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Generalized weighted composition operators on Bloch-type spaces

Xiangling Zhu*

*Correspondence: jyuzxl@163.com
Faculty of Information Technology, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau, China

Abstract
In this paper, we give three different characterizations for the boundedness and compactness of generalized weighted composition operators on Bloch-type spaces, especially we characterize them in terms of the sequence of Bloch-type norms of the generalized weighted composition operator applied to the functions $I^j(z) = z^j$.

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1 Introduction

Let $\mathbb{D}$ be an open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$, the Bloch-type space (or α-Bloch space) $\mathcal{B}^\alpha$ is the space that consists of all analytic functions f on $\mathbb{D}$ such that

$$B_\alpha(f) = \sup_{z\in\mathbb{D}}\big(1-|z|^2\big)^\alpha \big|f'(z)\big| < \infty.$$

$\mathcal{B}^\alpha$ becomes a Banach space under the norm $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B_\alpha(f)$. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well-known Bloch space. See [1, 2] for more information on Bloch-type spaces.

Throughout this paper, φ denotes a nonconstant analytic self-map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined by $C_\varphi f = f\circ\varphi$ for $f \in H(\mathbb{D})$. For a fixed $u \in H(\mathbb{D})$, define a linear operator uC_φ as follows:

$$uC_\varphi f = u(f\circ\varphi), \quad f \in H(\mathbb{D}).$$

The operator uC_φ is called the weighted composition operator. The weighted composition operator is a generalization of the composition operator and the multiplication operator defined by $M_u f = uf$.

A basic problem concerning composition operators on various Banach function spaces is to relate the operator theoretic properties of C_φ to the function theoretic properties of the symbol φ, which attracted a lot of attention recently; the reader can refer to [3].

The differentiation operator D is defined by $Df = f'$, $f \in H(\mathbb{D})$. For a nonnegative integer n, we define

$$\big(D^0 f\big)(z) = f(z), \qquad \big(D^n f\big)(z) = f^{(n)}(z), \quad n \ge 1, f \in H(\mathbb{D}).$$

Let φ be an analytic self-map of $\mathbb{D}$, $u \in H(\mathbb{D})$, and let n be a nonnegative integer. Define the linear operator $D^n_{\varphi,u}$, called the generalized weighted composition operator, by (see [4–6])

$$\left(D^n_{\varphi,u}f\right)(z) = u(z) \cdot \left(D^n f\right)\left(\varphi(z)\right), \quad f \in H(\mathbb{D}), z \in \mathbb{D}.$$

When $n = 0$ and $u(z) = 1$, $D^n_{\varphi,u}$ is the composition operator C_φ. If $n = 0$, then $D^n_{\varphi,u}$ is the weighted composition operator uC_φ. If $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$, which was studied in [7–10]. For $u(z) = 1$, $D^n_{\varphi,u} = C_\varphi D^n$, which was studied in [7, 11–14]. For the study of the generalized weighted composition operator on various function spaces, see, for example, [4–6, 15–19].

It is well known that the composition operator is bounded on the Bloch space by the Schwarz-Pick lemma. Composition operators and weighted composition operators on Bloch-type spaces were studied, for example, in [20–28]. The product-type operators on or into Bloch-type spaces have been studied in many papers recently, see [7–11, 13, 14, 18, 29–36] for example. In [27], Wulan *et al.* obtained a characterization for the compactness of the composition operators acting on the Bloch space as follows.

Theorem A *Let φ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi : \mathcal{B} \to \mathcal{B}$ is compact if and only if*

$$\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = 0.$$

In [14], Wu and Wulan obtained two characterizations for the compactness of the product of differentiation and composition operators acting on the Bloch space as follows.

