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Operator representation of sectorial linear relations and applications

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Abstract

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathcal{T}$ be a non-densely defined linear relation in $\mathcal{H}$ with domain $D(\mathcal{T})$. We prove that if $\mathcal{T}$ is sectorial then it can be expressed in terms of some sectorial operator A with domain $D(A) = D(\mathcal{T})$ and that $\mathcal{T}$ is maximal sectorial if and only if A is maximal sectorial in $\overline{D(\mathcal{T})}$. The operator A has the property that for every $u \in D(A)$ and every $v \in D(\mathcal{T})$ and any $u' \in \mathcal{T}(u)$, $\langle Au, v\rangle = \langle u', v\rangle$. We use this representation to show that every sectorial linear relation $\mathcal{T}$ is form closable, meaning that the form associated with $\mathcal{T}$ has a closed extension. We also prove a result similar to Kato's first representation theorem for sectorial linear relations.

Unlike the results available in the literature, we do not assume that the graph of the linear relation $\mathcal{T}$ is a closed subspace of $\mathcal{H} \times \mathcal{H}$.

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1 Introduction

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{S}$ be a closed linear subspace of $\mathcal{H} \times \mathcal{H}$, and consider the set $D(\mathcal{S}) = \{x \in \mathcal{H} : (x,u) \in \mathcal{S} \text{ for some } u \in \mathcal{H}\}$. If $s \in D(\mathcal{S})$, denote by $\mathcal{S}_s$ the collection of all elements $v \in \mathcal{H}$ such that the pair $(s,v) \in \mathcal{S}$. It is shown in [1, 2] and [3] that there exists some closed linear operator A in $\mathcal{H}$ with domain $D(A) = D(\mathcal{S})$ such that

$$\mathcal{S}_s = \mathcal{S}_0 + As \tag{1.1}$$

for every $s \in D(\mathcal{S})$, where the right-hand side of (1.1) is understood to represent the set $\{w + As : w \in \mathcal{S}_0\}$. Representation (1.1) easily follows from the decomposition of $\mathcal{S}$ as a direct orthogonal sum of two of its subspaces, a result that is based on the closed nature of $\mathcal{S}$. A natural question that arises is whether a representation of the form (1.1) exists for non-closed subspaces $\mathcal{S}$. In this case there is no guarantee that a decomposition of $\mathcal{S}$ that leads to this representation does exist.

Instead of considering non-closed subspaces $\mathcal{S}$ of $\mathcal{H} \times \mathcal{H}$ one can in general consider, as we do, linear relations in $\mathcal{H}$ whose graphs are not necessarily closed subspaces of $\mathcal{H} \times \mathcal{H}$.

The main objective of this paper is to show that an operator representation of the form (1.1) holds for maximal sectorial linear relations $\mathcal{T}$ whose graphs are not necessarily closed in $\mathcal{H} \times \mathcal{H}$. The key idea in obtaining this result is the fact that the numerical range of such a linear relation is a proper subset of the complex plane. We use this result to prove a theorem similar to Kato's first representation theorem.

Since every densely defined linear relation $\mathcal{T}$ in a Hilbert space $\mathcal{H}$ is an operator, all the relations $\mathcal{T}$ considered in here are assumed to be non-densely defined. We will use the term *operator* to mean a linear operator unless stated otherwise.

The paper is organized as follows. In Section 2 we recall some basic definitions and known results on sesquilinear forms. Most of the results given here can be found in [4]. Section 3 is devoted to some background information on linear relations, while Section 4 contains the main results. In particular we show that every maximal sectorial linear relation $\mathcal{T}$ in $\mathcal{H}$ has an operator representation of the form (1.1) and that every sectorial linear relation in $\mathcal{H}$ is form closable. Finally we show that a result similar to Kato's first representation theorem holds in the case of sectorial linear relations.

2 Sesquilinear forms and related results

We are concerned with sesquilinear forms $t(u,v)$ defined for both u and v belonging to a linear manifold D of a Hilbert $\mathcal{H}$. Hence $t(u,v)$ is complex-valued and linear in $u \in D$ for each fixed $v \in D$ and semilinear in $v \in D$ for each fixed $u \in D$. The linear manifold D will be called the domain of t, and we will denote it by $D(t)$. The form $t(u,u)$ is called the *quadratic form* associated with $t(u,v)$. We denote this form by $t(u)$. We shall refer to the sesquilinear form $t(u,v)$ as the form t.

Let $\mathcal{H}$ be a Hilbert space and let t be a form on $\mathcal{H}$. We say that t is *symmetric* if

$$t(u,v) = \overline{t(v,u)}$$

for all $u, v \in \mathcal{H}$. If A is a bounded operator on $\mathcal{H}$, the function

$$t(u,v) = \langle Au, v\rangle \tag{2.1}$$

represents a bounded sesquilinear form on $\mathcal{H}$. Conversely, an arbitrary bounded sesquilinear form t on $\mathcal{H}$ can be expressed in this form by a suitable choice of a bounded operator A on $\mathcal{H}$. The form defined by (2.1) is called the form associated with the operator A.

A symmetric sesquilinear form t on a Hilbert space $\mathcal{H}$ is called *nonnegative* (in symbols $t \geq 0$) if the associated quadratic form $t(u)$ is nonnegative ($t(u) \geq 0$) for all u, and *positive* if $t(u) > 0$ for all $u \neq 0$.

The *lower bound* γ of a symmetric form t is defined as the largest real number γ such that

$$t(u) \geq \gamma \|u\|^2.$$

The *upper bound* γ' is defined in a similar way. We say that the form t is *semi-bounded* if it is either bounded from below or from above. It can be shown (see [4]) that

$$\bigl|t(u,v)\bigr| \leq M\|u\|\|v\|, \quad M = \max\bigl(|\gamma|, \bigl|\gamma'\bigr|\bigr).$$

Hence a sesquilinear form t which is bounded both from below and above is bounded. Similarly, a symmetric operator A is said to be bounded from below if

$$\langle Au, u\rangle \geq \gamma \langle u, u\rangle$$

holds for all $u \in D(A)$ with $\|u\| = 1$. The largest γ with this property is called the lower bound of A. Notions of *boundedness from above* and *upper bound* are similarly defined. Like in the case of a symmetric form, a symmetric operator bounded from below or from above is said to be semi-bounded.

