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On the James type constant of $l_p - l_1$

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Abstract

For any $\tau \geq 0$, $t \geq 1$ and $p \geq 1$, the exact value of the James type constant $J_{X,t}(\tau)$ of the $l_p - l_1$ space is investigated. As an application, the exact value of the von Neuman-Jordan type constant of the $l_p - l_1$ space can also be obtained.

MSC: Primary 46B20; secondary 47H10

Keywords: James type constant; $l_p - l_1$ space; von Neuman-Jordan type constant

1 Introduction and preliminaries

Throughout this paper, we shall assume that X stands for a nontrivial Banach space, *i.e.*, $\dim X \geq 2$. We will use S_X and B_X to denote the unit sphere and unit ball of X, respectively.

A Banach space X is called uniformly non-square in the sense of James if there exists a positive number $\delta < 1$ such that $\frac{\|x+y\|}{2} \leq \delta$ or $\frac{\|x-y\|}{2} \leq \delta$, whenever $x, y \in S_X$. The non-square or James constant is defined by

$$J(X) = \sup\bigl\{\min\bigl(\|x+y\|, \|x-y\|\bigr), x, y \in S_X\bigr\}.$$

Obviously, X is uniformly non-square in the sense of James if and only if $J(X) < 2$ (see [1]).

The von Neumann-Jordan constant, introduced by Clarkson in [2], is defined as follows:

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}.$$

It is well known that the von Neumann-Jordan constant is not larger than the James constant. This result $C_{\mathrm{NJ}}(X) \leq J(X)$ was obtained by Takahashi-Kato in [3], Wang in [4] and Yang-Li in [5] almost at the same time.

Recently, as a generalization of the James constant and the von Neumann-Jordan constant, Takahashi in [6] introduced the James type constant $J_{X,t}(\tau)$ and the von Neumann-Jordan type constant $C_t(X)$, respectively, as follows:

$$J_{X,t}(\tau) = \sup\bigl\{\mu_t\bigl(\|x+\tau y\|, \|x-\tau y\|\bigr) : x, y \in S_X\bigr\},$$

where $\tau \geq 0$, $-\infty \leq t < +\infty$. Here, we denote $\mu_t(a,b) = (\frac{a^t+b^t}{2})^{\frac{1}{t}}$ $(t \neq 0)$ and $\mu_0(a,b) = \lim_{t\to 0} \mu_t(a,b) = \sqrt{ab}$ for two positive numbers a and b. It is well known that $\mu_t(a,b)$ is nondecreasing and $\mu_{-\infty}(a,b) = \lim_{t\to -\infty} \mu_t(a,b) = \min(a,b)$. Therefore, $J(X) = J_{X,-\infty}(1)$,

$$C_t(X) = \sup\left\{\frac{J_{X,t}(\tau)^2}{1+\tau^2} : 0 \leq \tau \leq 1\right\}.$$

It is obvious that $C_2(X) = C_{\mathrm{NJ}}(X)$ and the James type constants include some known constants such as Alonso-Llorens-Fuster's constant $T(X)$ in [7], Baronti-Casini-Papini's constant $A_2(X)$ in [8], Gao's constant $E(X)$ in [9] and Yang-Wang's modulus $\gamma_X(t)$ in [10]. These constants are defined by $T(X) = J_{X,0}(1)$, $A_2(X) = J_{X,1}(1)$, $E(X) = 2J^2_{X,2}(1)$ and $\gamma_X(t) = J^2_{X,2}(t)$.

Now let us list some known results of the constant $J_{X,t}(\tau)$; for more details, see [6, 11–14].

(1) If $-\infty \le t_1 \le t_2 < \infty$, then $J_{X,t_1}(\tau) \le J_{X,t_2}(\tau)$ for any $\tau \ge 0$.

