Lin *Journal of Inequalities and Applications* (2015) 2015:57
DOI 10.1186/s13660-015-0575-x

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Minimax problems under hierarchical structures

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Abstract
We discuss the minimax problems for set-valued mappings with several hierarchical structures, and scalar hierarchical minimax theorems, hierarchical minimax theorems, hierarchical minimax inequalities for set-valued mappings, and the existence of cone-saddle points.
Keywords: minimax theorems; minimax inequalities; cone-convexities; cone-saddle points

1 Introduction and preliminaries

Let U, V be two nonempty sets in two Hausdorff topological vector spaces, respectively, W be a Hausdorff topological vector space, $D \subset W$ a closed convex and pointed cone with apex at the origin and $\operatorname{int} D \neq \emptyset$. Let $D^\star = \{g \in W^\star : g(c) \geq 0 \text{ for all } c \in D\}$, where $W^\star$ is the set of all continuous linear functional on W. The scalar hierarchical minimax theorems are introduced and discussed by Lin [1] as follows: given three mappings $A, B, C : U \times V \rightrightarrows \mathbb{R}$, under suitable conditions the following relation holds:

$$\min \bigcup_{u \in U} \max \bigcup_{v \in V} A(u,v) \leq \max \bigcup_{v \in V} \min \bigcup_{u \in U} C(u,v). \tag{sH}$$

In [1], the three versions (H_1)-(H_3) of minimax theorems with hierarchical structures are also discussed: given three mappings $A, B, C : U \times V \rightrightarrows W$, under suitable conditions the following relation holds:

$$\operatorname{Max} \bigcup_{v \in V} \operatorname{Min}_w \bigcup_{u \in U} C(u,v) \subset \operatorname{Min}\left(\operatorname{co} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in V} A(u,v)\right) + D, \tag{H_1}$$

$$\operatorname{Max} \bigcup_{v \in V} \operatorname{Min}_w \bigcup_{u \in U} C(u,v) \subset \operatorname{Min} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in V} A(u,v) + D, \tag{H_2}$$

$$\operatorname{Min} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in V} A(u,v) \subset \operatorname{Max} \bigcup_{v \in V} \operatorname{Min}_w \bigcup_{u \in U} C(u,v) + W \setminus \big(D \setminus \{0\}\big). \tag{H_3}$$

In [2], given three mappings $A, B, C : U \times U \rightrightarrows W$, Lin *et al.* investigated the following two versions of minimax inequalities, the so-called hierarchical minimax inequalities:

$$\operatorname{Max} \bigcup_{u \in U} C(u,u) \subset \operatorname{Min}\left(\operatorname{co}\left(\bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in U} A(u,v)\right)\right) + D, \tag{Hi_1}$$

$$\operatorname{Max} \bigcup_{u \in U} C(u,u) \subset \operatorname{Min} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in U} A(u,v) + D. \tag{Hi_2}$$

In this paper, we propose new hierarchical structures relative to several non-continuous set-valued mappings which obey one of the following relations: (sH), (H_1), (H_2), (H_3), (Hi_1), and (Hi_2). As applications, the existence of saddle points for set-valued mappings is also discussed.

The fundamental concepts of maximal (minimal) point and weakly maximal (weakly minimal) point will be used in the sequel.

Definition 1 [3, 4] Let L be a nonempty subset of W. A point $w \in L$ is called a

(a) *minimal point* of L if $L \cap (w - D) = \{w\}$; $\operatorname{Min} L$ denotes the set of all minimal points of L;
(b) *maximal point* of L if $L \cap (w + D) = \{w\}$; $\operatorname{Max} L$ denotes the set of all maximal points of L;
(c) *weakly minimal point* of L if $L \cap (w - \operatorname{int} D) = \emptyset$; $\operatorname{Min}_w L$ denotes the set of all weakly minimal points of L;
(d) *weakly maximal point* of L if $L \cap (w + \operatorname{int} D) = \emptyset$; $\operatorname{Max}_w L$ denotes the set of all weakly maximal points of L.

Both Max and Max_w are denoted by max (both Min and Min_w by min) in $\mathbb{R}$ since both Max and Max_w (both Min and Min_w) are the same in $\mathbb{R}$. We note that for a nonempty compact set L, both sets $\operatorname{Max} L$ and $\operatorname{Min} L$ are nonempty. Furthermore, $\operatorname{Min} L \subset \operatorname{Min}_w L$, $\operatorname{Max} L \subset \operatorname{Max}_w L$, $L \subset \operatorname{Min} L + D$, and $L \subset \operatorname{Max} L - D$.

