Grinblat *Journal of Inequalities and Applications* (2015) 2015:116
DOI 10.1186/s13660-015-0578-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Families of sets not belonging to algebras and combinatorics of finite sets of ultrafilters

Leonid Š Grinblat*

*Correspondence:
grinblat@ariel.ac.il
Department of Computer Science and Mathematics, Ariel University, Ariel, Israel

Abstract

This article is a part of the theory developed by the author in which the following problem is solved under natural assumptions: to find necessary and sufficient conditions under which the union of at most countable family of algebras on a certain set X is equal to $\mathcal{P}(X)$. Here the following new result is proved. Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a finite collection of algebras of sets given on a set X with $\#(\Lambda) = n > 0$, and for each λ there exist at least $\frac{10}{3}n + \sqrt{\frac{2n}{3}}$ pairwise disjoint sets belonging to $\mathcal{P}(X) \setminus \mathcal{A}_\lambda$. Then there exists a family $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$ of pairwise disjoint subsets of X ($U^i_\lambda \cap U^j_{\lambda'} = \emptyset$ except the case $\lambda = \lambda'$, $i = j$); and for each λ the following holds: if $Q \in \mathcal{P}(X)$ and Q contains one of the two sets U^1_λ, U^2_λ, and its intersection with the other set is empty, then $Q \notin \mathcal{A}_\lambda$.

MSC: Primary 03E05; secondary 54D35

Keywords: algebra of sets; σ-algebra; ultrafilter; pairwise disjoint sets

1 Introduction

The present article is a further development of the theory formulated in [1–7]. The topic studied in these articles, as well as in the present paper, is sets not belonging to algebras of sets.

Definition 1.1 An *algebra* $\mathcal{A}$ on a set X is a non-empty family of subsets of X possessing the following properties: (1) if $M \in \mathcal{A}$, then $X \setminus M \in \mathcal{A}$; (2) if $M_1, M_2 \in \mathcal{A}$, then $M_1 \cup M_2 \in \mathcal{A}$.

It is clear that if $M_1, M_2 \in \mathcal{A}$, then $M_1 \cap M_2 \in \mathcal{A}$ and $M_1 \setminus M_2 \in \mathcal{A}$; also, it is clear that $X \in \mathcal{A}$.

1.1 Some notation and names

All algebras and measures are considered on some abstract set $X \neq \emptyset$. When it is clear from the context, we will not state explicitly that a set belongs to the family $\mathcal{P}(X)$ of all subsets of X. By $\mathbb{N}^+$ we denote the set of natural numbers. If $n_1, n_2 \in \mathbb{N}^+$ and $n_1 \leq n_2$, then $[n_1, n_2] = \{k \in \mathbb{N}^+ \mid n_1 \leq k \leq n_2\}$. Let ρ be a real number. By $\lfloor\rho\rfloor$ we denote the maximum integer $\leq \rho$. By $\lceil\rho\rceil$ we denote the minimum integer $\geq \rho$. The symbol $\#(M)$ denotes the cardinality of the set M. A set M is *countable* if $\#(M) = \aleph_0$.

The following concept was used in [5].

Definition 1.2 An algebra $\mathcal{A}$ has κ *lacunae*, where κ is a cardinal number, if there exist κ pairwise disjoint sets not belonging to $\mathcal{A}$.

Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of algebras and $\mathcal{A}_\lambda \neq \mathcal{P}(X)$ for each $\lambda \in \Lambda$. The following natural question arises: what are possible conditions that distinguish between the cases $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda \neq \mathcal{P}(X)$ and $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{P}(X)$? Let $\#(\Lambda) \leq \aleph_0$, and let us assume that $\mathcal{A}_\lambda$ are σ-algebras if $\#(\Lambda) = \aleph_0$. In [6] we obtained necessary and sufficient conditions for the equality $\bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda = \mathcal{P}(X)$ to hold. The first publication connected with this topic was that of Erdös [8] (this paper contains the well-known theorem of Alouglu-Erdös). Some information about the history of the subject after the publication of [8] and before the publication of [1] is presented in [2]. In fact, Alouglu and Erdös studied non-measurable sets with respect to families of measures. Let $\aleph_1 \leq \#(X) \leq 2^{\aleph_0}$. Let a σ-additive measure μ be defined on X. Here $\mu(X) = 1$, the measure of a one-point set equals 0, and the measure of each μ-measurable set equals 0 or 1. Such a measure μ is called a *σ-two-valued measure*. Clearly, there exist μ-non-measurable sets. The Alouglu-Erdös theorem states that if $\#(X) = \aleph_1$, then for any countable family of σ-two-valued measures $\mu_1, \dots, \mu_k, \dots$ there exists a set which is non-measurable with respect to all these measures. The proof of the Alouglu-Erdös theorem is very simple and is based on the possibility of constructing the well-known Ulam matrix (see [9]). The non-trivial Gitik-Shelah theorem (see [10]) asserts the validity of the Alouglu-Erdös theorem if $\#(X) = 2^{\aleph_0}$. Obviously, the Gitik-Shelah theorem is a generalization of the Alouglu-Erdös theorem. The Gitik-Shelah theorem can be reformulated in our language. As before, let us consider the σ-two-valued measures $\mu_1, \dots, \mu_k, \dots$. For each measure μ_k, we examine the algebra $\mathcal{A}_k$ of all μ_k measurable sets. The Gitik-Shelah theorem asserts that $\bigcup_{k\in\mathbb{N}^+} \mathcal{A}_k \neq \mathcal{P}(X)$. We note that here each algebra $\mathcal{A}_k$ has $\aleph_0$ lacunae. If $\#(X) = \aleph_1$, then the situation is much simpler: each algebra $\mathcal{A}_k$ has $\aleph_1$ lacunae. The Gitik-Shelah theorem is used in the proofs of our theorems for countable families of σ-algebras.

Definition 1.3 Let $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ be a family of algebras, and $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$ be a family of sets with the following properties:

(1) $U^i_\lambda \cap U^j_{\lambda'} = \emptyset$ except when $\lambda = \lambda'$, $i = j$;

(2) for any $\lambda \in \Lambda$, the following holds: if a set Q contains one of the two sets U^1_λ, U^2_λ and its intersection with the other set is empty, then $Q \notin \mathcal{A}_\lambda$.

Then we say that the family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ has *the full set of lacunae* $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$.

Now we give a simple proposition.

Proposition 1.4 *If a family of algebras $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ has the full set of lacunae $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$, then there exists a family of pairwise distinct sets $\{Q_\vartheta\}_{\vartheta\in\Theta}$ such that the following holds*:

(1) $Q_\vartheta \notin \bigcup_{\lambda\in\Lambda} \mathcal{A}_\lambda$ *for any* $\vartheta \in \Theta$;

(2) *any set Q_ϑ is a union of sets U^i_λ*;

(3) $Q_{\vartheta_1} \setminus Q_{\vartheta_2} \notin \bigcap_{\lambda\in\Lambda} \mathcal{A}_\lambda$ *for any pair* $\vartheta_1 \neq \vartheta_2$;

(4) $\#(\Theta) = 2^{\#(\Lambda)}$.

Proof Put $\Theta = \mathcal{P}(\Lambda)$. If $\vartheta \in \mathcal{P}(\Lambda)$, put

$$Q_\vartheta = \left(\bigcup_{\lambda\in\vartheta} U^1_\lambda\right) \cup \left(\bigcup_{\lambda\in\Lambda\setminus\vartheta} U^2_\lambda\right).$$

□

In this paper we deal mostly with the following problem: under which conditions a family of algebras $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ has a full set of lacunae. We assume that $\#(\Lambda) \leq \aleph_0$. This was studied in [1–3]. The proof of the two following theorems can be found in [2].

Theorem 1.5 *Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be a finite family of algebras, and assume that for each $k \in [1, n]$ the algebra $\mathcal{A}_k$ has $4k - 3$ lacunae. Then this family has a full set of lacunae.*

It is easy to prove (see [2], Chapter 14) that the estimate $4k - 3$ is the best possible in some sense.

Theorem 1.6 *Let $\{\mathcal{A}_k\}_{k\in\mathbb{N}^+}$ be a family of σ-algebras, and assume that for each k the algebra $\mathcal{A}_k$ has $4k - 3$ lacunae. Then this family has some full set of lacunae.*

Remark 1.7 Using the notion of absolute introduced by Gleason in [11], we can construct a family of algebras $\{\mathcal{B}_k\}_{k\in\mathbb{N}^+}$ with the following properties: each algebra $\mathcal{B}_k$ has $\aleph_0$ lacunae, is not a σ-algebra, and $\bigcup_{k\in\mathbb{N}^+} \mathcal{B}_k = \mathcal{P}(X)$ (see [2], Chapter 5). Hence, Theorem 1.6 and Theorem 2.4 below do not hold if we claim them for algebras which are not assumed to be σ-additive. Therefore, we suppose that *all algebras of a countable family of algebras are σ-algebras*.

The following definition was given in [2].

Definition 1.8 For each $n \in \mathbb{N}^+$, denote by $\mathfrak{v}(n)$ the minimal cardinal number such that if $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$, $\#(\Lambda) = n$, is a family of algebras, and for each $\lambda \in \Lambda$ the algebra $\mathcal{A}_\lambda$ has $\mathfrak{v}(n)$ lacunae, then the family $\{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ has a full set of lacunae.