For a nonsymmetric form t, the set of values of the quadratic form $t(u)$ for all $u \in D(t)$ such that $\|u\| = 1$ is called the *numerical range* of t, and we denote this set by $\Theta(t)$. If t is a symmetric form, then t is bounded from below if and only if $\Theta(t)$ is a finite or semi-infinite interval of the real line bounded from the left. Generalizing this, we say that a form t is bounded from the left if $\Theta(t)$ is a subset of a half plane of the form $\operatorname{Re} z \geq \gamma$, $z \in \mathbb{C}$, $\gamma \in \mathbb{R}$. We say that t is *sectorially bounded from the left*, or simply that t is *sectorial*, if its numerical range $\Theta(t)$ is contained as a sector of the form

$$\left|\arg(z-\gamma)\right| \leq \theta, \quad 0 \leq \theta < \frac{\pi}{2}, \gamma \in \mathbb{R}. \tag{2.2}$$

The numbers γ and θ are not uniquely determined. We call γ a *vertex* and θ a corresponding *semi-angle* of the form t.

For an operator A with domain $D(A)$ in a Hilbert space $\mathcal{H}$, the *numerical range* $\Theta(A)$ of A is the set of complex numbers

$$\Theta(A) = \left\{\langle Au, u\rangle : u \in D(A), \|u\| = 1\right\}.$$

Let us recall that an operator A in $\mathcal{H}$ is said to be sectorial if its numerical range is a subset of the sector of the form (2.2). If A is a sectorial operator in $\mathcal{H}$ with vertex γ and semi-angle θ, then the form t defined in $\mathcal{H}$ by

$$t(u,v) = \langle Au, v\rangle \quad \text{with } D(t) = D(A) \tag{2.3}$$

is obviously sectorial with the same vertex and semi-angle.

Let t be a sectorial form with domain $D(t)$ in a Hilbert space $\mathcal{H}$. A sequence $\{x_n\}$ of elements of $\mathcal{H}$ is said to be *t-convergent* to an element $x \in \mathcal{H}$, denoted by

$$x_n \underset{t}{\rightarrow} x \quad \text{as } n \rightarrow \infty,$$

if $x_n \in D(t)$ for all n, $x_n \rightarrow x$ and $t(x_n - x_m) \rightarrow 0$ as $m, n \rightarrow \infty$. Note that x may or may not belong to $D(t)$. We say that t is *closed* if $x_n \underset{t}{\rightarrow} x$ implies that $x \in D(t)$ and $t(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$ and that it is *closable* if it has a closed extension. If t is closable, its *closure* $\bar{t}$ is defined to be its smallest closed extension. This closure $\bar{t}$ is also defined in the following way. The domain $D(\bar{t})$ of $\bar{t}$ is the set consisting of all $x \in \mathcal{H}$ such that there exists a sequence x_n such that $x_n \underset{t}{\rightarrow} x$ and

$$\bar{t}(x,y) = \lim_{n\rightarrow\infty} t(x_n, y_n) \quad \text{for any } x_n \underset{t}{\rightarrow} x, y_n \underset{t}{\rightarrow} y.$$

Theorems 2.1, 2.2, and 2.4 below are taken from [4]. We note here that although we do not make much use of the next theorem, we feel it is worth mentioning it here.

Theorem 2.1 *Let t be a sectorial form and let $\bar{t}$ be its closure. Then*

(i) *$\bar{t}$ is sectorial,*
(ii) *the numerical range $\Theta(t)$ of t is a dense subset of the numerical range $\Theta(\bar{t})$ of $\bar{t}$,*
(iii) *a vertex α and a semi-angle θ for $\bar{t}$ can be chosen equal to the corresponding values for t.*

Theorem 2.2 *Let A be a sectorial operator in a Hilbert space $\mathcal{H}$. Then A is form-closable, that is, the form t defined by* (2.3) *above is closable.*

Let t be a closed sectorial form in a Hilbert space $\mathcal{H}$, D' be a linear subspace of $D(t)$, and let t' be the restriction of t to D'. The subspace D' is called a *core* of t if the closure of t' is t, that is, $\bar{t}' = t$.

See [4, p.317] for the following remark.

Remark 2.3 If t is bounded, then D' is a core of t if and only if D' is dense in $D(t)$.

Let $\mathcal{H}$ be a Hilbert space and let $A : D(A) \to \overline{D(A)}$ be a sectorial operator in $\mathcal{H}$. We say that A is *maximal sectorial* in $\overline{D(A)}$ if it has no proper sectorial extension A_1 in $\overline{D(A)}$. Maximal sectorial operators are useful in operator representations of sectorial forms as seen from the following theorem which is referred to in the literature as *Kato's first representation theorem*.

Theorem 2.4 *Let t be a densely defined closed sectorial sesquilinear form in a Hilbert space $\mathcal{H}$. There exists a maximal sectorial operator A in $\mathcal{H}$ such that*

(i) *$D(A) \subset D(t)$ and*

$$t(u,v) = \langle Au, v\rangle$$

for every $u \in D(A)$ and $v \in D(t)$;

(ii) *$D(A)$ is a core of t;*

(iii) *if $u \in D(t)$, $w \in \mathcal{H}$ and*

$$t(u,v) = \langle w, v\rangle$$

holds for every v belonging to a core of t, then $u \in D(A)$ and $Au = w$.

The maximal sectorial operator A is uniquely determined by condition (i).

3 Relations on sets

3.1 Preliminaries

Let $\mathcal{U}$ and $\mathcal{V}$ be two nonempty sets. By a *relation* $\mathcal{T}$ from $\mathcal{U}$ to $\mathcal{V}$ we mean a mapping whose domain $D(\mathcal{T})$ is a nonempty subset of $\mathcal{U}$, and taking values in $2^{\mathcal{V}}\backslash\emptyset$, the collection of all nonempty subsets of $\mathcal{V}$. Such a mapping $\mathcal{T}$ is also referred to as a *multi-valued* operator or at times as a *set-valued function*. If $\mathcal{T}$ maps the elements of its domain to singletons, then $\mathcal{T}$ is said to be a *single-valued* mapping or operator. Let $\mathcal{T}$ be a relation from $\mathcal{U}$ to $\mathcal{V}$, and let $\mathcal{T}(u)$ denote the image of an element $u \in \mathcal{U}$ under $\mathcal{T}$. If we define $\mathcal{T}(u) = \emptyset$ for $u \in \mathcal{U}$ and $u \notin D(\mathcal{T})$, then the domain $D(\mathcal{T})$ is given by

$$D(\mathcal{T}) = \{u \in \mathcal{U} : \mathcal{T}(u) \neq \emptyset\}.$$

Denote by $R(\mathcal{U},\mathcal{V})$ the class of all relations from $\mathcal{U}$ to $\mathcal{V}$. If $\mathcal{T}$ belongs to $R(\mathcal{U},\mathcal{V})$, the graph of $\mathcal{T}$, which we denote by $G(\mathcal{T})$, is the subset of $\mathcal{U}\times\mathcal{V}$ defined by

$$G(\mathcal{T}) = \{(u,v)\in\mathcal{U}\times\mathcal{V} : u\in D(\mathcal{T}), v\in\mathcal{T}(u)\}.$$

A relation $\mathcal{T}\in R(\mathcal{U},\mathcal{V})$ is uniquely determined by its graph, and conversely any nonempty subset of $\mathcal{U}\times\mathcal{V}$ uniquely determines a relation $\mathcal{T}\in R(\mathcal{U},\mathcal{V})$.

