right| \le \left[S_{n,p}(1;q;x) + \delta^{-2} S_{n,p}\left((t-x)^2;q;x\right)\right]\omega(f,\delta).$$

By Lemma 2 and Lemma 3, for $x \in [0,1]$, we have

$$\left|S_{n,p}(f;q;x) - f(x)\right| \le \left(1 + \delta^{-2}\delta_n^2(x)\right)\omega(f,\delta).$$

Taking $\delta = \delta_n(x)$, from the above inequality we obtain the desired result. □

Corollary 1 *Let* $M > 0$, $0 < \alpha \le 1$, $f \in \mathrm{Lip}_M^\alpha$ *on* $[0,1+p]$, $q \in (0,1)$, *we have*

$$\left\|S_{n,p}(f;q;\cdot) - f\right\|_{C[0,1]} \le 2M\eta_n^\alpha,$$

where

$$\eta_n = \left[\frac{4}{[n]_q}\left(1 + \frac{([p]_q + 1)^2}{[n]_q}\right)\right]^{1/2}. \tag{5}$$

Proof Let $M > 0$, $0 < \alpha \le 1$, $f \in \mathrm{Lip}_M^\alpha$ on $[0,1+p]$, we have $f \in C[0,1+p]$. For any $\delta > 0$, since $f \in \mathrm{Lip}_M^\alpha$ is equivalent to $\omega(f,\delta) \le M\delta^\alpha$, thus, by Theorem 2, for $x \in [0,1]$, we have $|S_{n,p}(f;q;x) - f(x)| \le 2\omega(f,\delta_n(x)) \le 2M\delta_n^\alpha(x) \le 2M\eta_n^\alpha$, where $\delta_n(x)$ and η_n are given in (4) and (5), respectively, which implies the proof is complete. □

Theorem 3 *Let* $f \in C[0,1+p]$, $x \in [0,1]$, $q \in (0,1)$, *we have*

$$\left|S_{n,p}(f;q;x) - f(x)\right| \le C\omega_2\left(f,\delta_n(x)\right) + \omega\left(f, \left|\frac{((2[p]_q - 1)q^{n+1} - 1)x + 1}{[2]_q[n+1]_q}\right|\right),$$

where C is a positive constant, $\delta_n(x)$ *is given by* (4).

Proof For $f \in C[0,1+p]$, $x \in [0,1]$, we define

$$\widehat{S}_{n,p}(f;q;x) = S_{n,p}(f;q;x) + f(x) - f(a_n x + b_n), \tag{6}$$

where $a_n = \frac{2q[n+p]_q}{[2]_q[n+1]_q}$, $b_n = \frac{1}{[2]_q[n+1]_q}$. By Lemma 2 we get $\widehat{S}_{n,p}(1;q;x) = 1$, $\widehat{S}_{n,p}(t;q;x) = x$. Let $g \in W^2$, $t \in [0,1+p]$, $x \in [0,1]$, by Taylor's formula, we obtain

$$\widehat{S}_{n,p}(g;q;x) = g(x) + \widehat{S}_{n,p}\left(\int_x^t (t-u)g''(u)\,du;q;x\right).$$

Using the inequality $(a-b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned} &S_{n,p}\left((t-x)^2;q;x\right) + (a_n x + b_n - x)^2 \\ &\quad\le \frac{4}{[n]_q}\left(x(1-x) + \frac{([p]_q+1)^2}{[n]_q}\right) + \frac{4(2 - 4[p]_q + 4[p]_q^2)}{[n]_q^2}x^2 + \frac{2}{[n]_q^2} \\ &\quad\le 5\delta_n^2(x), \end{aligned} \tag{7}$$

so, by the definition given by (6) and Lemma 4, for $x \in [0,1]$, we have

$$
\begin{aligned}
&\bigl|\widehat{S}_{n,p}(g;q;x) - g(x)\bigr| \\
&\quad\leq \left|S_{n,p}\left(\int_x^t (t-u)g''(u)\,du;q;x\right)\right| + \left|\int_x^{a_nx+b_n} (a_nx+b_n-u)g''(u)\,du\right| \\
&\quad\leq S_{n,p}\left(\left|\int_x^t |t-u|\bigl|g''(u)\bigr|\,du\right|;q;x\right) + \left|\int_x^{a_nx+b_n} |a_nx+b_n-u|\bigl|g''(u)\bigr|\,du\right| \\
&\quad\leq \bigl[S_{n,p}\bigl((t-x)^2;q;x\bigr) + (a_nx+b_n-x)^2\bigr]\bigl\|g''\bigr\| \\
&\quad\leq 5\delta_n^2(x)\bigl\|g''\bigr\|.
\end{aligned}
$$