Theorem B *Let φ be an analytic self-map of $\mathbb{D}$, $n \in \mathbb{N}$. Then the following statements are equivalent.*

(a) *$C_\varphi D^n : \mathcal{B} \to \mathcal{B}$ is compact.*
(b) *$\lim_{j\to\infty} \|C_\varphi D^n I^j\|_{\mathcal{B}} = 0$, where $I^j(z) = z^j$.*
(c) *$\lim_{|a|\to 1} \|C_\varphi D^n \sigma_a(z)\|_{\mathcal{B}} = 0$, where $\sigma_a(z) = (a - z)/(1 - \overline{a}z)$ is the Möbius map on $\mathbb{D}$.*

Motivated by Theorems A and B, in this work we show that $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded (respectively, compact) if and only if the sequence $(j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta})^\infty_{j=n}$ is bounded (respectively, convergent to 0 as $j \to \infty$), where $I^j(z) = z^j$. Moreover, we use two families of functions to characterize the boundedness and compactness of the operator $D^n_{\varphi,u}$.

Throughout the paper, we denote by C a positive constant which may differ from one occurrence to the next. In addition, we say that $A \preceq B$ if there exists a constant C such that $A \leq CB$. The symbol $A \approx B$ means that $A \preceq B \preceq A$.

2 Main results and proofs

In this section, we give our main results and proofs. First we characterize the boundedness of the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$.

Theorem 1 *Let n be a positive integer, $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements are equivalent.*

(a) *The operator* $D^n_{\varphi,u}:\mathcal{B}^\alpha\to\mathcal{B}^\beta$ *is bounded.*
(b) $\sup_{j\geq n} j^{\alpha-1}\|D^n_{\varphi,u}I^j(z)\|_{\mathcal{B}^\beta}<\infty$, *where* $I^j(z)=z^j$.
(c) $u\in\mathcal{B}^\beta$, $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)||\varphi'(z)|<\infty$ *and*

$$\sup_{a\in\mathbb{D}}\big\|D^n_{\varphi,u}f_a\big\|_{\mathcal{B}^\beta}<\infty,\qquad \sup_{a\in\mathbb{D}}\big\|D^n_{\varphi,u}h_a\big\|_{\mathcal{B}^\beta}<\infty,$$

where

$$f_a(z)=\frac{1-|a|^2}{(1-\overline{a}z)^\alpha}\quad \textit{and}\quad h_a(z)=\frac{(1-|a|^2)^2}{(1-\overline{a}z)^{\alpha+1}},\quad z\in\mathbb{D}.$$

(d)

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}}<\infty\quad \textit{and}\quad \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}<\infty.$$

Proof (a) $\Rightarrow$ (b) This implication is obvious, since for $j\in\mathbb{N}$, the function $j^{\alpha-1}I^j$ is bounded in $\mathcal{B}^\alpha$ and $j^{\alpha-1}\|I^j\|_{\mathcal{B}^\alpha}\approx 1$.

(b) $\Rightarrow$ (c) Assume that (b) holds and let $Q=\sup_{j\geq n} j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta}$. For any $a\in\mathbb{D}$, it is easy to see that f_a and h_a have bounded norms in $\mathcal{B}^\alpha$. It is clear that

$$f_a(z)=\big(1-|a|^2\big)\sum_{j=0}^\infty\frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)}\overline{a}^jz^j,$$

$$h_a(z)=\big(1-|a|^2\big)^2\sum_{j=0}^\infty\frac{\Gamma(j+1+\alpha)}{j!\Gamma(\alpha+1)}\overline{a}^jz^j.$$

By Stirling's formula, we have $\frac{\Gamma(j+\alpha)}{j!\Gamma(\alpha)}\approx j^{\alpha-1}$ as $j\to\infty$. Using linearity we get

$$\big\|D^n_{\varphi,u}f_a\big\|_{\mathcal{B}^\beta}\leq C\big(1-|a|^2\big)\sum_{j=0}^\infty|a|^jj^{\alpha-1}\big\|D^n_{\varphi,u}I^j\big\|_{\mathcal{B}^\beta}\preceq Q\quad\text{and}$$

$$\big\|D^n_{\varphi,u}h_a\big\|_{\mathcal{B}^\beta}\leq C\big(1-|a|^2\big)^2\sum_{j=0}^\infty(j+1)|a|^jj^{\alpha-1}\big\|D^n_{\varphi,u}I^j\big\|_{\mathcal{B}^\beta}\preceq Q.$$