For a relation $\mathcal{T}\in R(\mathcal{U},\mathcal{V})$, we define its inverse $\mathcal{T}^{-1}$ as the relation from $\mathcal{V}$ to $\mathcal{U}$ whose graph $G(\mathcal{T}^{-1})$ is given by

$$G(\mathcal{T}^{-1}) = \{(v,u)\in\mathcal{V}\times\mathcal{U} : (u,v)\in G(T)\}. \tag{3.1}$$

Let $\mathcal{T}\in R(\mathcal{U},\mathcal{V})$. Given a subset $\mathcal{M}$ of $\mathcal{U}$, we define the image of $\mathcal{M}$, $T(\mathcal{M})$ to be

$$\mathcal{T}(\mathcal{M}) = \bigcup\{T(m) : m\in\mathcal{M}\cap D(T)\}.$$

With this notation we define the range of $\mathcal{T}$ by

$$R(\mathcal{T}) := \mathcal{T}(\mathcal{U}).$$

We say that $\mathcal{T}$ is *surjective* if $R(\mathcal{T})=\mathcal{V}$ and that it is *injective* if $\mathcal{T}^{-1}$ is single-valued. It $\mathcal{T}$ is injective then we have the implication

$$\mathcal{T}(u_1)=\mathcal{T}(u_2) \quad\Rightarrow\quad u_1=u_2 \quad \text{for } u_1,u_2\in D(\mathcal{T}).$$

Let $\mathcal{N}$ be a nonempty subset of $\mathcal{V}$. The definition of $\mathcal{T}^{-1}$ given in (3.1) above implies that

$$\mathcal{T}^{-1}(\mathcal{N}) = \{u\in D(\mathcal{T}) : \mathcal{N}\cap\mathcal{T}(u)\neq\emptyset\}. \tag{3.2}$$

If in particular $v\in R(\mathcal{T})$, then

$$\mathcal{T}^{-1}(v) = \{u\in D(\mathcal{T}) : v\in\mathcal{T}(u)\}.$$

Let $\mathcal{T}\in R(\mathcal{U},\mathcal{V})$ and let $\mathcal{M}$ be a subset of $\mathcal{U}$ such that

$$\mathcal{M}\cap D(\mathcal{T})\neq\emptyset.$$

We define the *restriction* of $\mathcal{T}$ to $\mathcal{M}$ to be the relation $\mathcal{T}|_{\mathcal{M}}\in R(\mathcal{U},\mathcal{V})$ with domain $D(\mathcal{T}|_{\mathcal{M}}) = D(\mathcal{T})\cap\mathcal{M}$ given by

$$\mathcal{T}|_{\mathcal{M}}(w) = \mathcal{T}(w) \quad \text{for } w\in\mathcal{M}.$$

Given two relations $\mathcal{S}$ and $\mathcal{T}$ in $R(\mathcal{U},\mathcal{V})$, we say that $\mathcal{T}$ is an *extension* of $\mathcal{S}$ if

$$\mathcal{T}|_{D(\mathcal{S})} = \mathcal{S}.$$

If $\mathcal{T}$ is an extension of $\mathcal{S}$, then $G(\mathcal{S})\subset G(\mathcal{T})$. The converse is not necessarily true (see the remark following Theorem 3.3).

For a detailed study of relations, we refer to [1–3] and [5].

3.2 Linear relations

Let X and Y be linear spaces over a field $\mathbb{K} = \mathbb{R}$ (or $\mathbb{C}$) and let $\mathcal{T} \in R(X,Y)$. We say that $\mathcal{T}$ is a *linear relation* or a *multi-valued linear operator* if for all $x, z \in D(\mathcal{T})$ and any nonzero scalar α we have

(1) $T(x) + \mathcal{T}(z) = \mathcal{T}(x+z)$,

(2) $\alpha\mathcal{T}(x) = \mathcal{T}(\alpha x)$.

The equalities in (1) and (2) above are understood to be set equalities. These two conditions indirectly imply that the domain of a linear relation is a linear subspace. The class of linear relations in $R(X,Y)$ will be denoted by $LR(X,Y)$. If $X = Y$ then we denote $LR(X,X)$ by $LR(X)$. We say that $\mathcal{T}$ is a linear relation in X if $\mathcal{T} \in LR(X)$. We shall use the term *operator* to refer to a single-valued linear operator while a multi-valued linear operator will be generally referred to as a *linear relation*.

Let $\mathcal{T}$ be a linear relation in a Hilbert $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$. We define the adjoint $\mathcal{T}^*$ of $\mathcal{T}$ in $\mathcal{H}$ by

$$G(\mathcal{T}^*) = \{(s,w) \in \mathcal{H} \times \mathcal{H} : \langle v,s\rangle = \langle u,w\rangle \text{ for all } (u,v) \in G(\mathcal{T})\}.$$

We say that a linear relation $\mathcal{T}$ is *symmetric* if $G(\mathcal{T}) \subset G(\mathcal{T}^*)$ and that it is *selfadjoint* if $G(\mathcal{T}) = G(\mathcal{T}^*)$. Note that if $\mathcal{T}$ is symmetric then $\langle u,v\rangle$ is real for every $(u,v) \in G(\mathcal{T})$.

Let $\mathcal{T}$ be a symmetric linear relation in $\mathcal{H}$. We say that T is *semi-bounded* below by a real number α if $\langle k,h\rangle \geq \alpha\langle h,h\rangle$ for all $(h,k) \in G(\mathcal{T})$. It is said to be semi-bounded above by a real number β if $\langle k,h\rangle \leq \beta\langle h,h\rangle$ for all $(h,k) \in G(\mathcal{T})$. We say that $\mathcal{T}$ is semi-bounded if it is either bounded below or above.