On the other hand, by the definition given by (6) and Lemma 5, we have

$$
\bigl|\widehat{S}_{n,p}(f;q;x)\bigr| \leq \bigl|S_{n,p}(f;q;x)\bigr| + 2\|f\| \leq 3\|f\|. \tag{8}
$$

Thus, for $x \in [0,1]$, using Lemma 4(i), we have

$$
\begin{aligned}
&\bigl|S_{n,p}(f;q;x) - f(x)\bigr| \\
&\quad\leq \bigl|\widehat{S}_{n,p}(f-g;q;x)\bigr| + \bigl|\widehat{S}_{n,p}(g;q;x) - g(x)\bigr| + \bigl|g(x) - f(x)\bigr| + \bigl|f(a_nx+b_n) - f(x)\bigr| \\
&\quad\leq 4\|f-g\| + 5\delta_n^2(x)\bigl\|g''\bigr\| + \omega\left(f, \left|\frac{((2[p]_q-1)q^{n+1}-1)x+1}{[2]_q[n+1]_q}\right|\right).
\end{aligned}
$$

Hence, taking infimum on the right-hand side over all $g \in W^2$, we can get

$$
\bigl|S_{n,p}(f;q;x) - f(x)\bigr| \leq 5K_2\bigl(f,\delta_n^2(x)\bigr) + \omega\left(f, \left|\frac{((2[p]_q-1)q^{n+1}-1)x+1}{[2]_q[n+1]_q}\right|\right).
$$

By inequality (3), for every $q \in (0,1)$, we have

$$
\bigl|S_{n,p}(f;q;x) - f(x)\bigr| \leq C\omega_2\bigl(f,\delta_n(x)\bigr) + \omega\left(f, \left|\frac{((2[p]_q-1)q^{n+1}-1)x+1}{[2]_q[n+1]_q}\right|\right). \qquad \square
$$

Theorem 4 *Let $f \in C^1[0,1+p]$, $x \in [0,1]$, $q \in (0,1)$, we have*

$$
\begin{aligned}
\bigl|S_{n,p}(f;q;x) - f(x)\bigr| \leq{}& \bigl\|f'\bigr\|\left|\frac{(2[p]_q-1)q^{n+1}-1}{[2]_q[n+1]_q}x + \frac{1}{[2]_q[n+1]_q}\right| \\
&+ 2\delta_n(x)\omega\bigl(f',\delta_n(x)\bigr),
\end{aligned}
$$

where $\|f'\| = \max\{|f'(x)|; x \in [0,1+p]\}$, $\delta_n(x)$ *is given by* (4).

Proof Let $f \in C^1[0,1+p]$, for any $t \in [0,1+p]$, $x \in [0,1]$ and $\delta > 0$, we get

$$
\begin{aligned}
\bigl|f(t) - f(x) - f'(x)(t-x)\bigr| &\leq \left|\int_x^t \bigl|f'(u) - f'(x)\bigr|\,du\right| \\
&\leq \omega\bigl(f',|t-x|\bigr)|t-x| \\
&\leq \omega\bigl(f',\delta\bigr)\bigl(|t-x| + \delta^{-1}(t-x)^2\bigr),
\end{aligned}
$$

hence

$$\begin{aligned}&\left|S_{n,p}\big(f(t)-f(x)-f'(x)(t-x);q;x\big)\right|\\&\quad\le\omega\big(f',\delta\big)\big(S_{n,p}\big(|t-x|;q;x\big)+\delta^{-1}S_{n,p}\big((t-x)^2;q;x\big)\big).\end{aligned}$$