Therefore, by the arbitrariness of $a\in\mathbb{D}$,

$$\sup_{a\in\mathbb{D}}\big\|D^n_{\varphi,u}f_a\big\|_{\mathcal{B}^\beta}<\infty,\qquad \sup_{a\in\mathbb{D}}\big\|D^n_{\varphi,u}h_a\big\|_{\mathcal{B}^\beta}<\infty.$$

In addition, applying the operator $D^n_{\varphi,u}$ to I^j with $j=n,n+1$, we obtain

$$\big(D^n_{\varphi,u}I^n\big)'(z)=u'(z)n!\quad\text{and}$$

$$\big(D^n_{\varphi,u}I^{n+1}\big)'(z)=u'(z)(n+1)!\varphi(z)+u(z)(n+1)!\varphi'(z),$$

while for $j<n$, $(D^n_{\varphi,u}I^j)'(z)=0$. Thus, using the boundedness of the function φ, we have $u\in\mathcal{B}^\beta$ and $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)||\varphi'(z)|<\infty$.

(c) ⇒ (d) Assume that (c) holds. Let

$$C_1 := \sup_{a\in\mathbb{D}} \big\| D^n_{\varphi,u} f_a \big\|_{\mathcal{B}^\beta}, \qquad C_2 := \sup_{a\in\mathbb{D}} \big\| D^n_{\varphi,u} h_a \big\|_{\mathcal{B}^\beta}.$$

For $w \in \mathbb{D}$, set

$$g_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^\alpha} - \frac{\alpha}{\alpha+n}\frac{(1-|w|^2)^2}{(1-\overline{w}z)^{\alpha+1}}, \quad w \in \mathbb{D}.$$

It is easy to check that $g_w \in \mathcal{B}^\alpha$, $\|g_w\|_{\mathcal{B}^\alpha} < \infty$ for every $w \in \mathbb{D}$. Moreover,

$$\begin{aligned}\sup_{w\in\mathbb{D}} \big\| D^n_{\varphi,u} g_w \big\|_{\mathcal{B}^\beta} &\le \sup_{w\in\mathbb{D}} \big\| D^n_{\varphi,u} f_w \big\|_{\mathcal{B}^\beta} + \frac{\alpha}{\alpha+n} \sup_{w\in\mathbb{D}} \big\| D^n_{\varphi,u} h_w \big\|_{\mathcal{B}^\beta} \\ &\le C_1 + \frac{\alpha}{\alpha+n} C_2 < \infty.\end{aligned}$$

In addition,

$$g^{(n)}_{\varphi(\lambda)}\big(\varphi(\lambda)\big) = 0, \qquad \big| g^{(n+1)}_{\varphi(\lambda)}\big(\varphi(\lambda)\big)\big| = \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{|\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{\alpha+n}}.$$

It follows that

$$\begin{aligned}C_1 + \frac{\alpha}{\alpha+n} C_2 &> \big\| D^n_{\varphi,u} g_{\varphi(\lambda)} \big\|_{\mathcal{B}^\beta} \\ &\ge \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{\alpha+n}} \end{aligned} \tag{2.1}$$

for any $\lambda \in \mathbb{D}$. For any fixed $r \in (0,1)$, from (2.1) we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|>r} \frac{(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha+n}} &\le \sup_{|\varphi(\lambda)|>r} \frac{1}{r^{n+1}} \frac{(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{\alpha+n}} \\ &\le \frac{C_1 + \frac{\alpha}{\alpha+n} C_2}{r^{n+1}\alpha(\alpha+1)\cdots(\alpha+n-1)} < \infty. \end{aligned} \tag{2.2}$$

From the assumption that $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta |u(z)||\varphi'(z)| < \infty$, we get

$$\sup_{|\varphi(\lambda)|\le r} \frac{(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha+n}} \le \frac{\sup_{|\varphi(\lambda)|\le r}(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)|}{(1-r^2)^{\alpha+n}} < \infty. \tag{2.3}$$

Therefore, (2.2) and (2.3) yield the first inequality of (d).