For a linear relation $\mathcal{T}$ in $\mathcal{H}$, we define its numerical range $\Theta(\mathcal{T})$ by

$$\Theta(\mathcal{T}) = \{\langle v,u\rangle : (u,v) \in G(\mathcal{T}), \|u\| = 1\}.$$

By analogy with linear operators, see [4], we say that $\mathcal{T}$ is *accretive* if

$$\Theta(\mathcal{T}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$$

and that it is *sectorial* if it is accretive and $\Theta(\mathcal{T})$ is contained in the sector

$$\left|\arg(z-\gamma)\right| \leq \theta < \frac{\pi}{2}, \quad \gamma \in \mathbb{R}.$$

As before, the constants γ and θ, which are not unique, are referred to as a vertex and the corresponding angle, respectively. The theory of numerical ranges and sectoriality of operators and/or linear relations has been investigated by many authors. We mention [2, 4, 6, 7] and [8] where one can find a thorough account of some of these concepts.

We conclude this section with the following three theorems which are taken from [5].

Theorem 3.1 *Let $\mathcal{T} \in R(X,Y)$. The following properties are equivalent.*

(i) *$\mathcal{T}$ is a linear relation.*

(ii) *$G(\mathcal{T})$ is a linear subspace of $X \times Y$.*

(iii) *$\mathcal{T}^{-1}$ is a linear relation.*

(iv) *$G(\mathcal{T}^{-1})$ is a linear subspace of $Y \times X$.*

Corollary 3.2 *Let $\mathcal{T} \in R(X, Y)$.*

(i) *Then $\mathcal{T}$ is a linear relation if and only if*

$$\mathcal{T}(\alpha x_1 + \beta x_2) = \alpha \mathcal{T}(x_1) + \beta \mathcal{T}(x_2)$$

holds for all $x_1, x_2 \in D(\mathcal{T})$ and some nonzero scalars α and β.

(ii) *If $\mathcal{T}$ is a linear relation, then $\mathcal{T}(0)$ and $\mathcal{T}^{-1}(0)$ are linear subspaces.*

For a linear relation $\mathcal{T}$, the subspace $\mathcal{T}^{-1}(0)$ is called the *null space* (or *kernel*) of $\mathcal{T}$ and is denoted by $N(\mathcal{T})$.

Theorem 3.3 *Let $\mathcal{T}$ be a linear relation in a Hilbert space $\mathcal{H}$ and let $x \in D(\mathcal{T})$. Then $y \in \mathcal{T}(x)$ if and only if*

$$\mathcal{T}(x) = \mathcal{T}(0) + y.$$

Theorem 3.3 shows that

(i) $\mathcal{T}$ is single-valued if and only if $T(0) = \{0\}$;

(ii) if $\mathcal{S}$ and $\mathcal{T}$ are two linear relations in a Hilbert space $\mathcal{H}$ such that $G(\mathcal{S}) \subset G(\mathcal{T})$, then $\mathcal{T}$ is an extension of $\mathcal{S}$ if and only if $\mathcal{S}(0) = \mathcal{T}(0)$.

Theorem 3.4 *Let $\mathcal{T} \in R(X, Y)$. Then $\mathcal{T}$ is a linear relation if and only if for all $x_1, x_2 \in D(\mathcal{T})$ and all scalars α and β,*

$$\alpha \mathcal{T}(x_1) + \beta \mathcal{T}(x_2) \subset \mathcal{T}(\alpha x_1 + \beta x_2).$$

4 Operator representation of sectorial linear relations and applications

Let $\mathcal{T}$ be a sectorial linear relation in a Hilbert space $\mathcal{H}$. As in the case of sectorial operators, we say that $\mathcal{T}$ is *maximal sectorial* in $\mathcal{H}$ if there does not exist a sectorial linear relation $\mathcal{T}_1$ in $\mathcal{H}$ such that $G(\mathcal{T}) \subset G(\mathcal{T}_1)$.

If $\mathcal{H}$ is a Hilbert space and S is a closed linear subspace of $\mathcal{H} \times \mathcal{H}$, then (see [1, 2] and [3])

$$S = S_\infty \oplus S_1, \tag{4.1}$$

where

$$S_\infty = S \cap \bigl(\{0\} \oplus \mathcal{H}\bigr) = \bigl\{(0, s) : (0, s) \in S\bigr\}$$

and

$$S_1 = S \ominus S_\infty$$

are orthogonal closed linear subspaces with S_1 being the graph of some closed linear operator A. Let $\mathcal{S}$ be a linear relation in $\mathcal{H}$ whose graph is the linear subspace S, that is, $G(\mathcal{S}) = S$. If $s \in D(\mathcal{S})$ then (4.1) implies that

$$\mathcal{S}(s) = \mathcal{S}(0) + As. \tag{4.2}$$

If the linear subspace S is not closed, then there is no guarantee that decomposition (4.1) does exist, and therefore representation (4.2) may not exist. In this section, we show that representation (4.2) holds for maximal sectorial relations $\mathcal{T}$ without requiring that $G(\mathcal{T})$ be closed. We also show that the operator A is maximal sectorial in $\overline{D(A)}$ if and only if $\mathcal{T}$ is maximal sectorial.

For a sectorial linear relation $\mathcal{T}$ in a Hilbert space $\mathcal{H}$ and $s \in D(\mathcal{T})$, we show that there exists a sectorial operator A in $\mathcal{H}$ with $D(A) = D(\mathcal{T})$ such that

$$\mathcal{T}(s) \subset \overline{D(\mathcal{T})}^{\perp} + As$$

without the maximality assumption and use it to show that every sectorial linear relation $\mathcal{T}$ is form closable, meaning that the form associated with $\mathcal{T}$ has a closed extension.

Lemma 4.1 *Let $\mathcal{T}$ be a linear relation in a Hilbert space $\mathcal{H}$ with domain $D(\mathcal{T})$, and let h be an element of $\mathcal{H}$ such that $h \in \overline{D(\mathcal{T})}^{\perp}$ but $h \notin \mathcal{T}(0)$. Let $\mathfrak{C}$ be a linear subspace of $\mathbb{C}$. The relation $\widetilde{\mathcal{T}}$ defined on $D(\mathcal{T})$ by*

$$\widetilde{\mathcal{T}}(k) = \mathcal{T}(k) + \zeta h \quad \forall \zeta \in \mathfrak{C} \tag{4.3}$$

is a linear relation in $\mathcal{H}$ such that $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$ and $\Theta(\widetilde{\mathcal{T}}) = \Theta(\mathcal{T})$.