In [2] we proved that:

(1) $\mathfrak{v}(n) = 4n - 3$ for $n \leq 3$;

(2) $\mathfrak{v}(n) \leq 4n - 5$ for $n > 3$;

(3) $\mathfrak{v}(n) \leq 4n - \lfloor \frac{n+3}{2} \rfloor$ for any n;

(4) $3n - 2 \leq \mathfrak{v}(n)$ for any n.

In this paper we will improve the upper bound of $\mathfrak{v}(n)$.

From here until the end of Section 1 we present propositions and notions which form the method of proofs of our theorems. This method first appeared in [1] and was later used in [2–7]. Let βX be the Stone-Čech compactification of X with the discrete topology; βX is the family of all ultrafilters on X.

Consider an algebra $\mathcal{A}$. We say that $a, b \in \beta X$ are $\mathcal{A}$-equivalent iff $a \cap \mathcal{A} = b \cap \mathcal{A}$. Let $[b]_\mathcal{A}$ denote the $\mathcal{A}$-equivalence class of b, and define the *kernel* of the algebra $\mathcal{A}$:

$$\ker \mathcal{A} = \{ b \in \beta X \mid \#([b]_\mathcal{A}) > 1 \}.$$

If $\mathcal{A} = \mathcal{P}(X)$, then $\ker \mathcal{A} = \emptyset$. From now on, when we say a and b are $\mathcal{A}$-equivalent ultrafilters, we always assume that $a \neq b$. If a, b are $\mathcal{A}$-equivalent ultrafilters, then we say that a *has an $\mathcal{A}$-equivalent ultrafilter b*, or a is *$\mathcal{A}$-equivalent to b*.

Statement 1.9 Consider an algebra $\mathcal{A}$ and sets $U, V \in \mathcal{P}(X)$ such that $U \cap V = \emptyset$. The following two conditions are equivalent. (1) Each set Q containing one of the sets U, V and being disjoint from the other does not belong to $\mathcal{A}$. (2) There exist $\mathcal{A}$-equivalent ultrafilters a, b such that $U \in a$, $V \in b$.

Proof It is obvious that (1) follows from (2). Let us prove that (2) follows from (1). Let us assume the contrary. We fix an ultrafilter $q \ni U$. For any ultrafilter $r \ni V$, we choose a set $W(r) \in r$ such that $W(r) \in \mathcal{A}$ and $W(r) \notin q$. Since the set of all ultrafilters which contain V is a compact subset of βX, there exists a finite sequence of sets $W(r_1), \dots, W(r_m)$ with the following properties:

(1) $W(r_k) \in \mathcal{A}$ for any $k \in [1, m]$;
(2) $W(r_k) \notin q$ for any $k \in [1, m]$;
(3) $V \subseteq \bigcup_{k=1}^{m} W(r_k)$.

Let

$$\widetilde{W}(q) = X \Big\backslash \bigcup_{k=1}^{m} W(r_k).$$

It is clear that $\widetilde{W}(q) \in q$, $\widetilde{W}(q) \in \mathcal{A}$, and $\widetilde{W}(q) \cap V = \emptyset$. Since the set of all ultrafilters which contain U is a compact subset of βX, there exists a finite sequence of sets $\widetilde{W}(q_1), \dots, \widetilde{W}(q_n)$ such that $\widetilde{W}(q_k) \in \mathcal{A}$ for any $k \in [1, n]$, $\bigcup_{k=1}^{n} \widetilde{W}(q_k) = \widetilde{W} \supseteq U$, and $\widetilde{W} \cap V = \emptyset$. We have $\widetilde{W} \in \mathcal{A}$, a contradiction. □

The following crucial claim is a direct consequence of Statement 1.9.

Claim 1.10 *Consider an algebra $\mathcal{A}$ and $U \in \mathcal{P}(X)$. Then $U \notin \mathcal{A}$ if and only if there exist $\mathcal{A}$-equivalent ultrafilters p and q such that $U \in p$ and $U \notin q$.*

Proof The sufficiency is obvious. If $U \notin \mathcal{A}$, then the sets U and $V = X \setminus U$ satisfy the condition (1) of Statement 1.9. Therefore, there exist the corresponding ultrafilters p and q. □

It is clear that if $\mathcal{A} \neq \mathcal{P}(X)$, then $\#(\ker \mathcal{A}) \geq 2$. It is rather easy to show that an algebra $\mathcal{A}$ has k lacunae, where $2 \leq k \leq \aleph_0$, if and only if $\#(\ker \mathcal{A}) \geq k$.[a]

Definition 1.11 A set $M \subseteq \beta X$ is said to be $\mathcal{A}$-*equivalent* if $\#(M) > 1$, any two distinct ultrafilters in M are $\mathcal{A}$-equivalent, and there exist no $\mathcal{A}$-equivalent ultrafilters a, b such that $a \in M$, $b \notin M$.

Obviously, an $\mathcal{A}$-equivalent set has the form $[b]_{\mathcal{A}}$ (see above). Also it is obvious that an $\mathcal{A}$-equivalent set is closed in βX.

Remark 1.12 Consider algebras $\mathcal{A}, \mathcal{B}$. It is very easy to prove that the following statements are equivalent.

(1) $\mathcal{A} \supseteq \mathcal{B}$.
(2) If a and b are $\mathcal{A}$-equivalent ultrafilters, then a and b are $\mathcal{B}$-equivalent ultrafilters.
(3) If M is an $\mathcal{A}$-equivalent set, then M is contained in a certain $\mathcal{B}$-equivalent set.

Remark 1.13 If $M \subseteq \beta X$ (in particular, if $M \subseteq X$), then by $\overline{M}$ we denote the closure M in βX. The following arguments will be used later in this paper. Let $A \subseteq \beta X$, $2 \leq \#(A) < \aleph_0$.

The set A is divided into pairwise disjoint sets $A_1,\dots,A_m$ and $\#(A_k) > 1$ for each $k \in [1,m]$. Two different ultrafilters are called a-equivalent if and only if they belong to the same set A_k. We can construct the algebra $\mathcal{A}$ such that the a-equivalence relation is in fact the $\mathcal{A}$-equivalence relation, $\ker\mathcal{A} = A$, and $A_1,\dots,A_m$ are all $\mathcal{A}$-equivalent sets. Indeed, by definition $M \in \mathcal{A}$ if and only if for each $k \in [1,m]$ either $A_k \cap \overline{M} = \emptyset$, or $A_k \subseteq \overline{M}$.

Remark 1.14 Let us recall that an algebra which does not have $\aleph_0$ lacunae is called *ω-saturated*. So, an algebra $\mathcal{A}$ is ω-saturated if and only if $\#(\ker\mathcal{A}) < \aleph_0$. The algebra $\mathcal{A}$ from Remark 1.13 is ω-saturated.

Remark 1.15 Further we use two following very simple statements. (1) By Statement 1.9 a finite family of algebras $\mathcal{A}_1,\dots,\mathcal{A}_n$ has a full set of lacunae if and only if there exist $2n$ pairwise distinct ultrafilters $a_1,\dots,a_n,b_1,\dots,b_n$ such that a_k, b_k are $\mathcal{A}_k$-equivalent ultrafilters for each $k \in [1,n]$. (2) Let $\mathfrak{A} = \{\mathcal{A}_\lambda\}_{\lambda\in\Lambda}$ and $\mathfrak{A}' = \{\mathcal{A}'_\lambda\}_{\lambda\in\Lambda}$ be two non-empty families of algebras, and $\mathcal{A}'_\lambda \supseteq \mathcal{A}_\lambda$ for every $\lambda \in \Lambda$. Assume that the family $\mathfrak{A}'$ has a full set of lacunae $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$. Then the family $\mathfrak{A}$ has the same full set of lacunae $\{U^1_\lambda, U^2_\lambda\}_{\lambda\in\Lambda}$.

2 Main results. An open problem

The following result was announced in [3]: $\mathfrak{v}(n) \le \lceil \frac{10}{3}n + \frac{2}{\sqrt{3}}\sqrt{n}\rceil$ for any n. In this paper a stronger theorem is proved.

Theorem 2.1 $\mathfrak{v}(n) \le \lceil \frac{10}{3}n + \sqrt{\frac{2n}{3}}\rceil$.

Remark 2.2 The combinatorial nature of Theorem 2.1 is discussed in Section 4. Also in Section 4 the proof of Theorem 4.5 uses the classical Ramsey theorem.

Problem 2.3 We know that $\mathfrak{v}(n) \ge 3n-2$ for any n, and $\mathfrak{v}(n) > 3n-2$ if $n = 2,3$ since $\mathfrak{v}(2) = 5$, $\mathfrak{v}(3) = 9$ (see Section 1). Is it true that $\mathfrak{v}(n) = 3n-2$ for any $n \ne 2,3$? This result is obviously true for $n = 1$.

The final section of this article is devoted to the proof of the following theorem, which is a generalization of theorems of Alaouglu-Erdös and Gitik-Shelah.

Theorem 2.4 *It is possible to construct nondecreasing functions $\varphi : \mathbb{N}^+ \to \mathbb{N}^+$ such that the following conditions hold*:

(1) $\underline{\lim}_{n\to\infty} \frac{\varphi(n)-\frac{10}{3}n}{\sqrt{n}} = \sqrt{\frac{2}{3}}$;

(2) *if $\{\mathcal{A}_k\}_{k\in\mathbb{N}^+}$ is a family of σ-algebras and each algebra $\mathcal{A}_k$ has $\varphi(k)$ lacunae, then this family has a full set of lacunae.*

3 Finite families of algebras. Proof of Theorem 2.1

The following lemma is used in the proof of Lemma 3.2.