Next, note that

$$\begin{aligned}C_1 &\ge \big\| D^n_{\varphi,u} f_{\varphi(\lambda)} \big\|_{\mathcal{B}^\beta} \\ &\ge \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{(1-|\lambda|^2)^\beta |u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\alpha+n-1}} \\ &\quad - \alpha(\alpha+1)\cdots(\alpha+n)\frac{(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{\alpha+n}}\end{aligned}$$

for any $\lambda \in \mathbb{D}$. From (2.1) we get

$$\begin{aligned}
&\frac{(1-|\lambda|^2)^\beta |u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\alpha+n-1}} \\
&\quad \le \frac{\|D^n_{\varphi,u} f_{\varphi(\lambda)}\|_{\mathcal{B}^\beta}}{\alpha(\alpha+1)\cdots(\alpha+n-1)} + \frac{(\alpha+n)(1-|\lambda|^2)^\beta |u(\lambda)||\varphi'(\lambda)||\varphi(\lambda)|^{n+1}}{(1-|\varphi(\lambda)|^2)^{\alpha+n}} \\
&\quad \le \frac{C_1}{\alpha(\alpha+1)\cdots(\alpha+n-1)} + \frac{(\alpha+n)C_1 + \alpha C_2}{\alpha(\alpha+1)\cdots(\alpha+n-1)} \\
&\quad \le \frac{(\alpha+n+1)C_1 + \alpha C_2}{\alpha(\alpha+1)\cdots(\alpha+n-1)}.
\end{aligned}$$

By arbitrary $\lambda \in \mathbb{D}$, we get

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)^\beta |u'(\lambda)||\varphi(\lambda)|^n}{(1-|\varphi(\lambda)|^2)^{\alpha+n-1}} < \infty. \tag{2.4}$$

Combining (2.4) with the fact that $u \in \mathcal{B}^\beta$, similarly to the former proof, we get the second inequality of (d).

(d) $\Rightarrow$ (a) For any $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
&\left(1-|z|^2\right)^\beta \left|\left(D^n_{\varphi,u} f\right)'(z)\right| \\
&\quad = \left(1-|z|^2\right)^\beta \left|\left(f^{(n)}(\varphi)u\right)'(z)\right| \\
&\quad \le \left(1-|z|^2\right)^\beta |u(z)|\left|\varphi'(z)\right|\left|f^{(n+1)}\left(\varphi(z)\right)\right| + \left(1-|z|^2\right)^\beta \left|u'(z)\right|\left|f^{(n)}\left(\varphi(z)\right)\right| \\
&\quad \le C\frac{(1-|z|^2)^\beta |u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}}\|f\|_{\mathcal{B}^\alpha} + C\frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}\|f\|_{\mathcal{B}^\alpha},
\end{aligned} \tag{2.5}$$

where in the last inequality we used the fact that for $f \in \mathcal{B}^\alpha$ (see [2])

$$\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\alpha \left|f'(z)\right| \asymp \left|f'(0)\right| + \cdots + \left|f^{(n)}(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^{\alpha+n}\left|f^{(n+1)}(z)\right|.$$

Moreover

$$\left|\left(D^n_{\varphi,u} f\right)(0)\right| = \left|f^{(n)}\left(\varphi(0)\right)u(0)\right| \le \frac{|u(0)|}{(1-|\varphi(0)|^2)^{\alpha+n-1}}\|f\|_{\mathcal{B}^\alpha}.$$