(2) Let $t \ge 1$, $\tau \ge 0$ and $X = l_1 - l_2$, then

$$J_{X,t}(\tau) = \left(\frac{(1+\tau^2)^{\frac{t}{2}} + (1+\tau)^t}{2}\right)^{\frac{1}{t}}. \tag{1.1}$$

(3) Let X be an $l_\infty - l_1$ space. If $0 \le \tau \le 1$, then

$$J_{X,t}(\tau) = \begin{cases} (\frac{1+(1+\tau)^t}{2})^{\frac{1}{t}}, & t \ge 1, \\ 1 + \frac{\tau}{2}, & t \le 1. \end{cases}$$

(4) Let $1 \le t \le p \le \infty$, $2 \le p$ and $0 \le \tau \le 1$. Then

$$J_{X,t}(\tau) = 1 + 2^{-\frac{1}{p}}\tau,$$

where X is an $l_\infty - l_p$ space.

(5) Let $t_2 \ge t_1 \ge 1$ and $0 \le \tau \le 1$. Then, for any Banach space X,

$$J^{t_2}_{X,t_1}(\tau) \le J^{t_2}_{X,t_2}(\tau) \le \frac{(1+\tau)^{t_2} + \{2J^{t_1}_{X,t_1}(\tau) - (1+\tau)^{t_1}\}^{\frac{t_2}{t_1}}}{2}. \tag{1.2}$$

(6) $J_{X,t_1}(\tau) = 1+\tau$ if and only if $J_{X,t_2}(\tau) = 1+\tau$.

For $p \ge 1$, the $l_p - l_1$ space is defined by $X = \mathbf{R}^2$ with the norm

$$\|x\| = \|(x_1,x_2)\| = \begin{cases} \|x\|_p, & x_1x_2 \ge 0, \\ \|x\|_1, & x_1x_2 \le 0. \end{cases}$$

For any $\tau \ge 0$ and $p \ge 1$, we have calculated the exact value of the James type constant $J_{l_p-l_1,t}(\tau)$ for $t \ge 1$. As an application, we also give the exact value of the von Neumann-Jordan type constant $C_t(l_p - l_1)$ for $1 \le t \le 2$. In [11], for $1 < p \le 2$, it is known that $C_{\mathrm{NJ}}(l_p - l_1) = 1 + 2^{\frac{2}{p}-2}$ was given. In this paper, for $p \ge 2$, $(p-2)2^{\frac{2}{p}-2} \le 1$ and $p > 2$, $(p-2)2^{\frac{2}{p}-2} \ge 1$, the exact value of the von Neumann-Jordan constant $C_{\mathrm{NJ}}(l_p - l_1)$ is obtained.

2 Main results and their proofs

To give the value of $J_{X,t}(\tau)$ for $X = l_p - l_1$, we need the following lemmas.

Lemma 2.1 *Let $x_1, x_2, y_1, y_2 \ge 0$ and $p \ge 1$ such that*

$$x_1^p + x_2^p = 1 \quad \textit{and} \quad y_1^p + y_2^p = 1.$$

If $0 \le \tau \le 1$, $0 \le \tau y_1 \le x_1$ *and* $0 \le x_2 \le \tau y_2$, *then*

$$\left[(x_1+\tau y_1)^p + (x_2+\tau y_2)^p\right]^{\frac{1}{p}} + x_1 - \tau y_1 + \tau y_2 - x_2 \le 1 + \tau + \left(1+\tau^p\right)^{\frac{1}{p}}.$$

Proof It is readily seen that $0 \le x_1 - \tau y_1 + \tau y_2 - x_2 \le 1+\tau$. Let us now consider two possible cases.

Case 1. $0 \le x_1 - \tau y_1 + \tau y_2 - x_2 \le (1+\tau^p)^{1/p}$. Hence

$$\begin{aligned}
&\left[(x_1+\tau y_1)^p + (x_2+\tau y_2)^p\right]^{\frac{1}{p}} + x_1 - \tau y_1 + \tau y_2 - x_2 \\
&\quad \le \left[\left(x_1^p + x_2^p\right)^{1/p} + \left(\tau^p y_1^p + \tau^p y_2^p\right)^{1/p}\right] + \left(1+\tau^p\right)^{\frac{1}{p}} \\
&\quad = 1 + \tau + \left(1+\tau^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Case 2. $(1+\tau^p)^{1/p} \le x_1 - \tau y_1 + \tau y_2 - x_2 \le 1+\tau$. By Minkowski's inequality,