Definition 2 [5, 6] Let $\mathfrak{U}$, $\mathfrak{V}$ be two Hausdorff topological spaces. A set-valued mapping $F : \mathfrak{U} \rightrightarrows \mathfrak{V}$ with nonempty values is said to be

(a) *upper semicontinuous on* $\mathfrak{U}$ *if for any* $x_0 \in \mathfrak{U}$ and for every open set N containing $F(x_0)$, there exists a neighborhood M of x_0 such that $F(M) \subset N$;
(b) *lower semicontinuous on* $\mathfrak{U}$ *if for any* $x_0 \in \mathfrak{U}$ and any sequence $\{x_n\} \subset \mathfrak{U}$ such that $x_n \to x_0$ and any $y_0 \in F(x_0)$, there exists a sequence $y_n \in F(x_n)$ such that $y_n \to y_0$;
(c) *continuous on* $\mathfrak{U}$ *if F is both upper semicontinuous and lower semicontinuous at any* $x_0 \in \mathfrak{U}$.

Definition 3 [4, 7] The *Gerstewitz function* $\varphi_{kw} : W \to \mathbb{R}$ is defined by

$$\varphi_{kw}(u) = \min\{t \in \mathbb{R} : u \in w + tk - D\},$$

where $k \in \operatorname{int} D$ and $w \in W$.

Some properties of the scalarization function are as follows:

Proposition 1 [4, 7] *The Gerstewitz function* $\varphi_{kw} : W \to \mathbb{R}$ *has the following properties*:

(a) $\varphi_{kw}(w) > r \Leftrightarrow w \notin w + rk - D$;
(b) $\varphi_{kw}(w) \geq r \Leftrightarrow w \notin w + rk - \operatorname{int} D$;
(c) $\varphi_{kw}(\cdot)$ *is a convex function*;
(d) $\varphi_{kw}(\cdot)$ *is an increasing function, that is*, $w_2 - w_1 \in \operatorname{int} D \Rightarrow \varphi_{kw}(w_1) < \varphi_{kw}(w_2)$;
(e) $\varphi_{kw}(\cdot)$ *is a continuous function.*

We also need the following cone-convexities for set-valued mappings.

Definition 4 [3] Let U be a nonempty convex subset of a topological vector space. A set-valued mapping $F : U \rightrightarrows W$ is said to be

(a) *above-D-convex* (respectively, *above-D-concave*) on W if for all $u_1, u_2 \in U$ and all $\alpha \in [0, 1]$,

$$F\bigl(\alpha u_1 + (1-\alpha)u_2\bigr) \subset \alpha F(u_1) + (1-\alpha)F(u_2) - D$$
$$\bigl(\text{respectively, } \alpha F(u_1) + (1-\alpha)F(u_2) \subset F\bigl(\alpha u_1 + (1-\alpha)u_2\bigr) - D\bigr);$$

(b) *above-naturally D-quasi-convex* on W if for all $u_1, u_2 \in U$ and all $\alpha \in [0, 1]$,

$$F\bigl(\alpha u_1 + (1-\alpha)u_2\bigr) \subset \operatorname{co}\bigl\{F(u_1) \cup F(u_2)\bigr\} - D,$$

where $\operatorname{co} A$ denotes the convex hull of a set A; and

(c) *above-D-quasi-convex* on W if for each $w \in W$, the set $\{u \in U : F(u) \subset w - D\}$ is a convex subset of U.

By definition, the above-D-convex mapping is also an above-naturally D-quasi-convex on U. The following whole intersection theorem is a variant form of Ha [8].

Lemma 1 *Let U be a nonempty convex subset of a real Hausdorff topological space, V be a nonempty compact convex subset of a real Hausdorff topological space. Let the three mappings $L, M, N : U \rightrightarrows V$ with $L(u) \subset M(u) \subset N(u)$ for all $u \in U$ satisfy*

(a) *$L(u)$, $N(u)$ are open in V for each $u \in U$, $L^{-1}(v)$, $N^{-1}(v)$ are convex in U for each $v \in V$; and*

(b) *$V \setminus M(u)$ is convex for each $u \in U$, and $M^{-1}(v)$ is open in U for each $v \in V$.*

Then either there is an $v_0 \in V$ such that $L^{-1}(v_0)$ is a empty set, or the whole intersection $\bigcap_{v \in V} N^{-1}(v)$ is nonempty.

In the sequel we also need the following proposition.