From (d) we see that

$$\left\|D^n_{\varphi,u} f\right\|_{\mathcal{B}^\beta} = \left|\left(D^n_{\varphi,u} f\right)(0)\right| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^\beta \left|\left(D^n_{\varphi,u} f\right)'(z)\right| < \infty.$$

Therefore the operator $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. The proof is complete. □

For the study of the compactness of $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, we need the following lemma, which can be proved in a standard way; see, for example, Proposition 3.11 in [3].

Lemma 2 *Let n be a positive integer, $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and for any* *bounded sequence* $(f_j)_{j\in\mathbb{N}}$ *in* $\mathcal{B}^\alpha$ *which converges to zero uniformly on compact subsets of* $\mathbb{D}$, $\|D^n_{\varphi,u}f_j\|_{\mathcal{B}^\beta} \to 0$ *as* $j \to \infty$.

Theorem 3 *Let* n *be a positive integer*, $0 < \alpha, \beta < \infty$, $u \in H(\mathbb{D})$ *and* φ *be an analytic self-map of* $\mathbb{D}$ *such that* $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is bounded. Then the following statements are equivalent.*

(a) $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ *is compact.*
(b) $\lim_{j\to\infty} j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta} = 0$, *where* $I^j(z) = z^j$.
(c) $\lim_{|\varphi(a)|\to1} \|D^n_{\varphi,u}f_{\varphi(a)}\|_{\mathcal{B}^\beta} = 0$ *and* $\lim_{|\varphi(a)|\to1} \|D^n_{\varphi,u}h_{\varphi(a)}\|_{\mathcal{B}^\beta} = 0$.
(d)

$$\lim_{|\varphi(z)|\to1} \frac{(1-|z|^2)^\beta|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+\alpha}} = 0 \quad \textit{and} \quad \lim_{|\varphi(z)|\to1} \frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{n+\alpha-1}} = 0.$$

Proof (a) ⇒ (b) Assume that $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact. Since the sequence $\{j^{\alpha-1}I^j\}$ is bounded in $\mathcal{B}^\alpha$ and converges to 0 uniformly on compact subsets, by Lemma 2 it follows that $j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta} \to 0$ as $j \to \infty$.

(b) ⇒ (c) Suppose that (b) holds. Fix $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta} < \varepsilon$ for all $j \ge N$. Let $z_k \in \mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Arguing as in the proof of Theorem 1, we have

$$\begin{aligned}
&\bigl\|D^n_{\varphi,u}f_{\varphi(z_k)}\bigr\|_{\mathcal{B}^\beta} \\
&\quad\le C\bigl(1-\bigl|\varphi(z_k)\bigr|^2\bigr)\sum_{j=0}^{\infty}\bigl|\varphi(z_k)\bigr|^j j^{\alpha-1}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta} \\
&\quad= C\bigl(1-\bigl|\varphi(z_k)\bigr|^2\bigr)\left(\sum_{j=0}^{N-1}\bigl|\varphi(z_k)\bigr|^j j^{\alpha-1}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta} + \sum_{j=N}^{\infty}\bigl|\varphi(z_k)\bigr|^j j^{\alpha-1}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta}\right) \\
&\quad\le CQ\bigl(1-\bigl|\varphi(z_k)\bigr|^N\bigr) + C\varepsilon,
\end{aligned}$$

where $Q = \sup_{j\ge n} j^{\alpha-1}\|D^n_{\varphi,u}I^j\|_{\mathcal{B}^\beta}$. Since $|\varphi(z_k)| \to 1$ as $k \to \infty$, from the last inequality and the arbitrariness of ε, we get $\lim_{k\to\infty} \|D^n_{\varphi,u}f_{\varphi(z_k)}\|_{\mathcal{B}^\beta} = 0$, *i.e.*, $\lim_{|\varphi(a)|\to1} \|D^n_{\varphi,u}f_{\varphi(a)}\|_{\mathcal{B}^\beta} = 0$.