$$\begin{aligned}
&\left[(x_1+\tau y_1)^p + (x_2+\tau y_2)^p\right]^{1/p} + x_1 - \tau y_1 + \tau y_2 - x_2 \\
&\quad \le \left(x_1^p + \tau^p y_2^p\right)^{1/p} + \left(\tau^p y_1^p + x_2^p\right)^{1/p} + x_1 - \tau y_1 + \tau y_2 - x_2 \\
&\quad \le \left(x_1^p + \tau^p y_2^p\right)^{1/p} + \tau y_1 + x_2 + x_1 - \tau y_1 + \tau y_2 - x_2 \\
&\quad \le (1+\tau) + \left(1+\tau^p\right)^{1/p},
\end{aligned}$$

where the second inequality follows from the fact $\|\cdot\|_p \le \|\cdot\|_1$. Consequently, the proof is complete. □

Lemma 2.2 *Let* $\tau \in (0,1)$, $t \in [1,2]$ *and* $p \ge 2$. *Then*

(a) $2\tau^p + p - 2 - p\tau^2 \ge 0$;

(b) $1 - \tau^{2p-2} - (p-1)(\tau^{p-2} - \tau^p) \ge 0$;

(c) *the function*

$$f(\tau) = \frac{\tau - \tau^{p-1}}{(1-\tau)(1+\tau)^{t-1}}\left(1+\tau^p\right)^{\frac{t}{p}-1}$$

is nondecreasing; *moreover*, $0 \le f(\tau) \le (p-2)2^{\frac{t}{p}-t}$.

Proof (a) Letting $h(\tau) = 2\tau^p + (p-2) - p\tau^2$, we have $h'(\tau) = 2p(\tau^{p-1} - \tau) \le 0$, and $h(\tau) \ge h(1) = 0$.

(b) Letting $g(\tau) = 1 - \tau^{2p-2} - (p-1)(\tau^{p-2} - \tau^p)$, we have

$$g'(\tau) = -(p-1)\tau^{p-3}\left(2\tau^p + p - 2 - p\tau^2\right).$$

Hence, $g'(\tau) \le 0$ by (a) and $g(\tau) \ge g(1) = 0$.

(c) By a basic calculation, then by use of (b), we have

$$\begin{aligned}
f'(\tau) = {}& \frac{1}{[(1-\tau)(1+\tau)^{t-1}]^2}\Big\{(1-\tau)(1+\tau)^{t-1}\Big[\left(1-(p-1)\tau^{p-2}\right)\left(1+\tau^p\right)^{\frac{t}{p}-1} \\
&+ \left(\tau - \tau^{p-1}\right)(t-p)\tau^{p-1}\left(1+\tau^p\right)^{\frac{t}{p}-2}\Big]
\end{aligned}$$

$$
\begin{aligned}
&- \big(\tau - \tau^{p-1}\big)\big(1 + \tau^{p}\big)^{\frac{t}{p}-1}\big[-(1+\tau)^{t-1} + (1-\tau)(t-1)(1+\tau)^{t-2}\big]\big\} \\
&= \frac{(1+\tau^p)^{\frac{t}{p}-2}(1+\tau)^{t-2}}{[(1-\tau)(1+\tau)^{t-1}]^2}\big\{(1+\tau)\big(1+\tau^p\big)\big[1-(p-1)\tau^{p-2} - \tau + (p-1)\tau^{p-1} \\
&\quad + \tau - \tau^{p-1}\big] + (1-\tau)\big(\tau - \tau^{p-1}\big)\big[(t-p)(1+\tau)\tau^{p-1} - \big(1+\tau^p\big)(t-1)\big]\big\} \\
&= \frac{(1+\tau^p)^{\frac{t}{p}-2}(1+\tau)^{t-2}}{[(1-\tau)(1+\tau)^{t-1}]^2}\big\{\big(1+\tau^2\big)\big[1-\tau^{2p-2} - (p-1)\tau^{p-2}\big(1-\tau^2\big)\big] \\
&\quad + (2-t)(1-\tau)\big(\tau - \tau^{p-1}\big)\big(1-\tau^{p-1}\big)\big\} \geq 0.
\end{aligned}
$$