Lemma 3.1 *Consider an algebra $\mathcal{A}$ which is not ω-saturated;*[b] *let a number $\xi \in \mathbb{N}^+$ be given. Then it is possible to construct an ω-saturated algebra $\mathcal{A}'$ such that $\#(\ker\mathcal{A}') \ge \xi$ and $\mathcal{A}' \supset \mathcal{A}$.*

Proof Take two distinct $\mathcal{A}$-equivalent ultrafilters s_1, t_1. Consider two distinct ultrafilters $a_1, a_2 \in \ker \mathcal{A} \setminus \{s_1, t_1\}$. If a_1 has an $\mathcal{A}$-equivalent ultrafilter in $\{s_1, t_1\}$, and a_2 has an $\mathcal{A}$-equivalent ultrafilter in $\{s_1, t_1\}$, then a_1 and a_2 are $\mathcal{A}$-equivalent ultrafilters. Denote $s_2 = a_1$, $t_2 = a_2$. If, for example, a_1 does not have an $\mathcal{A}$-equivalent ultrafilter in $\{s_1, t_1\}$, then take an ultrafilter c such that $a_1 \neq c$ and a_1, c are $\mathcal{A}$-equivalent ultrafilters. In this case denote $s_2 = a_1$, $t_2 = c$. Now take three pairwise disjoint ultrafilters $b_1, b_2, b_3 \in \ker \mathcal{A} \setminus \{s_1, t_1, s_2, t_2\}$. If every ultrafilter b_i has an $\mathcal{A}$-equivalent ultrafilter in $\{s_1, t_1, s_2, t_2\}$, then in the set $\{b_1, b_2, b_3\}$ we can choose two distinct $\mathcal{A}$-equivalent ultrafilters, for example, b_1 and b_2. Put $s_3 = b_1$, $t_3 = b_2$. If, for example, b_1 does not have an $\mathcal{A}$-equivalent ultrafilter in $\{s_1, t_1, s_2, t_2\}$, then take an ultrafilter d such that $b_1 \neq d$ and b_1, d are $\mathcal{A}$-equivalent ultrafilters. Denote $s_3 = b_1$, $t_3 = d$. It is clear that for every $\ell \in \mathbb{N}^+$ it is possible to construct a sequence of pairwise distinct ultrafilters $s_1, t_1, \dots, s_\ell, t_\ell$ such that s_i and t_i are $\mathcal{A}$-equivalent ultrafilters for all $i \in [1, \ell]$. Let $2\ell \geq \xi$. Define $M_1 = \{s_1, t_1\}, \dots, M_\ell = \{s_\ell, t_\ell\}$. By Remark 1.13 it is possible to construct an algebra $\mathcal{A}'$ such that $\ker \mathcal{A}' = \bigcup_{i=1}^{\ell} M_i$ and $M_1, \dots, M_\ell$ are $\mathcal{A}'$-equivalent sets. □

The following lemma is given in [2] without proof.

Lemma 3.2 $\mathfrak{v}(n) \in \mathbb{N}^+$, *and* $\mathfrak{v}(n+1) - \mathfrak{v}(n) \leq 4$.

Proof It is obvious that $\mathfrak{v}(1) = 1$. Let $n \in \mathbb{N}^+$ and assume that $\mathfrak{v}(n) \in \mathbb{N}^+$. Consider a family of algebras $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ with $\#(\ker \mathcal{A}_k) \geq \mathfrak{v}(n) + 4$ for each $k \in [1, n+1]$. We must prove that this family has a full set of lacunae. By Lemma 3.1 and the arguments in Remark 1.15 we can assume that the algebras $\mathcal{A}_1, \dots, \mathcal{A}_{n+1}$ are ω-saturated. We choose $\mathcal{A}_{n+1}$-equivalent ultrafilters $s_{n+1}^{(1)}$, $s_{n+1}^{(2)}$. Put $B_k = \ker \mathcal{A}_k \setminus \{s_{n+1}^{(1)}, s_{n+1}^{(2)}\}$ for each $k \in [1, n]$. Put

$$\begin{aligned} B'_k = \bigl\{ & q \in B_k \mid q \text{ does not have an } A_k\text{-equivalent ultrafilter in } B_k \\ & \text{and has an } A_k\text{-equivalent ultrafilter in } \bigl\{ s_{n+1}^{(1)}, s_{n+1}^{(2)} \bigr\} \bigr\}. \end{aligned}$$

It is clear that $\#(B'_k) \leq 2$. Put $B''_k = B_k \setminus B'_k$. Clearly, each ultrafilter in B''_k has an $\mathcal{A}_k$-equivalent ultrafilter in B''_k. Therefore, by Remark 1.13, we can construct an algebra $\mathcal{A}'_k$ such that $\ker \mathcal{A}'_k = B''_k$ and the $\mathcal{A}'_k$-equivalent relation in $\ker \mathcal{A}'_k$ is in fact the $\mathcal{A}_k$-equivalent relation. We have $\#(\ker \mathcal{A}'_k) \geq \mathfrak{v}(n)$ for each $k \in [1, n]$. Therefore, there exist $2n$ pairwise distinct ultrafilters $s_1^{(1)}, s_1^{(2)}, \dots, s_n^{(1)}, s_n^{(2)}$, and $s_k^{(1)}$, $s_k^{(2)}$ are $\mathcal{A}_k$-equivalent ultrafilters from $\ker \mathcal{A}'_k$. We have pairwise distinct ultrafilters $s_1^{(1)}, s_1^{(2)}, \dots, s_{n+1}^{(1)}, s_{n+1}^{(2)}$, and $s_k^{(1)}$, $s_k^{(2)}$ are $\mathcal{A}_k$-equivalent ultrafilters for each $k \in [1, n+1]$. □

Remark 3.3 It is obvious that $\mathfrak{v}(1) = 1$. Therefore, by Lemma 3.2 we have $\mathfrak{v}(n) \leq 4n - 3$ for any n. In Chapter 14, [2], we proved that $\mathfrak{v}(4) \leq 11$. Therefore, by Lemma 3.2, we have that $\mathfrak{v}(n) \leq 4n - 5$ for any $n \geq 4$.

We now turn to the proof of Theorem 2.1. This proof is a strong improvement of the proposition $\mathfrak{v}(n) \leq 4n - \lfloor \frac{n+3}{2} \rfloor$ mentioned above (see [2], Chapter 14).

Proof of Theorem 2.1 (1) By Remark 3.3 our theorem is true for all $n \leq 13$. (This can be verified by a simple computation.) Fix a natural number $n \geq 14$ and a real number

$$\omega(n) \geq \sqrt{\frac{2n}{3}}.$$

Let $\mathcal{A}_1, \dots, \mathcal{A}_n$ be algebras such that

$$\#(\ker \mathcal{A}_k) \geq \frac{10}{3}n + \omega(n)$$

for each $k \in [1, n]$. By Lemma 3.1 and arguments in Remark 1.15, we can assume that the algebras $\mathcal{A}_1, \dots, \mathcal{A}_n$ are ω-saturated. We will prove that there exist pairwise distinct ultrafilters

$$a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$$

such that a_k^*, b_k^* are $\mathcal{A}_k$-equivalent ultrafilters for each $k \in [1, n]$. Our goal is to contradict *the assumption that ultrafilters* $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ *do not exist*. Inductively assume that

$$\mathfrak{v}(n-1) \leq \left\lceil \frac{10}{3}(n-1) + \sqrt{\frac{2n-2}{3}} \right\rceil.$$

Then there exists a set of pairwise distinct ultrafilters

$$\mathfrak{F} = \{a_1, \dots, a_{n-1}, b_1, \dots, b_{n-1}\}$$

such that a_k, b_k are $\mathcal{A}_k$-equivalent ultrafilters for each $k \in [1, n-1]$. Consider $\ker \mathcal{A}_n$. It is clear that

$$\#(\ker \mathcal{A}_n \setminus \mathfrak{F}) \geq \frac{10}{3}n + \omega(n) - 2n + 2 = \frac{4}{3}n + \omega(n) + 2.$$

If there exist two $\mathcal{A}_n$-equivalent ultrafilters in $\ker \mathcal{A}_n \setminus \mathfrak{F}$, we immediately obtain the required construction yielding the existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$. Therefore, each ultrafilter from $\ker \mathcal{A}_n \setminus \mathfrak{F}$ has an $\mathcal{A}_n$-equivalent ultrafilter in $\mathfrak{F}$. Therefore, there exist distinct ultrafilters $c_n, d_n \in \ker \mathcal{A}_n \setminus \mathfrak{F}$ and $k_1 \in [1, n-1]$ such that (a_{k_1}, c_n) and (b_{k_1}, d_n) are two pairs of $\mathcal{A}_n$-equivalent ultrafilters. For simplicity, say $k_1 = 1$. Now consider $\ker \mathcal{A}_1$. It is clear that

$$\#\bigl(\ker \mathcal{A}_1 \setminus \bigl(\mathfrak{F} \cup \{c_n, d_n\}\bigr)\bigr) \geq \frac{10}{3}n + \omega(n) - 2n = \frac{4}{3}n + \omega(n).$$