By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}&\left|S_{n,p}\big(f(t)-f(x)-f'(x)(t-x);q;x\big)\right|\\&\quad\le\omega\big(f',\delta\big)\Big(\sqrt{S_{n,p}(1;q;x)}+\delta^{-1}\sqrt{S_{n,p}\big((t-x)^2;q;x\big)}\Big)\sqrt{S_{n,p}\big((t-x)^2;q;x\big)}.\end{aligned}$$

Thus, by Lemma 2 and Lemma 4, for $x\in[0,1]$, we can get

$$\begin{aligned}\left|S_{n,p}(f;q;x)-f(x)\right|\le\|f'\|\left|\frac{(2[p]_q-1)q^{n+1}-1}{[2]_q[n+1]_q}x+\frac{1}{[2]_q[n+1]_q}\right|\\+\omega\big(f',\delta\big)\big(1+\delta^{-1}\delta_n(x)\big)\delta_n(x).\end{aligned}$$

Taking $\delta=\delta_n(x)$, then from the above inequality we obtain the desired result.

Finally we give the global approximation for the sequence $\{S_{n,p}(f;q)\}$. For the next theorem we shall use some notations.

For $f\in C[0,1+p]$ and $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0,1]$, let

$$\omega_2^{\varphi}(f,\sqrt{\eta})=\sup_{0<h\le\sqrt{\eta}}\sup_{x\pm h\varphi(x)\in[0,1+p]}\left|f\big(x+h\varphi(x)\big)-2f(x)+f\big(x-h\varphi(x)\big)\right|$$

be the second order Ditzian-Totik modulus of smoothness, and let

$$\overline{K}_{2,\varphi}(f,\eta)=\inf\big\{\|f-g\|+\eta\left\|\varphi^2g''\right\|+\eta^2\left\|g''\right\|:g\in W^2(\varphi)\big\}$$

be the corresponding K-functional, where $W^2(\varphi)=\{g\in C[0,1+p]:g'\in AC_{\mathrm{loc}}[0,1+p],\varphi^2g''\in C[0,1+p]\}$ and $g'\in AC_{\mathrm{loc}}[0,1+p]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b]\subseteq[0,1+p]$. It is well known (see [5], p.24, Theorem 1.3.1) that

$$\overline{K}_{2,\varphi}(f,\eta)\le C\omega_2^{\varphi}(f,\sqrt{\eta})\tag{9}$$

for some absolute constant $C>0$.

Furthermore, the Ditzian-Totik modulus of first order is given by

$$\vec{\omega}_{\psi}(f,\eta)=\sup_{0<h\le\eta}\sup_{x,x+h\psi\in[0,1+p]}\left|f\big(x+h\psi(x)\big)-f(x)\right|,\tag{10}$$

where ψ is an admissible step-weight function on $[0,1]$. □

Now we state our next main result.

Theorem 5 *Let $\{S_{n,p}(f;q)\}$ be defined by* (1). *Then there exists an absolute constant $C > 0$ such that*

$$\|S_{n,p}(f;q;\cdot) - f\| \le C\omega_2^{\varphi}\left(f, \sqrt{\frac{1}{[n]_q}}\right) + \overrightarrow{\omega}_{\psi}\left(f; \frac{1}{[n]_q}\right),$$

where $f \in C[0,1+p]$, $0 < q < 1$, $\varphi(x) = \sqrt{x(1-x)}$ and $\psi(x) = (1 + |2[p]_q - 1|)x + 1$, $x \in [0,1]$.