Now from $\lim_{\tau\to 1^-} f(\tau) = (p-2)2^{\frac{t}{p}-t}$, we have $0 \leq f(\tau) \leq (p-2)2^{\frac{t}{p}-t}$. □

Theorem 2.3 *Let $t \geq 1$, $p \geq 1$, $\tau \geq 0$ and $X = l_p - l_1$ space. Then*

$$
J_{X,t}(\tau) = \left(\frac{(1+\tau^p)^{\frac{t}{p}} + (1+\tau)^t}{2}\right)^{\frac{1}{t}}. \tag{2.1}
$$

Proof As $J_{X,t}(\tau) = \tau J_{X,t}(\frac{1}{\tau})$ is valid for any $\tau > 0$, we only consider the case $0 \leq \tau \leq 1$. We claim that the following inequality is valid for any $x, y \in S_{l_p - l_1}$:

$$
\|x+\tau y\| + \|x - \tau y\| \leq \big(1+\tau^p\big)^{\frac{1}{p}} + 1 + \tau. \tag{2.2}
$$

In fact, by the convexity of norm, we only need to show that this inequality is valid for any $x, y \in \operatorname{ext}(S_{l_p-l_1})$, where $\operatorname{ext}(S_{l_p-l_1})$ denotes the set of extreme points of $S_{l_p-l_1}$. From $\operatorname{ext}(S_{l_p-l_1}) = \{(x_1, x_2) : x_1^p + x_2^p = 1, x_1x_2 \geq 0\}$, we may assume that $x = (a, b)$, $y = (c, d)$, where $a, b, c, d \geq 0$ with $a^p + b^p = c^p + d^p = 1$.

(I) If $(a - c\tau)(b - d\tau) \geq 0$,

$$
\begin{aligned}
\|x+\tau y\| + \|x-\tau y\| &= \|x+\tau y\|_p + \|x-\tau y\|_p \\
&\leq 1 + \tau + \big[|a-c\tau|^p + |b - d\tau|^p\big]^{\frac{1}{p}} \\
&\leq 1 + \tau + \max\big\{\big[a^p + b^p\big]^{\frac{1}{p}}, \big[(c\tau)^p + (d\tau)^p\big]^{\frac{1}{p}}\big\} \\
&\leq 2 + \tau \\
&\leq \big(1+\tau^p\big)^{\frac{1}{p}} + 1 + \tau.
\end{aligned}
$$

(II) If $(a - c\tau)(b - d\tau) \leq 0$.

We may assume that $a - c\tau > 0$ and $b - d\tau \leq 0$. Then, by use of Lemma 2.1, we also have

$$
\|x+\tau y\| + \|x-\tau y\| = \|x+\tau y\|_p + \|x-\tau y\|_1 \leq \big(1+\tau^p\big)^{\frac{1}{p}} + 1 + \tau.
$$

Thus (2.2) is valid.

Now, by taking $x = (1, 0)$ and $y = (0, 1)$, we have $2J_{l_p-l_1,1}(\tau) = (1+\tau^p)^{\frac{1}{p}} + 1 + \tau$. Therefore by (1.2) we have

$$
J_{X,t}^t(\tau) \leq \frac{(1+\tau)^t + [2J_{X,1}(\tau) - (1+\tau)]^t}{2} = \frac{(1+\tau)^t + (1+\tau^p)^{\frac{t}{p}}}{2}.
$$