If there exist two $\mathcal{A}_1$-equivalent ultrafilters in $\ker \mathcal{A}_1 \setminus (\mathfrak{F} \cup \{c_n, d_n\})$, we immediately obtain the required construction yielding the existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$. Similarly, if an ultrafilter in $\ker \mathcal{A}_1 \setminus (\mathfrak{F} \cup \{c_n, d_n\})$ has an $\mathcal{A}_1$-equivalent ultrafilter in $\{a_1, b_1, c_n, d_n\}$, then the construction which contradicts the non-existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ is yielded immediately. So, each ultrafilter in $\ker \mathcal{A}_1 \setminus (\mathfrak{F} \cup \{c_n, d_n\})$ has an $\mathcal{A}_1$-equivalent ultrafilter in the set $\mathfrak{F} \setminus \{a_1, b_1\}$. Therefore, there exist distinct ultrafilters $c_1, d_1 \in \ker \mathcal{A}_1 \setminus (\mathfrak{F} \cup \{c_n, d_n\})$ and $k_2 \in [2, n-1]$ such that (a_{k_2}, c_1) and (b_{k_2}, d_1) are two pairs of $\mathcal{A}_1$-equivalent ultrafilters. For simplicity, say $k_2 = 2$. This process can be continued. Suppose that there exists a natural number η such that

$$3 \le \eta \le \frac{1}{3}n + \omega(n) + 2.$$

Suppose also that there exists a set of pairwise distinct ultrafilters

$$\mathfrak{E} = \{c_1, \dots, c_{\eta-1}, c_n, d_1, \dots, d_{\eta-1}, d_n\}$$

and the following holds:

(A) (a_{i+1}, c_i) and (b_{i+1}, d_i) are two pairs of $\mathcal{A}_i$-equivalent ultrafilters for each $i \in [2, \eta - 1]$;

(B) $\mathfrak{F} \cap \mathfrak{E} = \emptyset$.

Let us recall what we have said above: (a_1, c_n) and (b_1, d_n) are two pairs of $\mathcal{A}_n$-equivalent ultrafilters; (a_2, c_1) and (b_2, d_1) are two pairs of $\mathcal{A}_1$-equivalent ultrafilters.

Define $L_\eta = \ker \mathcal{A}_\eta \setminus (\mathfrak{F} \cup \mathfrak{E})$. It is clear that

$$\#(L_\eta) \ge \frac{10}{3}n + \omega(n) - (2n - 2) - 2\eta = \frac{4}{3}n + \omega(n) - 2\eta + 2.$$

If there exist two $\mathcal{A}_\eta$-equivalent ultrafilters in L_η, we immediately obtain the required construction yielding the existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$. Similarly, if an ultrafilter in L_η has an $\mathcal{A}_\eta$-equivalent ultrafilter in $\{a_1, \dots, a_\eta, b_1, \dots, b_\eta\} \cup \mathfrak{E}$, then the construction which contradicts the non-existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ is yielded immediately. Therefore, every ultrafilter from L_η has an $\mathcal{A}_\eta$-equivalent ultrafilter in $\{a_{\eta+1}, \dots, a_{n-1}, b_{\eta+1}, \dots, b_{n-1}\}$. We have

$$\#(L_\eta) - \#\bigl([\eta + 1, n - 1]\bigr) \ge \frac{4}{3}n + \omega(n) - 2\eta + 2 - n + \eta + 1 = \frac{1}{3}n + \omega(n) + 3 - \eta > 0.$$

Therefore, there exist distinct ultrafilters $c_\eta, d_\eta \in L_\eta$ and $k_{\eta+1} \in [\eta + 1, n - 1]$ such that $(a_{k_{\eta+1}}, c_\eta)$ and $(b_{k_{\eta+1}}, d_\eta)$ are two pairs of $\mathcal{A}_\eta$-equivalent ultrafilters. For simplicity, say $k_{\eta+1} = \eta + 1$. We have that (a_{i+1}, c_i) and (b_{i+1}, d_i) are two pairs of $\mathcal{A}_i$-equivalent ultrafilters for each $i \in [1, \eta]$.

Put $\rho = \lfloor \frac{n}{3} \rfloor$. In view of the above, we can assume that there exist pairwise distinct ultrafilters $c_1, \dots, c_{\rho-1}, c_n, d_1, \dots, d_{\rho-1}, d_n$ such that the following holds:

(a) (a_1, c_n) and (b_1, d_n) are two pairs of $\mathcal{A}_n$-equivalent ultrafilters;

(b) (a_{i+1}, c_i) and (b_{i+1}, d_i) are two pairs of $\mathcal{A}_i$-equivalent ultrafilters for each $i \in [1, \rho - 1]$;

(c) $\mathfrak{F} \cap \{c_1, \dots, c_{\rho-1}, c_n, d_1, \dots, d_{\rho-1}, d_n\} = \emptyset$, see Figure 1.

(2) Put

$$Z_\rho = \{a_1, \dots, a_\rho, b_1, \dots, b_\rho, c_1, \dots, c_{\rho-1}, c_n, d_1, \dots, d_{\rho-1}, d_n\},$$

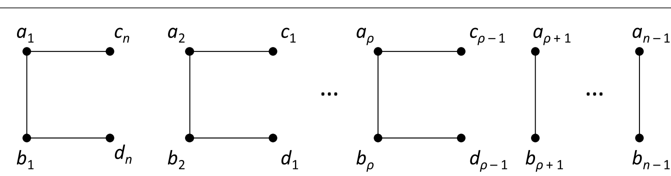

Figure 1 Ultrafilters a_i, b_i, c_i, and d_i.

$$\begin{aligned}
&Z'_\rho = \{a_1,\dots,a_{n-1},b_1,\dots,b_{n-1},c_1,\dots,c_{\rho-1},c_n,d_1,\dots,d_{\rho-1},d_n\},\\
&Z''_\rho = \{a_{\rho+1},\dots,a_{n-1},b_{\rho+1},\dots,b_{n-1}\},\\
&L_\rho = \ker\mathcal{A}_\rho \setminus Z'_\rho.
\end{aligned}$$

Clearly,

$$\begin{aligned}
\#(L_\rho) &\geq \frac{10}{3}n + \omega(n) - 4\cdot\left\lfloor\frac{n}{3}\right\rfloor - 2\left(n-1-\left\lfloor\frac{n}{3}\right\rfloor\right)\\
&= \frac{4}{3}n - 2\cdot\left\lfloor\frac{n}{3}\right\rfloor + \omega(n) + 2 \geq \frac{2}{3}n + \omega(n) + 2,\\
\#(L_\rho) - \#\bigl([\rho+1,n-1]\bigr) &\geq \frac{2}{3}n + \omega(n) + 2 - \left(n-1-\left\lfloor\frac{n}{3}\right\rfloor\right)\\
&= \left\lfloor\frac{n}{3}\right\rfloor - \frac{n}{3} + \omega(n) + 3 > 0.
\end{aligned}$$

The above arguments show that the following assumption should be made: for each ultrafilter $q \in L_\rho$, there exists an ultrafilter $\tilde{q} \in Z''_\rho$ such that q and $\tilde{q}$ are $\mathcal{A}_\rho$-equivalent ultrafilters. In general, there can be such q for which the number of corresponding $\tilde{q}$ is greater than 1. We choose in an arbitrary way only one $\tilde{q}$ for each $q \in L_\rho$. We obtain the mapping $f : L_\rho \to Z''_\rho, f(q) = \tilde{q}$. The map f is one-to-one. (If $f(q_1) = f(q_2)$ and $q_1 \neq q_2$, then q_1, q_2 are $\mathcal{A}_\rho$-similar ultrafilters, and the construction which contradicts the non-existence of ultrafilters $a_1^*,\dots,a_n^*,b_1^*,\dots,b_n^*$ is yielded immediately.) Put

$$\begin{aligned}
\mathfrak{I}_1^\rho = \bigl\{&k \in [\rho+1,n-1] \mid \text{there exist ultrafilters } q_k^a, q_k^b \in L_\rho\\
&\text{such that } f\bigl(q_k^a\bigr) = a_k, f\bigl(q_k^b\bigr) = b_k\bigr\},\\
\mathfrak{I}_2^\rho = \bigl\{&k \in [\rho+1,n-1] \setminus \mathfrak{I}_1^\rho \mid \text{there exists an ultrafilter } q_k^* \in L_\rho\\
&\text{such that } f\bigl(q_k^*\bigr) \in \{a_k,b_k\}\bigr\}.
\end{aligned}$$

Obviously, $\mathfrak{I}_1^\rho \cap \mathfrak{I}_2^\rho = \emptyset$. Since $\#(L_\rho) - \#([\rho+1,n-1]) > 0$, we have $\#(\mathfrak{I}_1^\rho) = \tau > 0$. Clearly,

$$\#\bigl(\mathfrak{I}_2^\rho\bigr) = \#(L_\rho) - 2\tau \geq \frac{2}{3}n + \omega(n) + 2 - 2\tau.$$

Put

$$L_n = \ker\mathcal{A}_n \setminus Z'_\rho.$$

We have obtained above the estimate $\#(L_\rho) \geq \frac{2}{3}n + \omega(n) + 2$. In exactly the same way, the following estimate can be obtained:

$$\#(L_n) \geq \frac{2}{3}n + \omega(n) + 2.$$

If there exist two $\mathcal{A}_n$-equivalent ultrafilters from L_n, we immediately obtain the required construction regarding the existence of ultrafilters $a_1^*,\dots,a_n^*,b_1^*,\dots,b_n^*$. Similarly, if an ultrafilter in L_n has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_1,b_1,c_1,\dots,c_{\rho-1},c_n,d_1,\dots,d_{\rho-1},d_n\}$, then it is easy to find the corresponding ultrafilters $a_1^*,\dots,a_n^*,b_1^*,\dots,b_n^*$.