Note that the equality in (4.3) is an equality of sets. The plus sign on the right-hand side of this equality is understood to mean that to each element of $T(k)$ we add ζh for all $\zeta \in \mathfrak{C}$. In other words,

$$R(\widetilde{\mathcal{T}}) = \{k' + \zeta h : k' \in R(\mathcal{T}), \zeta \in \mathfrak{C}\}.$$

Proof First we prove the linearity of $\widetilde{\mathcal{T}}$. To do this we let

$$k_1, k_2 \in D(\widetilde{\mathcal{T}}) = D(\mathcal{T}).$$

Then

$$\begin{aligned}\widetilde{\mathcal{T}}(k_1 + k_2) &= \mathcal{T}(k_1 + k_2) + \zeta h \quad \forall \zeta \in \mathfrak{C} \\ &= [\mathcal{T}(k_1) + \mathcal{T}(k_2)] + \zeta h \quad \forall \zeta \in \mathfrak{C} \\ &= [\mathcal{T}(k_1) + \alpha h \,\forall \alpha \in \mathfrak{C}] + [\mathcal{T}(k_2) + \beta h \,\forall \beta \in \mathfrak{C}] \\ &= \widetilde{\mathcal{T}}(k_1) + \widetilde{\mathcal{T}}(k_2).\end{aligned}$$

We also have that for $\gamma \in \mathbb{C}$,

$$\begin{aligned}\widetilde{\mathcal{T}}(\gamma k_1) &= \mathcal{T}(\gamma k_1) + \zeta h \quad \forall \zeta \in \mathfrak{C} \\ &= \gamma \mathcal{T}(k_1) + \zeta h \quad \forall \zeta \in \mathfrak{C} \\ &= \gamma [\mathcal{T}(k_1) + \zeta' h] \quad \forall \zeta' \in \mathfrak{C} \\ &= \gamma \widetilde{T}(k_1).\end{aligned}$$

The condition $h \notin \mathcal{T}(0)$ implies that $\mathcal{T}(0)$ is a proper subset of $\widetilde{\mathcal{T}}(0)$. It therefore follows from Theorem 3.3 that $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$.

To prove the equality of the numerical ranges, we let $k' \in \widetilde{\mathcal{T}}(k)$. Then there exist elements $l \in \mathcal{T}(k)$ and $\delta \in \mathfrak{C}$ such that $k' = l + \delta h$. Hence

$$\langle k', k \rangle = \langle l + \delta h, k \rangle = \langle l, k \rangle + \delta \langle h, l \rangle = \langle l, k \rangle.$$

Hence, given $k' \in \widetilde{\mathcal{T}}(k)$, there exists an element $l \in \mathcal{T}(k)$ such that

$$\langle k', k \rangle = \langle l, k \rangle. \tag{4.4}$$

Similarly, given an element $l \in \mathcal{T}(k)$, there exists an element $k' \in \widetilde{\mathcal{T}}(k)$ such that (4.4) holds. This shows that

$$\Theta(\widetilde{\mathcal{T}}) = \Theta(\mathcal{T}). \qquad \square$$

We know that the numerical range $\Theta(\mathcal{T})$ of a sectorial linear relation $\mathcal{T}$ satisfies the condition $\Theta(\mathcal{T}) \neq \mathbb{C}$. The following theorem is particularly useful when dealing with this type of linear relations.

Theorem 4.2 *Let $\mathcal{T}$ be a linear relation in a Hilbert space $\mathcal{H}$ with domain $D(\mathcal{T})$ and numerical range $\Theta(\mathcal{T})$.*

(i) *If $\Theta(\mathcal{T}) \neq \mathbb{C}$ then $\mathcal{T}(0) \perp \overline{D(\mathcal{T})}$.*

(ii) *If $\mathcal{T}$ is a maximal sectorial linear relation, then $\mathcal{T}(0) = \overline{D(\mathcal{T})}^{\perp}$.*

Proof (i) Assume that $\Theta(\mathcal{T}) \neq \mathbb{C}$. Then there is at least one element $\alpha \in \mathbb{C}$ such that $\alpha \notin \Theta(\mathcal{T})$. Since $\mathcal{T}$ is not single-valued, it follows that $\mathcal{T}(0) \neq \{0\}$. Hence there exists a nonzero element h of $\mathcal{T}(0)$. Let $k \in D(\mathcal{T})$ with $\|k\| = 1$. Such k exists since $D(\mathcal{T})$ is a linear subspace of $\mathcal{H}$. Let $k' \in \mathcal{T}(k)$. Then $k' + \xi h \in \mathcal{T}(k)$ for every $\xi \in \mathbb{C}$. This follows from Theorem 3.3 and the fact that $\mathcal{T}(0)$ is a linear subspace of $\mathcal{H}$ (see Corollary 3.2). Hence

$$\langle k' + \xi h, k \rangle \in \Theta(\mathcal{T}) \quad \forall \xi \in \mathbb{C},$$

that is,

$$\langle k', k \rangle + \xi \langle h, k \rangle \in \Theta(\mathcal{T}) \quad \forall \xi \in \mathbb{C}.$$

If $\langle h, k \rangle \neq 0$ then

$$\langle k', k \rangle + \xi \langle h, k \rangle = \alpha$$

for some suitable choice of ξ, contradicting the fact that $\alpha \notin \Theta(\mathcal{T})$. Hence $\langle h, k \rangle = 0$ for every $h \in \mathcal{T}(0)$ and every $k \in D(\mathcal{T})$ with $\|k\| = 1$. The result then follows from the linearity of $D(\mathcal{T})$ and the continuity of the inner product.

(ii) Since $\Theta(\mathcal{T}) \neq \mathbb{C}$, it follows from part (i) that $\mathcal{T}(0) \subset \overline{D(\mathcal{T})}^{\perp}$.

Now assume that there exists $h \in \overline{D(\mathcal{T})}^{\perp}$ such that $h \notin \mathcal{T}(0)$. Lemma 4.1 implies that there exists a sectorial linear relation $\widetilde{\mathcal{T}}$ in $\mathcal{H}$ such that $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$, contradicting the maximality of $\mathcal{T}$. Hence $\overline{D(\mathcal{T})}^{\perp} \subset \mathcal{T}(0)$. It therefore follows that $\mathcal{T}(0) = \overline{D(\mathcal{T})}^{\perp}$. $\square$

Part (i) of the preceding theorem also implies that if $T(0) \nsubseteq \overline{D(T)}^{\perp}$ then

$$\Theta(T) = \mathbb{C}.$$

Corollary 4.3 *Let $\mathcal{T}$ be a densely defined sectorial linear relation in a Hilbert space $\mathcal{H}$. Then $\mathcal{T}$ is an operator.*

The lemma below is helpful in defining the sectorial form associated with a sectorial linear relation $\mathcal{T}$.

Lemma 4.4 *Let $\mathcal{T}$ be a sectorial linear relation in a Hilbert space $\mathcal{H}$ with domain $D(\mathcal{T})$, and let $x, y \in D(\mathcal{T})$. The equality*

$$\langle x_1, y\rangle = \langle x_2, y\rangle$$

holds for all $x_1, x_2 \in \mathcal{T}(x)$.