Proof Let

$$\widehat{S}_{n,p}(f;q;x) = S_{n,p}(f;q;x) + f(x) - f(a_n x + b_n),$$

where $f \in C[0,1+p]$, $a_n = \frac{2q[n+p]_q}{[2]_q[n+1]_q}$, $b_n = \frac{1}{[2]_q[n+1]_q}$. Let $g \in W^2(\varphi)$, $t \in [0,1+p]$, $x \in [0,1]$, by using Taylor's formula, we have

$$\widehat{S}_{n,p}(g;q;x) = g(x) + S_{n,p}\left(\int_x^t (t-u)g''(u)\,du;q;x\right) - \int_x^{a_n x + b_n} (a_n x + b_n - u)g''(u)\,du.$$

Hence

$$\begin{aligned} \left|\widehat{S}_{n,p}(g;q;x) - g(x)\right| &\le \left|S_{n,p}\left(\int_x^t (t-u)g''(u)\,du;q;x\right)\right| \\ &\quad + \left|\int_x^{a_n x+b_n} (a_n x + b_n - u)g''(u)\,du\right|. \end{aligned} \tag{11}$$

Let $\lambda_n^2(x) = \varphi^2(x) + \frac{([p]_q+1)^2}{[n]_q}$, because the function λ_n is concave on $[0,1]$, we have for $u = t + \tau(x-t)$, $\tau \in [0,1]$, the estimate

$$\frac{|t-u|}{\lambda_n^2(u)} \le \frac{\tau|x-t|}{\lambda_n^2(t) + \tau(\lambda_n^2(x) - \lambda_n^2(t))} \le \frac{|x-t|}{\lambda_n^2(x)}.$$

Hence, by (11) we have

$$\begin{aligned} &\left|\widehat{S}_{n,p}(g;q;x) - g(x)\right| \\ &\quad \le \frac{1}{\lambda_n^2(x)} S_{n,p}\left((t-x)^2;q;x\right)\left\|\lambda_n^2 g''\right\| + \frac{1}{\lambda_n^2(x)}(a_n x + b_n - x)^2\left\|\lambda_n^2 g''\right\|. \end{aligned}$$

In view of (7) and $\|\lambda_n^2 g''\| \le \|\varphi^2 g''\| + \frac{([p]_q+1)^2}{[n]_q}\|g''\|$, for $x \in [0,1]$, we have

$$\left|\widehat{S}_{n,p}(g;q;x) - g(x)\right| \le \frac{20}{[n]_q}\left(\left\|\varphi^2 g''\right\| + \frac{([p]_q+1)^2}{[n]_q}\left\|g''\right\|\right).$$

Using (8), for $f \in C[0,1+p]$, we find

$$\begin{aligned} &\left|S_{n,p}(f;q;x) - f(x)\right| \\ &\quad \le 4\|f-g\| + \frac{20}{[n]_q}\left(\left\|\varphi^2 g''\right\| + \frac{([p]_q+1)^2}{[n]_q}\left\|g''\right\|\right) + \left|f(a_n x + b_n) - f(x)\right|. \end{aligned}$$

Taking the infimum on the right-hand side over all $g \in W^2(\varphi)$, we obtain

$$\left|S_{n,p}(f;q;x) - f(x)\right| \le 20\left([p]_q + 1\right)^2 \overline{K}_{2,\varphi}\left(f, \frac{1}{[n]_q}\right) + \left|f(a_n x + b_n) - f(x)\right|.$$

On the other hand, by (10) we have

$$\begin{aligned}
&\left|f(a_n x + b_n) - f(x)\right| \\
&\quad = \left|f\left(x + \psi(x)\left(\frac{(a_n - 1)x + b_n}{\psi(x)}\right)\right) - f(x)\right| \\
&\quad \le \sup_{t, t+\psi(t)(\frac{(a_n-1)x+b_n}{\psi(x)}) \in [0,1+p]} \left|f\left(t + \psi(t)\left(\frac{(a_n - 1)x + b_n}{\psi(x)}\right)\right) - f(t)\right| \\
&\quad \le \overrightarrow{\omega}_\psi\left(f; \left|\frac{(a_n - 1)x + b_n}{\psi(x)}\right|\right) \\
&\quad \le \overrightarrow{\omega}_\psi\left(f; \frac{1}{[n]_q}\right),
\end{aligned}$$

so, using (9) we obtain

$$\left\|S_{n,p}(f;q) - f\right\| \le C\omega_2^\varphi\left(f, \sqrt{\frac{1}{[n]_q}}\right) + \overrightarrow{\omega}_\psi\left(f; \frac{1}{[n]_q}\right).$$

□

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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