We are interested in the following situation: let $q \in L_n$, and q has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_2, \dots, a_\rho, b_1, \dots, b_\rho\}$. Let, for instance, q and a_2 be $\mathcal{A}_n$-equivalent ultrafilters. Then let us consider d_1. For d_1 there are four possible cases:

⟨1⟩ $d_1 \notin \ker \mathcal{A}_n$;
⟨2⟩ b_2, d_1 are $\mathcal{A}_n$-equivalent ultrafilters;
⟨3⟩ d_1 has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_3, \dots, a_\rho, b_3, \dots, b_\rho\}$;
⟨4⟩ d_1 has an $\mathcal{A}_n$-equivalent ultrafilter in Z''_ρ.

In case ⟨2⟩ let us consider c_1. For c_1 the possible corresponding cases are:

⟨i⟩ $c_1 \notin \ker \mathcal{A}_n$;
⟨ii⟩ c_1 has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_3, \dots, a_\rho, b_3, \dots, b_\rho\}$;
⟨iii⟩ c_1 has an $\mathcal{A}_n$-equivalent ultrafilter in Z''_ρ.

Consider case ⟨3⟩ for d_1. Let d_1, b_3 be $\mathcal{A}_n$-equivalent ultrafilters. Let us consider c_2. For c_2 there are four possible cases:

⟨1⟩ $c_2 \notin \ker \mathcal{A}_n$;
⟨2⟩ a_3, c_2 are $\mathcal{A}_n$-equivalent ultrafilters;
⟨3⟩ c_2 has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_4, \dots, a_\rho, b_2, b_4, \dots, b_\rho\}$;
⟨4⟩ c_2 has an $\mathcal{A}_n$-equivalent ultrafilter in Z''_ρ.

Consider case ⟨3⟩ for c_2. Let b_2, c_2 be $\mathcal{A}_n$-equivalent ultrafilters. Let us consider c_1. For c_1 the possible corresponding cases are:

⟨i⟩ $c_1 \notin \ker \mathcal{A}_n$;
⟨ii⟩ c_1 has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_3, \dots, a_\rho, b_4, \dots, b_\rho\}$;
⟨iii⟩ c_1 has an $\mathcal{A}_n$-equivalent ultrafilter in Z''_ρ.

Continuing these constructions in an obvious way, we find an ultrafilter

$$q_* \in \{c_1, \dots, c_{\rho-1}, d_1, \dots, d_{\rho-1}\}$$

such that one of the following two statements is true: (1) $q_* \notin \ker \mathcal{A}_n$; (2) q_* has an $\mathcal{A}_n$-equivalent ultrafilter in Z''_ρ. Let us put

$$\begin{aligned}
&\alpha = \#\bigl(\bigl\{q \in L_n \mid q \text{ has an } \mathcal{A}_n\text{-equivalent ultrafilter in } \{a_2, \dots, a_\rho, b_2, \dots, b_\rho\}\bigr\}\bigr),\\
&\beta = \#\bigl(\{c_1, \dots, c_{\rho-1}, d_1, \dots, d_{\rho-1}\} \setminus \ker \mathcal{A}_n\bigr),\\
&\gamma = \#\bigl(\bigl\{q_* \in \{c_1, \dots, c_{\rho-1}, d_1, \dots, d_{\rho-1}\} \mid q_* \text{ has an } \mathcal{A}_n\text{-equivalent}\\
&\qquad \text{ultrafilter in } Z''_\rho\bigr\}\bigr).
\end{aligned}$$

The above constructions clearly show that $\alpha \leq \beta + \gamma$. Put

$$\hat{L} = \bigl\{q \in L_n \cup \{c_1, \dots c_{\rho-1}, d_1, \dots, d_{\rho-1}\} \mid q \text{ has } \mathcal{A}_n\text{-similar ultrafilter in } Z''_\rho\bigr\}.$$

Clearly,

$$\#(\hat{L}) \geq \#(L_n) + \gamma - \alpha \geq \frac{2}{3}n + \omega(n) + 2 + \beta + \gamma - \alpha \geq \frac{2}{3}n + \omega(n) + 2.$$

So, for every ultrafilter $q \in \hat{L}$, there exists an ultrafilter $\bar{q} \in Z''_\rho$ such that q and $\bar{q}$ are $\mathcal{A}_n$-similar ultrafilters. In general it can happen that for some q there exist more than one corresponding $\bar{q}$. Choose arbitrarily only one ultrafilter $\bar{q}$ for each $q \in \hat{L}$. We obtain a mapping $\hat{f} : \hat{L} \to Z''_\rho, \hat{f}(q) = \bar{q}$. Consider the corresponding map $\hat{f} : \hat{L}_\rho \to Z''_\rho$. It is one-to-one. Indeed, if $\hat{f}(q_1) = \hat{f}(q_2)$ and $q_1 \neq q_2$, then q_1, q_2 are $\mathcal{A}_n$-similar ultrafilters, and the construction which contradicts the non-existence of ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ is yielded immediately. Put

$$\begin{aligned}\hat{\mathfrak{I}}_1 = \bigl\{k \in [\rho+1, n-1] \mid &\text{there exist ultrafilters } \mathfrak{q}_k^a, \mathfrak{q}_k^b \in \hat{L}\\ &\text{such that } \hat{f}\bigl(\mathfrak{q}_k^a\bigr) = a_k, \hat{f}\bigl(\mathfrak{q}_k^b\bigr) = b_k\bigr\},\\ \hat{\mathfrak{I}}_2 = \bigl\{k \in [\rho+1, n-1] \setminus \hat{\mathfrak{I}}_1 \mid &\text{there exists an ultrafilter } \mathfrak{q}_k^* \in \hat{L}\\ &\text{such that } \hat{f}\bigl(\mathfrak{q}_k^*\bigr) \in \{a_k, b_k\}\bigr\}.\end{aligned}$$

Obviously, $\hat{\mathfrak{I}}_1 \cap \hat{\mathfrak{I}}_2 = \emptyset$. Since $\#(\hat{L}) - \#([\rho+1, n-1]) > 0$, we have $\#(\hat{\mathfrak{I}}_1) = \hat{\tau} > 0$. Clearly,

$$\#(\hat{\mathfrak{I}}_2) = \#(\hat{L}) - 2\hat{\tau} \geq \frac{2}{3}n + \omega(n) + 2 - 2\hat{\tau}.$$

If $\tau \geq \hat{\tau}$, put

$$\mathfrak{I} = (\hat{\mathfrak{I}}_1 \cup \hat{\mathfrak{I}}_2) \cap \mathfrak{I}_1^\rho.$$

If $\tau < \hat{\tau}$, put

$$\mathfrak{I} = \bigl(\mathfrak{I}_1^\rho \cup \mathfrak{I}_2^\rho\bigr) \cap \hat{\mathfrak{I}}_1.$$

Clearly,

$$\#(\mathfrak{I}) \geq \frac{2}{3}n + \omega(n) + 2 - n + 1 + \left\lfloor \frac{n}{3} \right\rfloor > \omega(n) + 2.$$

(3) We fix $\nu \in \mathfrak{I}$. A number $k \in [1, \rho]$ is called *ν-marked* if the following is true:
for $k = 1$: (a_1, d_n) and (b_1, c_n) are pairs of $\mathcal{A}_\nu$-equivalent ultrafilters;
for $k > 1$: (a_k, d_{k-1}) and (b_k, c_{k-1}) are pairs of $\mathcal{A}_\nu$-equivalent ultrafilters.
Put

$$\chi_\nu = \#\bigl(\bigl\{k \in [1, \rho] \mid k \text{ is a } \nu\text{-marked number}\bigr\}\bigr).$$

Our aim is to prove that

$$\chi_\nu > \frac{\omega(n)}{2}.$$

We have the following options:

⟨1⟩ There exist ultrafilters $q_\nu^a, q_\nu^b \in \mathfrak{I}_1^\rho$ and an ultrafilter $\mathfrak{q}_\nu^* \in \hat{\mathfrak{I}}_2$.
⟨2⟩ There exists an ultrafilter $q_\nu^* \in \mathfrak{I}_2^\rho$ and ultrafilters $\mathfrak{q}_\nu^a, \mathfrak{q}_\nu^b \in \hat{\mathfrak{I}}_1$.
⟨3⟩ There exist ultrafilters $q_\nu^a, q_\nu^b \in \mathfrak{I}_1^\rho$ and ultrafilters $\mathfrak{q}_\nu^a, \mathfrak{q}_\nu^b \in \hat{\mathfrak{I}}_1$.

Denote q_ν^* by q_ν. Denote $\mathfrak{q}_\nu^*$ by q'_ν. Choose one of two ultrafilters q_ν^a, q_ν^b and denote it by q_ν; at this step we do not consider the second ultrafilter. Choose one of two ultrafilters $\mathfrak{q}_\nu^a$, $\mathfrak{q}_\nu^b$ and denote it by q'_ν; at this step we do not consider the second ultrafilter. *If possible, the ultrafilter q'_ν is taken from $\hat{L} \setminus L_n$.* Let $\nu = \rho + 1$. We know that there exists a corresponding ultrafilter $q_{\rho+1} \in L_\rho$ which has an $\mathcal{A}_\rho$-equivalent ultrafilter in $\{a_{\rho+1}, b_{\rho+1}\}$. We also know that there exists a corresponding ultrafilter $q'_{\rho+1} \in \hat{L}$ which has $\mathcal{A}_n$-equivalent ultrafilter in $\{a_{\rho+1}, b_{\rho+1}\}$.