Proof Let $x, y \in D(\mathcal{T})$ and let $x_1, x_2 \in \mathcal{T}(x)$. Theorem 3.3 implies that

$$x_2 = x_1 + z$$

for some $z \in \mathcal{T}(0)$. Hence

$$\langle x_2, y\rangle = \langle x_1 + z, y\rangle = \langle x_1, y\rangle + \langle z, y\rangle = \langle x_1, y\rangle,$$

where the last equality follows from the fact that $y \perp T(0)$ by Theorem 4.2. □

Lemma 4.5 *Let $\mathcal{T}$ be a sectorial linear relation in a Hilbert space $\mathcal{H}$ and assume that there exists a sectorial operator A in $\mathcal{H}$ with $D(A) = D(\mathcal{T})$ and $R(A) \subset \overline{D(A)}$ such that*

$$\mathcal{T}(x) = \overline{D(\mathcal{T})}^{\perp} + Ax \tag{4.5}$$

for all $x \in D(\mathcal{T}) = D(A)$. If $\widetilde{\mathcal{T}}$ is another sectorial linear relation in $\mathcal{H}$ such that $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$ and if (y, y') is a pair such that $(y, y') \in G(\widetilde{\mathcal{T}})$ but $(y, y') \notin G(\mathcal{T})$, then $y \in \overline{D(\mathcal{T})} \backslash D(\mathcal{T})$, the orthogonal complement of $D(\mathcal{T})$ in $\overline{D(\mathcal{T})}$.

Proof Since $\mathcal{T}$ is sectorial, Theorem 4.2 implies that $\mathcal{T}(0) \perp \overline{D(\mathcal{T})}$. Equality (4.5) together with the condition $A(0) = 0$ (A is linear) imply that

$$\mathcal{T}(0) = \overline{D(\mathcal{T})}^{\perp}. \tag{4.6}$$

Let (y, y') be a pair such that $(y, y') \in G(\widetilde{\mathcal{T}})$ but $(y, y') \notin G(\mathcal{T})$ and decompose y and y' as

$$y = y_1 + y_2, \qquad y' = y_1' + y_2',$$

where $y_1, y_1' \in \overline{D(\mathcal{T})}^{\perp}$ and $y_2, y_2' \in \overline{D(\mathcal{T})}$. Equality (4.6) implies that

$$y_1, y_1' \in \mathcal{T}(0) \subset \widetilde{\mathcal{T}}(0).$$

The linearity of $\widetilde{\mathcal{T}}(0)$ implies that $-y_1' \in \widetilde{\mathcal{T}}(0)$. Since $\widetilde{\mathcal{T}}(y) = \widetilde{\mathcal{T}}(0) + y'$ (see Theorem 3.3) and $-y_1' \in \widetilde{T}(0)$, it follows that $-y_1' + y' = y_2' \in \widetilde{\mathcal{T}}(y)$, that is,

$$\big(y, Py'\big) \in G(\widetilde{\mathcal{T}}), \tag{4.7}$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\overline{D(\mathcal{T})}$.

The sectoriality of $\widetilde{\mathcal{T}}$ combined with Theorem 4.2 imply that

$$D(\widetilde{\mathcal{T}}) \perp \widetilde{\mathcal{T}}(0) \supset \mathcal{T}(0),$$

and so

$$\langle y, y_1 \rangle = 0. \tag{4.8}$$

Since $y_1 \in \mathcal{T}(0)$ and $\mathcal{T}(0) \perp \overline{D(\mathcal{T})}$, equality (4.8) implies that $\langle y_1, y_1 \rangle = 0$, and so $y_1 = 0$. Hence

$$y = y_2 \in \overline{D(\mathcal{T})}. \tag{4.9}$$

It remains to show that

$$y \notin D(\mathcal{T}) = D(A). \tag{4.10}$$

We do this by contradiction.

First we note that (4.9) implies that $D(\widetilde{T}) \subset \overline{D(\mathcal{T})}$ and therefore $\overline{D(\widetilde{T})} \subset \overline{D(\mathcal{T})}$. Since $D(\mathcal{T}) \subset D(\widetilde{\mathcal{T}})$ we see that $\overline{D(\mathcal{T})} \subset \overline{D(\widetilde{\mathcal{T}})}$. Hence

$$\overline{D(\widetilde{\mathcal{T}})} = \overline{D(\mathcal{T})}. \tag{4.11}$$

To arrive at a contradiction, assume that (4.10) does not hold. Then $y \in D(\mathcal{T})$. Equality (4.5) means that

$$Ay \in T(y) \tag{4.12}$$

since $0 \in \overline{D(\mathcal{T})}^{\perp}$. Furthermore, we have the situation

$$Py' \neq Ay, \tag{4.13}$$

where as before P denotes the orthogonal projection of $\mathcal{H}$ onto $\overline{D(\mathcal{T})}$. Otherwise y' would be decomposed as

$$y' = \tilde{y} + Ay, \tag{4.14}$$

where $\tilde{y} \in \overline{D(\mathcal{T})}^{\perp}$. Equality (4.14) would then mean that $(y, y') \in G(\mathcal{T})$ (see (4.5)), which is not the case.

Now, condition (4.12) means that $(y, Ay) \in G(\mathcal{T})$ and that $(y, Ay) \in G(\widetilde{\mathcal{T}})$ since $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$. Since we also have that $(y, Py') \in G(\widetilde{T})$ (see (4.7)), the linearity of $\widetilde{\mathcal{T}}$ implies that

$$Ay - Py' \in \widetilde{\mathcal{T}}(y) - \widetilde{\mathcal{T}}(y) = \widetilde{\mathcal{T}}(y - y) = \widetilde{\mathcal{T}}(0).$$

It follows from (4.13) that $Ay - Py' \neq 0$. Since $R(A) \subset \overline{D(\mathcal{T})}$ it follows that the two conditions

$$Ay - Py' \in \widetilde{\mathcal{T}}(0) \tag{4.15}$$

and

$$0 \neq Ay - Py' \in \overline{D(\mathcal{T})} = \overline{D(\widetilde{\mathcal{T}})} \tag{4.16}$$

hold at the same time. The equality in (4.16) is obtained from (4.11). Theorem 4.2 implies that (4.15) and (4.16) cannot hold simultaneously since $\widetilde{\mathcal{T}}$ is assumed to be a sectorial linear relation. This contradiction shows that (4.10) holds and this concludes the proof. □

Theorem 4.6 *Let $\mathcal{T}$ be a linear relation in a Hilbert space $\mathcal{H}$ with domain $D(\mathcal{T})$.*

(i) *$\mathcal{T}$ is sectorial if and only if there exists a sectorial operator A in $\mathcal{H}$ with $D(A) = D(\mathcal{T})$ and $R(A) \subset \overline{D(A)}$ such that*

$$\mathcal{T}(x) \subset \overline{D(\mathcal{T})}^{\perp} + Ax \tag{4.17}$$

for all $x \in D(\mathcal{T})$.