Notice that

$$\sum_{j=0}^{N-1}(j+1)r^j = \frac{1-r^N-Nr^N(1-r)}{(1-r)^2}, \quad 0 \le r < 1,$$

arguing as in the proof of Theorem 1, we get

$$\begin{aligned}
\bigl\|D^n_{\varphi,u}h_{\varphi(z_k)}\bigr\|_{\mathcal{B}^\beta} &\le C\bigl(1-\bigl|\varphi(z_k)\bigr|^2\bigr)^2\sum_{j=0}^{\infty}\bigl|\varphi(z_k)\bigr|^j j^{\alpha}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta} \\
&\le C\bigl(1-\bigl|\varphi(z_k)\bigr|^2\bigr)^2\sum_{j=0}^{N-1}(j+1)\bigl|\varphi(z_k)\bigr|^j j^{\alpha-1}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta} \\
&\quad + C\bigl(1-\bigl|\varphi(z_k)\bigr|^2\bigr)^2\sum_{j=N}^{\infty}(j+1)\bigl|\varphi(z_k)\bigr|^j j^{\alpha-1}\bigl\|D^n_{\varphi,u}I^j\bigr\|_{\mathcal{B}^\beta} \\
&\le C(1-\bigl|\varphi(z_k)\bigr|^N - N\bigl|\varphi(z_k)\bigr|^N\bigl(1-\bigl|\varphi(z_k)\bigr|\bigr) + C\varepsilon.
\end{aligned}$$

Therefore, $\lim_{k\to\infty} \|D^n_{\varphi,u} h_{\varphi(z_k)}\|_{\mathcal{B}^\beta} \le C\varepsilon$. By the arbitrariness of ε, we obtain the desired result.

(c) ⇒ (d) To prove (d) we only need to show that if $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$, then

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n}} = 0, \qquad \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}} = 0.$$

Let $(z_k)_{k\in\mathbb{N}}$ be such a sequence that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Arguing as in the proof of Theorem 1, we obtain

$$\lim_{k\to\infty} \left\| D^n_{\varphi,u} g_{\varphi(z_k)} \right\|_{\mathcal{B}^\beta} \le \lim_{k\to\infty} \left\| D^n_{\varphi,u} f_{\varphi(z_k)} \right\|_{\mathcal{B}^\beta} + \frac{\alpha}{n+\alpha} \lim_{k\to\infty} \left\| D^n_{\varphi,u} h_{\varphi(z_k)} \right\|_{\mathcal{B}^\beta} = 0.$$

Hence $\lim_{k\to\infty} \|D^n_{\varphi,u} g_{\varphi(z_k)}\|_{\mathcal{B}^\beta} = 0$. Similarly to the proof of Theorem 1, we have

$$\frac{n!(1-|z_k|^2)^\beta |u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{\alpha+n}} \le \left\| D^n_{\varphi,u} g_{\varphi(z_k)} \right\|_{\mathcal{B}^\beta} \to 0 \quad \text{as } k \to \infty,$$

which implies

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u(z_k)||\varphi'(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha+n}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{\alpha+n}} = 0. \tag{2.6}$$

In addition,

$$\begin{aligned} &\left\| D^n_{\varphi,u} f_{\varphi(z_k)} \right\|_{\mathcal{B}^\beta} + \frac{(n+1)!(1-|z_k|^2)^\beta |u(z_k)||\varphi'(z_k)||\varphi(z_k)|^{n+1}}{(1-|\varphi(z_k)|^2)^{\alpha+n}} \\ &\quad \ge \frac{n!(1-|z_k|^2)^\beta |u'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}}. \end{aligned}$$

From (2.6) and the assumption that $\|D^n_{\varphi,u} f_{\varphi(z_k)}\|_{\mathcal{B}^\beta} \to 0$ as $k \to \infty$, we have

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u'(z_k)|}{(1-|\varphi(z_k)|^2)^n} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |u'(z_k)||\varphi(z_k)|^n}{(1-|\varphi(z_k)|^2)^{\alpha+n-1}} = 0,$$

as desired.