When the number $\chi_{\rho+1}$ attains its minimal value, we must assume the following: there exist pairwise distinct ultrafilters

$$a'_{\rho+2}, \dots, a'_{n-1}, b'_{\rho+2}, \dots, b'_{n-1} \in \ker \mathcal{A}_{\rho+1} \Big\backslash \Big(Z'_\rho \cup \big\{ q_{\rho+1}, q'_{\rho+1} \big\} \Big),$$

and (a_k, a'_k), (b_k, b'_k) are pairs of $\mathcal{A}_{\rho+1}$-equivalent ultrafilters for each $k \in [\rho + 2, n - 1]$. We will only consider the cases where *finding ultrafilters $a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^*$ is not immediate.*

Case 1. $q'_{\rho+1} \in L_n$.

Case 1-1. $q_{\rho+1} = q'_{\rho+1}$.

We consider only two subcases of Case 1-1.

Case 1-1-1. There exists an ultrafilter $q^* \notin Z_\rho$ such that q^*, $q_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters.

Case 1-1-2. There exists an ultrafilter $q^* \in Z_\rho$ such that q^*, $q_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters.

Case 1-2. $q_{\rho+1} \neq q'_{\rho+1}$.

We consider only two subcases of Case 1-2.

Case 1-2-1. $q_{\rho+1}$, $q'_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters.

Case 1-2-2. There exists an ultrafilter $q^* \in \{a_1, \dots, a_\rho, b_1, \dots, b_\rho\}$ such that q^*, $q_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters.

Before we consider these cases, let us denote $\mathcal{R}_1 = \{a_1, b_1, c_n, b_n\}$, and $\mathcal{R}_k = \{a_k, b_k, c_{k-1}, d_{k-1}\}$ if $k \in [2, \rho]$.

First we consider Cases 1-1-1 and 1-2-1. For the situation where the number $\chi_{\rho+1}$ attains its the minimum value, we have the following options for the set $\mathcal{R}_1$:

(1) a_1, b_1 are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_1) = 2$;
(2) a_1, d_n are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_1) = 2$;
(3) b_1, c_n are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_1) = 2$;
(4) the number 1 is $(\rho + 1)$-marked.

If $k \in [2, \rho]$, by analogy, we have the following options for the set $\mathcal{R}_k$:

(1*) a_k, b_k are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_k) = 2$;
(2*) a_k, d_{k-1} are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_k) = 2$;
(3*) b_k, c_{k-1} are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters and $\#(\ker \mathcal{A}_{\rho+1} \cap \mathcal{R}_k) = 2$;
(4*) the number k is $(\rho + 1)$-marked.

So we have

$$4(n - 1 - \rho) + 4 \cdot \chi_{\rho+1} + 2(\rho - \chi_{\rho+1}) = \#(\ker \mathcal{A}_{\rho+1}) \geq \frac{10}{3} n + \omega(n).$$

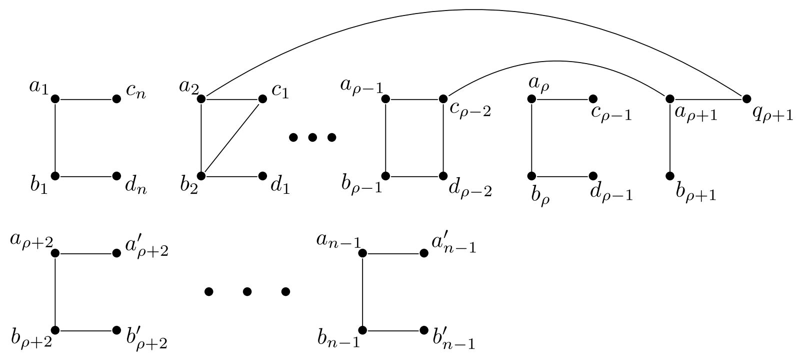

Figure 2 Illustration for Case 2.

Recall that $\rho = \lfloor \frac{n}{3} \rfloor$. Therefore we have

$$\chi_{\rho+1} > \frac{\omega(n)}{2} + 1.$$

Now consider Case 1-1-2. The situation is as follows:

⟨a⟩ $c_{\rho-1}, q_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters;
⟨b⟩ $a_\rho, d_{\rho-1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters;
⟨c⟩ for $\mathcal{R}_1$ one of the options (1)-(4) is fulfilled;
⟨d⟩ for $\mathcal{R}_k$, where $k \in [2, \rho-1]$, one of the options (1*)-(4*) is fulfilled.

Now consider Case 1-2-2. The situation is as follows: $b_\rho, q_{\rho+1}$ are $\mathcal{A}_{\rho+1}$-equivalent ultrafilters, and the conditions ⟨b⟩, ⟨c⟩, ⟨d⟩ are fulfilled. It is clear that in Cases 1-1-2 and 1-2-2 we have

$$\chi_{\rho+1} > \frac{\omega(n)}{2} + 1.$$

It is clear that in Case 1 there may be subcases which we have not considered. But always

$$\chi_{\rho+1} > \frac{\omega(n)}{2} + 1.$$

Case 2. $q'_{\rho+1} \in \hat{L} \setminus L_n$. Suppose that $q'_{\rho+1} = c_{\rho-2}$ and $c_{\rho-2}, a_{\rho+1}$ are $\mathcal{A}_n$-equivalent ultrafilters. For the number $\chi_{\rho+1}$ to be minimal and the situation to be nontrivial, we assume the following:

(i) $(a_{\rho-1}, b_{\rho-1})$, $(c_{\rho-2}, d_{\rho-2})$, $(a_2, q_{\rho+1})$, (b_2, c_1) are pairs of $\mathcal{A}_{\rho+1}$-equivalent ultrafilters;
(ii) $a_{\rho+1}, q_{\rho+1}$ are $\mathcal{A}_\rho$-equivalent ultrafilters;
(iii) $\ker \mathcal{A}_{\rho+1} \subset Z'_\rho \cup \{a'_{\rho+2}, \dots, a'_{n-1}, b'_{\rho+2}, \dots, b'_{n-1}\} \cup \{q_{\rho+1}\}$, see Figure 2.

We assume that one of the cases (1)-(4) holds for $\mathcal{R}_1$ and that one of the cases (1*)-(4*) holds for $\mathcal{R}_k$, where $k \in [3, \rho] \setminus \{\rho-1\}$. We have

$$4(n-1-\rho) + 4 \cdot \chi_{\rho+1} + 2(\rho - 2 - \chi_{\rho+1}) + 6 = \#(\ker \mathcal{A}_{\rho+1}) \geq \frac{10}{3} n + \omega(n).$$

Recall that $\rho = \lfloor \frac{n}{3} \rfloor$. Therefore we have

$$\chi_{\rho+1} > \frac{\omega(n)}{2}.$$

Analyzing the other situations in Case 2, we come to the same conclusion: $\chi_{\rho+1} > \frac{\omega(n)}{2}$; and we can assume that if k is a $(\rho+1)$-marked number, then $q'_{\rho+1} \notin \mathcal{R}_k$. It is obvious that the same conclusion is true in Case 1.

(4) It is obvious that for each $\nu \in \mathfrak{I}$ we have $\chi_\nu > \frac{\omega(n)}{2}$, and $q'_k \notin \mathcal{R}_k$ if k is a ν-marked number. We know that

$$\omega(n) \geq \sqrt{\frac{2n}{3}}, \qquad \#(\mathfrak{I}) > \omega(n) + 2, \qquad \rho = \left\lfloor \frac{n}{3} \right\rfloor.$$

Therefore we have

$$\frac{\omega(n)}{2} \cdot \#(\mathfrak{I}) > \frac{\omega(n)}{2} \cdot \bigl(\omega(n) + 2\bigr) > \rho.$$

Therefore there exist distinct numbers $\nu_1, \nu_2 \in \mathfrak{I}$ and $k_0 \in [1, \rho]$ such that k_0 is a ν_1-marked number and ν_2-marked number. Let $\nu_1 = \rho + 1$, $\nu_2 = \rho + 2$. Consider the ultrafilters $q_{\rho+1}$, $q'_{\rho+2}$. If $q_{\rho+1} \neq q'_{\rho+2}$, put $z_{\rho+1} = q_{\rho+1}$, $z'_{\rho+2} = q'_{\rho+2}$. Let $q_{\rho+1} = q'_{\rho+2}$. There are two possible cases.

I. *There exist the ultrafilters $q^a_{\rho+1}$, $q^b_{\rho+1}$*, and assume that $q_{\rho+1} = q^b_{\rho+1}$. Put $z_{\rho+1} = q^a_{\rho+1}$, $z'_{\rho+2} = q'_{\rho+2}$.