(ii) *$\mathcal{T}$ is maximal sectorial if and only if the operator A is maximal sectorial in $\overline{D(\mathcal{T})}$ and*

$$\mathcal{T}(x) = \mathcal{T}(0) + Ax \tag{4.18}$$

for all $x \in D(\mathcal{T}) = D(A)$.

Proof (i) Let $\mathcal{T}$ be a sectorial linear relation in $\mathcal{H}$ with domain $D(\mathcal{T})$ and decompose $\mathcal{H}$ as

$$\mathcal{H} = \overline{D(\mathcal{T})}^{\perp} \oplus \overline{D(\mathcal{T})}. \tag{4.19}$$

Let $k \in D(\mathcal{T})$ and let $k' \in \mathcal{T}(k)$. Then k' can be decomposed as

$$k' = k_1' + k_2',$$

where $k_1' \in \overline{D(T)}^{\perp}$ and $k_2' \in \overline{D(T)}$. Theorem 3.3 implies that

$$T(k) = T(0) + k' = \left[T(0) + k_1'\right] + k_2'. \tag{4.20}$$

Let P be the orthogonal projection of $\mathcal{H}$ onto $\overline{D(\mathcal{T})}$. Since $\mathcal{T}(0) \subset \overline{D(\mathcal{T})}^{\perp}$ (see part (i) of Theorem 4.2) and $\overline{D(\mathcal{T})}^{\perp}$ is a linear subspace of $\mathcal{H}$, equality (4.20) shows that Px' is irrespective of the choice of $x' \in \mathcal{T}(k)$, that is, if $x_1', x_2' \in \mathcal{T}(k)$ then $Px_1' = Px_2'$. Armed with this fact, we define an operator A on $D(\mathcal{T})$ by

$$Ak = Pk', \quad k' \in \mathcal{T}(k). \tag{4.21}$$

Equality (4.21) implies that $R(A) \subset \overline{D(\mathcal{T})} = \overline{D(\mathcal{A})}$. The linearity of both $\mathcal{T}$ and P implies that A is a linear operator. Equality (4.20) then implies that (4.17) holds.

To show that A is sectorial, we show that the numerical ranges satisfy the equality

$$\Theta(A) = \Theta(\mathcal{T}). \tag{4.22}$$

The validity of (4.22) implies that the reverse implication holds. Now let $k \in D(\mathcal{T})$ with $\|k\| = 1$ and let $\tilde{k} \in \mathcal{T}(k)$. Relation (4.17) implies that $\tilde{k}$ can be decomposed as

$$\tilde{k} = \hat{k} + Ak \tag{4.23}$$

for some $\hat{k} \in \overline{D(\mathcal{T})}^{\perp}$. Hence

$$\langle \tilde{k}, k\rangle = \langle \hat{k} + Ak, k\rangle = \langle Ak, k\rangle. \tag{4.24}$$

The equality $\Theta(\mathcal{T}) = \Theta(A)$ then follows from (4.24) and Lemma 4.4. This shows that the operator A is sectorial.

(ii) Now assume that $\mathcal{T}$ is a maximal sectorial linear relation in $\mathcal{H}$ with domain $D(\mathcal{T})$. Theorem 4.2 implies that $\mathcal{T}(0) = \overline{D(\mathcal{T})}^{\perp}$. In this case decomposition (4.19) becomes

$$\mathcal{H} = \mathcal{T}(0) \oplus \overline{D(\mathcal{T})}. \tag{4.25}$$

Equality (4.20) can now be reformulated as

$$\mathcal{T}(k) = \mathcal{T}(0) + k' = \left[\mathcal{T}(0) + k_1'\right] + k_2' = \mathcal{T}(0) + k_2'. \tag{4.26}$$

The last equality in (4.26) is a result of the fact that $\mathcal{T}(0)$ is a linear subspace of $\mathcal{H}$. With A as defined in part (i) above, we see that (4.18) holds in this case.

We now consider the maximality question. If the operator A would have a proper sectorial extension $\widetilde{A}$ in $\overline{D(A)}$, then $\widetilde{A}$ would generate a proper sectorial extension $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ with $\Theta(\widetilde{\mathcal{T}}) = \Theta(\widetilde{A})$ defined by

$$\widetilde{\mathcal{T}}(x) = \mathcal{T}(0) + \widetilde{A}(x), \quad x \in \overline{D(\mathcal{T})},$$

contradicting the maximality of $\mathcal{T}$. Hence if $\mathcal{T}$ is a maximal sectorial linear relation in $\mathcal{H}$, then the operator A is maximal sectorial in $\overline{D(\mathcal{T})}$.

Now let us assume that an operator A with the stated properties does exist and suppose that $\mathcal{T}$ is not maximal sectorial in $\mathcal{H}$. Then there exists a sectorial linear relation $\widetilde{\mathcal{T}}$ in $\mathcal{H}$ such that $G(\mathcal{T}) \subset G(\widetilde{\mathcal{T}})$. Let (y, y') be a pair such that $(y, y') \in G(\widetilde{\mathcal{T}})$ but $(y, y') \notin G(\mathcal{T})$. As in the proof of Lemma 4.5, it can be shown that

$$\left(y, Py'\right) \in G(\widetilde{\mathcal{T}}), \tag{4.27}$$

where P is the orthogonal projection of $\mathcal{H}$ onto $\overline{D(\mathcal{T})}$. Theorem 3.3 implies that we can express $\widetilde{\mathcal{T}}(y)$ as

$$\widetilde{\mathcal{T}}(y) = \widetilde{\mathcal{T}}(0) + Py'. \tag{4.28}$$

Since $\widetilde{\mathcal{T}}$ is sectorial, it follows that $\widetilde{\mathcal{T}}(0) \perp D(\widetilde{\mathcal{T}})$ and so (4.28) implies that

$$\langle y', y \rangle = \langle Py', y \rangle. \tag{4.29}$$

Let $\overline{D(\mathcal{T})} \backslash D(\mathcal{T})$ denote the complement of $D(\mathcal{T})$ in $\overline{D(\mathcal{T})}$. Lemma 4.5 implies that

$$y \in \overline{D(\mathcal{T})} \backslash D(\mathcal{T}). \tag{4.30}$$

Condition (4.30) shows that $D(\widetilde{\mathcal{T}}) \subset \overline{D(\mathcal{T})}$. Define an operator $B : D(\widetilde{\mathcal{T}}) \to \overline{D(\mathcal{T})}$ by setting $By = Py'$. Since $R(A) \subset \overline{D(\mathcal{T})}$ and $\mathcal{T}(0) \perp \overline{D(\mathcal{T})}$, (4.18) ensures that B is well defined on $D(\mathcal{T})$ and coincides with A on this domain while (4.28) ensures that it is well defined on the whole of $D(\widetilde{\mathcal{T}})$. Since $\widetilde{\mathcal{T}}$ is sectorial, (4.29) implies that the operator B is also sectorial. Hence B is a sectorial extension of A. This contradicts the maximality of A. This contradiction implies that $\mathcal{T}$ is a maximal sectorial linear relation in $\mathcal{H}$. □