(d) ⇒ (a) Assume that $(f_k)_{k\in\mathbb{N}}$ is a bounded sequence in $\mathcal{B}^\alpha$ converging to 0 uniformly on compact subsets of $\mathbb{D}$. By the assumption, for any $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that

$$\frac{(1-|z|^2)^\beta |\varphi'(z)||u(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}} < \varepsilon \quad \text{and} \quad \frac{(1-|z|^2)^\beta |u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}} < \varepsilon \tag{2.7}$$

when $\delta < |\varphi(z)| < 1$. Suppose that $D^n_{\varphi,u} : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, by Theorem 1, we have

$$C_3 = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left|u'(z)\right| < \infty \tag{2.8}$$

and

$$C_4 = \sup_{z\in\mathbb{D}} \left(1-|z|^2\right)^\beta \left|u(z)\right| \left|\varphi'(z)\right| < \infty. \tag{2.9}$$

Let $K = \{z \in \mathbb{D} : |\varphi(z)| \leq \delta\}$. Then by (2.8) and (2.9) we have that

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\bigl(1-|z|^2\bigr)^\beta\bigl|\bigl(D^n_{\varphi,u}f_k\bigr)'(z)\bigr| \\
&\quad\leq \sup_{z\in K}\bigl(1-|z|^2\bigr)^\beta\bigl|u(z)\bigr|\bigl|\varphi'(z)\bigr|\bigl|f_k^{(n+1)}\bigl(\varphi(z)\bigr)\bigr| + \sup_{z\in K}\bigl(1-|z|^2\bigr)^\beta\bigl|u'(z)\bigr|\bigl|f_k^{(n)}\bigl(\varphi(z)\bigr)\bigr| \\
&\qquad + C\sup_{z\in\mathbb{D}\setminus K}\frac{(1-|z|^2)^\beta|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n}}\|f_k\|_{\mathcal{B}^\alpha} + C\sup_{z\in\mathbb{D}\setminus K}\frac{(1-|z|^2)^\beta|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+n-1}}\|f_k\|_{\mathcal{B}^\alpha} \\
&\quad\leq C_4\sup_{z\in K}\bigl|f_k^{(n+1)}\bigl(\varphi(z)\bigr)\bigr| + C_3\sup_{z\in K}\bigl|f_k^{(n)}\bigl(\varphi(z)\bigr)\bigr| + C\varepsilon\|f_k\|_{\mathcal{B}^\alpha},
\end{aligned}$$

i.e., we get

$$\begin{aligned}
\bigl\|D^n_{\varphi,u}f_k\bigr\|_{\mathcal{B}^\beta} &= C_4\sup_{|w|\leq\delta}\bigl|f_k^{(n+1)}(w)\bigr| + C_3\sup_{|w|\leq\delta}\bigl|f_k^{(n)}(w)\bigr| \\
&\quad + C\varepsilon\|f_k\|_{\mathcal{B}^\alpha} + \bigl|u(0)\bigr|\bigl|f_k^{(n)}\bigl(\varphi(0)\bigr)\bigr|. \qquad (2.10)
\end{aligned}$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, Cauchy's estimate gives that $f_k^{(n)} \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Hence, letting $k \to \infty$ in (2.10) and using the fact that ε is an arbitrary positive number, we obtain $\|D^n_{\varphi,u}f_k\|_{\mathcal{B}^\beta} \to 0$ as $k \to \infty$. Applying Lemma 2 the result follows. □

Competing interests
The author declares that they have no competing interests.

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