II. *The ultrafilters $q^a_{\rho+1}$, $q^b_{\rho+1}$ do not exist.* Then there exist the ultrafilters $\mathfrak{q}^a_{\rho+1}$, $\mathfrak{q}^b_{\rho+1}$, and assume that $q'_{\rho+1} = \mathfrak{q}^b_{\rho+1}$. Put $z_{\rho+2} = q_{\rho+2}$. If $q'_{\rho+1} \neq z_{\rho+2}$, put $z'_{\rho+1} = q'_{\rho+1}$. Otherwise we have $\mathfrak{q}^a_{\rho+1} \in L_n$ since $q'_{\rho+1} = q_{\rho+2} \in L_n$ (see in the part (3) of our proof how we have chosen the ultrafilter q'_ν); and put $z'_{\rho+1} = \mathfrak{q}^a_{\rho+1}$.

Thus, we consider either the pair of ultrafilters $z_{\rho+1}$, $z'_{\rho+2}$, or the pair of ultrafilters $z'_{\rho+1}$, $z_{\rho+2}$. These two pairs have the same properties. We will consider the pair $z_{\rho+1}$, $z'_{\rho+2}$. We have the following:

1° $z_{\rho+1}$ has an $\mathcal{A}_\rho$-equivalent ultrafilter in $\{a_{\rho+1}, b_{\rho+1}\}$;
2° $z'_{\rho+2}$ has an $\mathcal{A}_n$-equivalent ultrafilter in $\{a_{\rho+2}, b_{\rho+2}\}$;
3° $z_{\rho+1} \neq z'_{\rho+2}$;
4° $z_{\rho+1} \notin Z'_\rho$;
5° $z'_{\rho+2} \notin \mathfrak{F} \cup \mathcal{R}_{k_0}$.

Suppose that $a_{\rho+1}$ and $z_{\rho+1}$ are $\mathcal{A}_\rho$-equivalent ultrafilters, $a_{\rho+2}$ and $z'_{\rho+2}$ are $\mathcal{A}_n$-equivalent ultrafilters, and $k_0 = 3$. It is possible that

$$z'_{\rho+2} \in \{c_3, \ldots, c_{\rho-1}, d_3, \ldots, d_{\rho-1}\}.$$

Suppose that

$$z'_{\rho+2} \notin \{d_3, \ldots, d_{\rho-1}\}.$$

Now it is easy to construct the corresponding ultrafilters $a^*_1, \ldots, a^*_n, b^*_1, \ldots, b^*_n$. Let us list them in pairs: $(a^*_1, b^*_1) = (a_1, b_1)$, $(a^*_2, b^*_2) = (a_2, b_2)$, $(a^*_3, b^*_3) = (b_4, d_3), \ldots, (a^*_{\rho-1}, b^*_{\rho-1}) =$

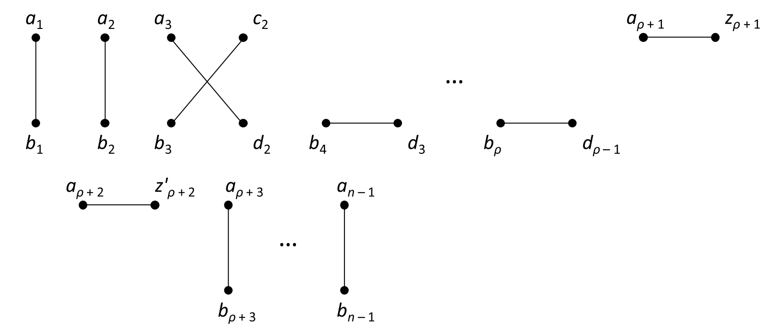

Figure 3 Construction of ultrafilters $a_1^*, \dots, a_n^*$, $b_1^*, \dots, b_n^*$.

$(b_\rho, d_{\rho-1})$, $(a_\rho^*, b_\rho^*) = (a_{\rho+1}, z_{\rho+1})$, $(a_{\rho+1}^*, b_{\rho+1}^*) = (a_3, d_2)$, $(a_{\rho+2}^*, b_{\rho+2}^*) = (b_3, c_2)$, $(a_{\rho+3}^*, b_{\rho+3}^*) = (a_{\rho+3}, b_{\rho+3}), \dots, (a_{n-1}^*, b_{n-1}^*) = (a_{n-1}, b_{n-1})$, $(a_n^*, b_n^*) = (a_{\rho+2}, z'_{\rho+2})$, see Figure 3. □

4 Combinatorial theorems

In this section we consider for each $n \in \mathbb{N}^+$ a matrix $\mathfrak{M}(n)$ which has n rows and $\aleph_0$ columns. We denote by α_i^k the element of $\mathfrak{M}(n)$ in the ith row and the kth column. The following holds:

(1) $\alpha_i^k \in \mathbb{N}$;

(2) for any $\alpha_i^k > 0$, there exists $\alpha_i^{k'}$ such that $\alpha_i^k = \alpha_i^{k'}$ and $k \neq k'$.

We denote by $w(\mathfrak{M}(n), i)$ the number of nonzero elements in the ith row of $\mathfrak{M}(n)$. It is clear that

$$0 \leq w\bigl(\mathfrak{M}(n), i\bigr) \leq \aleph_0.$$

Definition 4.1 A matrix $\mathfrak{M}(n)$ is said to be *saturated* if there exist pairwise distinct natural numbers $k_1, k_1', \dots, k_n, k_n'$ such that $\alpha_i^{k_i} = \alpha_i^{k_i'} > 0$ for each $i \in [1, n]$.

Definition 4.2 For each $n \in \mathbb{N}^+$, denote by $\mathfrak{v}'(n)$ the minimal natural number such that if for some matrix $\mathfrak{M}(n)$ we have $w(\mathfrak{M}(n), i) \geq \mathfrak{v}'(n)$ for each $i \in [1, n]$, then $\mathfrak{M}(n)$ is saturated.

We suppose that $\mathfrak{v}'(n) \in \mathbb{N}^+$ since, obviously, $\mathfrak{v}'(n) < \aleph_0$.

It is easy to prove that $\mathfrak{v}(n) = \mathfrak{v}'(n)$. Therefore, by Theorem 2.1, the following theorem is true.

Theorem 4.3 *If for some matrix $\mathfrak{M}(n)$ we have*

$$w\bigl(\mathfrak{M}(n), i\bigr) \geq \frac{10}{3} n + \sqrt{\frac{2n}{3}}$$

for each $i \in [1, n]$, then $\mathfrak{M}(n)$ is saturated.

The following theorem is a particular case of the well-known theorem of Ramsey [12].

Theorem 4.4 *Consider a set S, $\#(S) = n \in \mathbb{N}^+$, and let T be the family of all two-element subsets of S. We divide T into two disjoint sub-families T_1, T_2. Fix a natural number $\mu \geq 2$.*

We claim that there exists the minimal number $R(\mu) \in \mathbb{N}^+$ such that if $n \geq R(\mu)$, then there exists a set $S' \subset S$, $\#(S') = \mu$, and either all two-element subsets of S' belong to T_1 or they all belong to T_2.

In the formulation of the following theorem, we use the number $R(\mu)$ from Theorem 4.4.

Theorem 4.5 *Consider a matrix $\mathfrak{M}(n)$, and fix a natural number $\mu \geq 2$. Let*

$$w\bigl(\mathfrak{M}(n), i\bigr) \geq \frac{10}{3}n + \sqrt{\frac{2n}{3}}$$

for any $i \in [1, n]$, and $n \geq R(\mu)$. Then

(1) *there exist pairwise distinct natural numbers*

$$k_1, k_1', \ldots, k_n, k_n'$$

such that $\alpha_i^{k_i} = \alpha_i^{k_i'} > 0$ and $k_i < k'$ for each $i \in [1, n]$;

(2) *there exists a family of segments*

$$D \subset \bigl\{[k_i, k_i']\bigr\}_{i \leq n},$$

$\#(D) = \mu$, and one of the following two cases holds;

(a) *if $I_1, I_2 \in D$ are distinct, then $I_1 \cap I_2 = \emptyset$;*

(b) *$\cap D \neq \emptyset$.*[c]

Proof Let us use the notation of Theorem 4.4. By Theorem 4.3 there exists a corresponding family of segments

$$S = \bigl\{[k_i, k_i']\bigr\}_{i \leq n}.$$

Let T be the family of all subsets of S with the exact two elements. Divide T into two disjoint sub-families T_1, T_2. Let T_1 be the family of pairs of disjoint segments. Let T_2 be the family of pairs of distinct joint segments. By Theorem 4.4 there exists a family $D \subset S$ such that $\#(D) = \mu$ and all pairs of distinct segments from D belong either to T_1 or to T_2. If all pairs of distinct segments belong to T_2, then it is easy to see that $\cap D \neq \emptyset$. □

Remark 4.6 The following well-known result is given, for example, in [13]:

$$R(\mu) \leq \binom{2\mu - 2}{\mu - 1}.$$

Therefore Theorem 4.5 is true if the condition $n \geq R(\mu)$ will be exchanged by $n \geq \binom{2\mu-2}{\mu-1}$.

5 Countable families of σ-algebras

In the first nine subsections we present facts from [1] and [2].

Definition 5.1 A point $a \in \beta X$ is said to be *irregular* if for any countable sequence of sets $M_1, \ldots, M_k, \ldots \subset \beta X$ such that $a \notin \overline{M}_k$ for all k, we have $a \notin \overline{\cup M_k}$.

Since a point of βX is an ultrafilter on X and, *vice versa*, an ultrafilter on X is a point of βX, we will also call an irregular point an *irregular ultrafilter*. All points of X are irregular.