Theorem 4.7 *Let $\mathcal{T}$ be a sectorial linear relation in a Hilbert space $\mathcal{H}$ with domain $D(\mathcal{T})$. Then $\mathcal{T}$ is form closable, that is, the form*

$$t(u,v) = \langle u', v \rangle, \quad u' \in \mathcal{T}(u), u, v \in D(\mathcal{T}) \tag{4.31}$$

is closable.

Note that the form t given by (4.31) is well defined since the inner product $\langle u', v \rangle$ is independent of the choice of the vector $u' \in \mathcal{T}(u)$ by Lemma 4.4.

Proof Let $\mathcal{T}$ be a sectorial linear relation in $\mathcal{H}$ and let $u, v \in D(\mathcal{T})$. Theorem 4.6 guarantees the existence of a sectorial operator

$$A : D(\mathcal{T}) \to \overline{D(\mathcal{T})}$$

such that for every $u \in D(\mathcal{T})$ and every $u' \in \mathcal{T}(u)$, there exists a vector $w \in \overline{D(\mathcal{T})}^{\perp}$ such that

$$u' = w + Au. \tag{4.32}$$

Equality (4.32) implies that for every $v \in D(\mathcal{T})$,

$$\langle u', v \rangle = \langle Au, v \rangle.$$

The conclusion that the form t is closable then follows from Theorem 2.2. □

In the next theorem we show that every closed sectorial sesquilinear form in a Hilbert space $\mathcal{H}$ has an associated sectorial linear relation in $\mathcal{H}$.

Theorem 4.8 *Let t be a non-densely defined closed sectorial sesquilinear form in a Hilbert space $\mathcal{H}$ with domain $D(t)$.*

(a) (i) *There exists a sectorial linear relation $\mathcal{T}$ in $\mathcal{H}$ such that $D(\mathcal{T}) \subset D(t)$, $\mathcal{T}(0) = \overline{D(t)}^{\perp}$ and*

$$t(u,v) = \langle u', v\rangle \tag{4.33}$$

for every $u \in D(\mathcal{T})$ and $v \in D(t)$, where u' is an arbitrary vector in $\mathcal{T}(u)$.

(ii) *If $u \in D(t)$, $w \in \overline{D(t)}$ and $t(u,v) = \langle w, v\rangle$ holds for every v belonging to a core of t, then $u \in D(\mathcal{T})$ and*

$$\mathcal{T}(u) = \overline{D(t)}^{\perp} + w.$$

(iii) *The linear relation $\mathcal{T}$ in* (*i*) *is unique.*

(b) *If t is bounded, the relation $\mathcal{T}$ in* (a) *is maximal sectorial.*

Proof Let $\mathcal{H}$ be a Hilbert space and let t be a closed sesquilinear form defined in $\mathcal{H}$ with domain $D(t)$. Since t is densely defined in $\overline{D(t)}$, Theorem 2.4(i) implies that there exists a maximal sectorial operator A in $\overline{D(t)}$ such that $D(A) \subset D(t)$ and $t(u,v) = \langle Au, v\rangle$ for every $u \in D(A)$ and every $v \in D(t)$. Define a linear relation $\mathcal{T}$ on $D(\mathcal{T}) = D(A)$ by

$$\mathcal{T}(x) = \overline{D(t)}^{\perp} + Ax, \quad x \in D(A). \tag{4.34}$$

The linearity of the relation $\mathcal{T}$ follows from the fact that $\overline{D(t)}^{\perp}$ is a linear subspace of $\mathcal{H}$ and that A is linear. The linearity of A also implies that $\mathcal{T}(0) = \overline{D(t)}^{\perp}$. Definition (4.34) shows that $\mathcal{T}$ inherits the sectoriality from A while the condition $D(\mathcal{T}) \subset D(t)$ is a consequence of the inclusion $D(\mathcal{T}) = D(A) \subset D(t)$. Equality (4.33) is obtained using the fact that for $u \in D(\mathcal{T}) = D(A)$, $u' \in \mathcal{T}(u)$ and $v \in D(t)$,

$$\langle u', v\rangle = \langle Au, v\rangle = t(u,v). \tag{4.35}$$

The first equality in (4.35) is a consequence of (4.34). This proves (a)(i).

Part (a)(ii) follows immediately from Theorem 2.4(iii) and (4.34). To prove the uniqueness assertion in part (a)(iii), we let $\mathcal{S}$ be another sectorial linear relation in $\mathcal{H}$ with $D(\mathcal{S}) \subset D(t)$, $\mathcal{S}(0) = \overline{D(t)}^{\perp}$ and such that

$$t(u,v) = \langle u', v\rangle \tag{4.36}$$

for every $u \in D(\mathcal{S})$ and $v \in D(t)$, where u' is an arbitrary vector in $\mathcal{S}(u)$. Part (a)(ii) of the theorem then implies that $u \in D(\mathcal{T})$ and

$$\mathcal{T}(u) = \overline{D(t)}^{\perp} + u'. \tag{4.37}$$

Equality (4.37) implies that $(u,u') \in G(\mathcal{T})$ and that $G(\mathcal{S}) \subset G(\mathcal{T})$. Since $\mathcal{S}(0) = \mathcal{T}(0)$ it follows that $\mathcal{T}$ is an extension of $\mathcal{S}$.

To prove part (b) of the theorem, note that if t is bounded then Remark 2.3 implies that $\overline{D(A)} = D(t)$ since $D(A)$ is a core of t by part (ii) of Theorem 2.4. In this case (4.34) can be written in the form

$$\mathcal{T}(x) = \overline{D(A)}^{\perp} + Ax, \quad x \in D(A). \tag{4.38}$$

The maximality of $\mathcal{T}$ defined by (4.38) follows from part (ii) of Theorem 4.6. This concludes the proof of the theorem. □

Competing interests
The author declares that he has no competing interests.

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