On the other hand, by taking $x = (1,0)$, $y = (0,1)$, we have

$$\|x + \tau y\| = \big(1 + \tau^p\big)^{\frac{1}{p}}, \qquad \|x - \tau y\| = 1 + \tau,$$

so

$$J^t_{X,t}(\tau) \geq \frac{(1+\tau)^t + (1+\tau^p)^{\frac{t}{p}}}{2}.$$

Therefore, (2.1) is valid for $t \geq 1$. □

Theorem 2.4 *Let $p = 2$, $t \geq 1$ or $p > 2$, $t \in [1,2]$, and X be an $l_p - l_1$ space.*
For p and t such that $(p-2)2^{\frac{t}{p}-t} \leq 1$, then

$$C_t(X) = \left(\frac{2^{\frac{t}{p}-\frac{t}{2}} + 2^{\frac{t}{2}}}{2}\right)^{\frac{2}{t}}. \tag{2.3}$$

For p and t such that $(p-2)2^{\frac{t}{p}-t} > 1$, then

$$C_t(X) = \frac{1}{1+\tau_0^2}\left(\frac{(1+\tau_0)^t + (1+\tau_0^p)^{\frac{t}{p}}}{2}\right)^{\frac{2}{t}},$$

where τ_0 is the unique solution of the equation

$$\frac{(\tau - \tau^{p-1})(1+\tau^p)^{\frac{t}{p}-1}}{(1-\tau)(1+\tau)^{t-1}} = 1. \tag{2.4}$$

Proof By (2.1), we have

$$C_t(X) = \big[\sup\big\{h(\tau) : 0 \leq \tau \leq 1\big\}\big]^{\frac{2}{t}}, \quad \text{where } h(\tau) = \frac{(1+\tau)^t + (1+\tau^p)^{\frac{t}{p}}}{2(1+\tau^2)^{\frac{t}{2}}}.$$

A simple computation yields

$$h'(\tau) = \frac{t(1-\tau)(1+\tau)^{t-1}}{2(1+\tau^2)^{\frac{t}{2}+1}}\left[1 - \frac{(\tau-\tau^{p-1})(1+\tau^p)^{\frac{t}{p}-1}}{(1-\tau)(1+\tau)^{t-1}}\right].$$

If $p = 2$, $t \geq 1$ or $p > 2$, $t \in [1,2]$ such that $(p-2)2^{\frac{t}{p}-t} \leq 1$, Lemma 2.2 implies $h'(\tau) \geq 0$, so that h is nondecreasing. Hence

$$C_t(X) = h(1)^{\frac{2}{t}} = \left(\frac{2^{\frac{t}{p}-\frac{t}{2}} + 2^{\frac{t}{2}}}{2}\right)^{\frac{2}{t}}.$$

Otherwise, let $\tau_0 \in (0,1)$ be the unique solution to equation (2.4). It then follows from Lemma 2.2 that $h'(\tau) \geq 0$ for $\tau \in [0,\tau_0]$ and $h'(\tau) \leq 0$ for $\tau \in [\tau_0,1]$. In other words, h attains its maximum at τ_0. Hence

$$C_t(X) = \frac{1}{1+\tau_0^2}\left(\frac{(1+\tau_0)^t + (1+\tau_0^p)^{\frac{t}{p}}}{2}\right)^{\frac{2}{t}}.$$ □

For $1 < p \le 2$, $C_{\mathrm{NJ}}(l_p - l_1) = 1 + 2^{\frac{2}{p}-2}$ (see [11]). Now, by taking $t = 2$ in Theorem 2.3, as a generalization, we can obtain the following corollary on the von Neumann-Jordan constant of $l_p - l_1$ space.

Corollary 2.5 *Let X be the $l_p - l_1$ space.*

(a) *If $p \ge 2$ and $(p-2)2^{\frac{2}{p}-2} \le 1$, then $C_{\mathrm{NJ}}(X) = 1 + 2^{\frac{2}{p}-2}$.*

(b) *If $p > 2$ and $(p-2)2^{\frac{2}{p}-2} \ge 1$, then*

$$C_{\mathrm{NJ}}(X) = \frac{1}{2} + \frac{1-\tau_0^p}{2(\tau_0 - \tau_0^{p-1})},$$

where $\tau_0 \in (0,1)$ is the unique solution to the equation

$$\frac{(\tau - \tau^{p-1})(1+\tau^p)^{\frac{2}{p}-1}}{1-\tau^2} = 1.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper, and read and approved the final manuscript.

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