Definition 5.2 An algebra $\mathcal{A}$ is said to be *simple* if there exists $Z \subseteq \beta X$ such that:
(1) $\#(Z) \leq \aleph_0$;
(2) if $Z \neq \emptyset$, all points of Z are irregular;
(3) $\ker \mathcal{A} \subseteq \overline{Z}$.

The proof of the following theorem is in [2], Chapter 17.

Theorem 5.3 *Let $\mathcal{A}_1, \dots, \mathcal{A}_k, \dots$ and $\mathcal{B}_1, \dots, \mathcal{B}_k, \dots$ be two countable families of σ-algebras. Let all algebras $\mathcal{A}_k$ be simple, and among the algebras $\mathcal{B}_k$ let there be no simple algebras. Then there exist pairwise disjoint sets $W, U_1, \dots, U_k, \dots, V_1, \dots, V_k, \dots$ such that*:
(1) $\ker \mathcal{A}_k \subseteq \overline{W}$ *for each* k;
(2) *for each $k \in \mathbb{N}^+$, the following holds*: *if a set Q contains one of the two sets U_k, V_k and intersection with the other set is empty, then $Q \notin \mathcal{B}_k$.*

Remark 5.4 The Gitik-Shelah theorem is essentially used in the proof of Theorem 5.3. Under the assumption that the continuum hypothesis ($\aleph_1 = 2^{\aleph_0}$) is true, the proof of Theorem 5.3 essentially uses not the nontrivial Gitik-Shelah theorem but the rather simple Alaoglu-Erdös theorem.

Definition 5.5 The set $\{a \in \ker \mathcal{A} \mid a \text{ is an irregular point}\}$ is called the *spectrum* of an algebra $\mathcal{A}$ and is denoted $sp\mathcal{A}$.

It is clear that if $\mathcal{A}$ is a simple algebra, then $\#(sp\mathcal{A}) \leq \aleph_0$.

The proof of the lemma below is in [2], Chapter 7.

Lemma 5.6 *If $\mathcal{A}$ is a simple σ-algebra, then* $\ker \mathcal{A} \subseteq \overline{sp\mathcal{A}}$.

The proof of the lemma below is in [2], Chapter 7.

Lemma 5.7 *If $\mathcal{A}$ is a simple σ-algebra and $a \in sp\mathcal{A}$, then*

$$\{b \in sp\mathcal{A} \mid a \textit{ is } \mathcal{A}\textit{-equivalent to } b\} \neq \emptyset.$$

Remark 5.8 If an ω-saturated algebra $\mathcal{A}$ is a σ-algebra, then $\mathcal{A}$ is simple and $\ker \mathcal{A} = sp\mathcal{A}$.

The proof of the following lemma is easily derived from Lemma 5.7 and arguments in Remark 1.13.

Lemma 5.9 *Let $\mathcal{A}$ be a simple but not ω-saturated σ-algebra $\mathcal{A}$ and let $\nu \in \mathbb{N}^+$. We can construct an ω-saturated σ-algebra $\mathcal{A}'$ such that* $\ker \mathcal{A}' \subset sp\mathcal{A}$, $\#(\ker \mathcal{A}') \geq \nu$, *and two ultrafilters are $\mathcal{A}'$-equivalent if and only if they are $\mathcal{A}$-equivalent.*[d]

Proof of Theorem 2.4 Consider a sequence of integers $n_0 = 0 < n_1 < n_2 < \cdots < n_m < \cdots$. Construct the function $\varphi : \mathbb{N}^+ \to \mathbb{N}^+$ as follows: if $k \in [n_{m-1} + 1, n_m]$, where $m \in \mathbb{N}^+$, then

$$\varphi(k) = 4 \cdot n_{m-1} + \left\lceil \frac{10}{3}(n_m - n_{m-1}) + \sqrt{\frac{2(n_m - n_{m-1})}{3}} \right\rceil.$$

We can choose numbers $n_1, n_2, \dots, n_m, \dots$ such that condition (1) of our theorem is true. By Theorem 5.3 and Lemma 5.9 we can suppose that all algebras $\mathcal{A}_k$ are ω-saturated σ-algebras. Put $\mathcal{A}'_k = \mathcal{A}_k$ if $k \in [1, n_1]$. By Theorem 2.1 there exists a set of pairwise distinct irregular ultrafilters $G_1 = \{s_1, t_1, \dots, s_{n_1}, t_{n_1}\}$, and s_k, t_k are $\mathcal{A}'_k$-equivalent ultrafilters for each $k \in [1, n_1]$. Let $k \in [n_1 + 1, n_2]$ and

$$E_k = \{a \in \ker \mathcal{A}_k \setminus G_1 \mid a \text{ has } A_k\text{-equivalent ultrafilter in } \ker \mathcal{A}_k \setminus G_1\}.$$

We can construct (see Remark 1.13) ω-saturated σ-algebra $\mathcal{A}'_k$ and

(1) $\ker \mathcal{A}'_k = E_k$;

(2) two ultrafilters are $\mathcal{A}'_k$-equivalent if and only if they are $\mathcal{A}_k$-equivalent.

In view of Remark 1.12, $\mathcal{A}'_k \supseteq \mathcal{A}_k$. It is clear that

$$\#\bigl(\ker \mathcal{A}'_k\bigr) \ge \left\lceil \frac{10}{3}(n_2 - n_1) + \sqrt{\frac{2(n_2 - n_1)}{3}} \right\rceil.$$

By Theorem 2.1 there exist pairwise distinct irregular ultrafilters $s_{n_1+1}, t_{n_1+1}, \dots, s_{n_2}, t_{n_2}$, and s_k, t_k are $\mathcal{A}'_k$-equivalent ultrafilters for each $k \in [n_1 + 1, n_2]$. Put

$$G_2 = \{s_1, t_1, \dots, s_{n_2}, t_{n_2}\}.$$

It is clear that $\#(G_2) = 2n_2$. Consider algebras $\mathcal{A}_{n_2+1}, \dots, \mathcal{A}_{n_3}$. We can construct corresponding algebras $\mathcal{A}'_{n_2+1}, \dots, \mathcal{A}'_{n_3}$, and

$$\ker \mathcal{A}'_k \cap G_2 = \emptyset,$$
$$\#\bigl(\ker \mathcal{A}'_k\bigr) \ge \left\lceil \frac{10}{3}(n_3 - n_2) + \sqrt{\frac{2(n_3 - n_2)}{3}} \right\rceil$$

for each $k \in [n_2 + 1, n_3]$ and so on. Further, we consider algebras $\mathcal{A}_{n_3+1}, \dots, \mathcal{A}_{n_4}$ and so on. So we can construct pairwise distinct irregular ultrafilters

$$s_1, t_1, \dots, s_k, t_k, \dots,$$

such that s_k, t_k are $\mathcal{A}_k$-equivalent ultrafilters for each $k \in \mathbb{N}^+$. We can construct a corresponding family of sets $\{U_k^1, U_k^2\}_{k \in \mathbb{N}^+}$ (see Definition 1.3). □

Competing interests

The author declares that they have no competing interests.

Endnotes

[a] If $\#(\ker \mathcal{A}) \ge \aleph_0$, then, as it is shown in [2], $\#(\ker \mathcal{A}) \ge 2^{2^{\aleph_0}}$.

[b] In footnote a we already noticed that in this case $\#(\ker \mathcal{A}) \ge 2^{2^{\aleph_0}}$.

[c] It is clear that if $\cap D \neq \emptyset$, then $\#(\cap D) \ge 2$.

[d] It is clear that $\mathcal{A}' \supset \mathcal{A}$ (see Remark 1.12).

Received: 20 November 2014 Accepted: 27 January 2015 Published online: 01 April 2015

References

1. Grinblat, LŠ: On sets not belonging to algebras of subsets. Mem. Am. Math. Soc. **100**, 480 (1992)
2. Grinblat, LŠ: Algebras of Sets and Combinatorics. Translations of Mathematical Monographs, vol. 214. Am. Math. Soc., Providence (2002)
3. Grinblat, LŠ: Theorems on sets not belonging to algebras. Electron. Res. Announc. Am. Math. Soc. **10**, 51-57 (2004)
4. Grinblat, LŠ: On sets not belonging to algebras. J. Symb. Log. **72**, 483-500 (2007)
5. Grinblat, LŠ: Theorems with uniform conditions on sets not belonging to algebras. In: Topology Proceedings, vol. 33, pp. 361-380 (2009)
6. Grinblat, LŠ: Finite and countable families of algebras of sets. Math. Res. Lett. **17**, 613-623 (2010)
7. Grinblat, LŠ: Limitations on representing $\mathcal{P}(X)$ as a union of proper subalgebras. Proc. Am. Math. Soc. **143**, 859-868 (2015)
8. Erdös, P: Some remarks on set theory. Proc. Am. Math. Soc. **1**, 127-141 (1950)
9. Ulam, S: Zur Masstheorie in der allgemeinen Mengenlehre. Fundam. Math. **16**, 140-150 (1930)
10. Gitik, M, Shelah, S: Forcing with ideals and simple forcing notions. Isr. J. Math. **68**, 129-160 (1989)
11. Gleason, AM: Projective topological spaces. Ill. J. Math. **2**, 482-489 (1958)
12. Ramsey, FP: On a problem of formal logic. Proc. Lond. Math. Soc.. **30**, 264-286 (1930)
13. van Lint, JH, Wilson, RM: A Course in Combinatorics. Cambridge University Press, Cambridge (1992)