Proposition 2 *Let U be a nonempty set, $k \in \operatorname{int} D$ and $w \in W$. Suppose that the set-valued mappings $F, G : U \rightrightarrows W$ come with nonempty compact values and, for some $u \in U$, $\operatorname{Max}_w F(u) \subset \operatorname{Max}_w G(u) - D$. We have the following two results*:

(a) *for any $\varphi \in D^\star$, the inequality*

$$\max \varphi F(u) \le \max \varphi G(u)$$

holds;

(b) *for the Gerstewitz function $\varphi_{kw} : W \to \mathbb{R}$, the inequality*

$$\max \varphi_{kw} F(u) \le \max \varphi_{kw} G(u)$$

holds.

Proof For the proof of (a), we refer to Proposition 1.2 [9]. We omit the proof of (b) since it is quite similar to the proof of (a). □

2 Scalar hierarchical minimax theorems

We first establish the following scalar hierarchical minimax theorem.

Theorem 1 *Let U be a nonempty convex subset of a real Hausdorff topological space, V be a nonempty compact convex subset of a real Hausdorff topological space. Suppose that the set-valued mappings $A, B, C : U \times V \rightrightarrows \mathbb{R}$ with nonempty compact values satisfy the following conditions*:

(i) *the mappings $u \mapsto A(u, v)$ and $u \mapsto C(u, v)$ are above-$\mathbb{R}_+$-quasi-convex on U for each $v \in V$, and the mappings $v \mapsto A(u, v)$ and $v \mapsto C(u, v)$ are upper semicontinuous on V for each $u \in U$*;

(ii) *the mapping $u \mapsto B(u, v)$ is upper semicontinuous on U for each $v \in V$, and the mapping $v \mapsto B(u, v)$ is above-$\mathbb{R}_+$-concave for each $u \in U$*; *and*

(iii) *for all $(u, v) \in U \times V$, $\max A(u, v) \leq \max B(u, v) \leq \max C(u, v)$.*

Then, for each $t \in \mathbb{R}$, either there is $v_0 \in V$ such that

$$C(u, v_0) \cap (t + \mathbb{R}_+) \neq \emptyset$$

for all $u \in U$, or there is $u_0 \in U$ such that

$$A(u_0, v) \subset t - \operatorname{int} \mathbb{R}_+$$

for all $v \in V$.

Proof Give any $t \in \mathbb{R}$. Define three mappings $L, M, N : U \rightrightarrows V$ by

$$L(u) = \{v \in V : \forall h \in C(u, v), h < t\},$$
$$M(u) = \{v \in V : \forall g \in B(u, v), g < t\},$$

and

$$N(u) = \{v \in V : \forall f \in A(u, v), f < t\}$$

for all $u \in U$. By (iii), $L(u) \subset M(u) \subset N(u)$ for all $u \in U$.

Since the mapping $u \mapsto C(u, v)$ is above-$\mathbb{R}_+$-quasi-convex on U for each $v \in V$, the set $L^{-1}(v)$ is convex for each $v \in V$. Similarly, the set $N^{-1}(v)$ is convex for each $v \in V$. Next, we claim that the set $L(u)$ is open in V, or the set $V \setminus L(u) = \{v \in V : \exists h \in C(u, v), h \geq t\}$ is closed for each $u \in U$. For any net $\{v_\nu\} \subset V \setminus L(u)$ that converges to some point $v_0 \in V$, there exists $h_\nu \in C(u, v_\nu)$ such that $h_\nu \geq t$. By the upper semicontinuity of H at v, $C(u, v_0)$ is compact. By Lemma 2.2 [10], there exist $h_0 \in C(u, v_0)$ and a subnet $\{h_{\nu_\alpha}\}$ that converges to h_0. Since $h_{\nu_\alpha} \geq t$, we have $h_0 \geq t$, and hence $v_0 \in V \setminus L(u)$. This proves that the set $V \setminus L(u)$ is closed, and the set $L(u)$ is open for each $u \in U$. Similarly, by the upper semicontinuity of A and B, the sets $M^{-1}(v)$ and $N(u)$ are open for each $u \in U$ and $v \in V$.

Next, we claim that the set $V \setminus M(u)$ is convex in V for each $u \in U$. For each $u \in U$, for any $v_1, v_2 \in V \setminus M(u)$ and any $\tau \in [0, 1]$. There exist $g_1 \in B(u, v_1)$ with $g_1 \geq t$ and $g_2 \in B(u, v_2)$ with $g_2 \geq t$, $\tau g_1 + (1 - \tau) g_2 \geq t$. By the above-$\mathbb{R}_+$-concavity of B,

$$\tau g_1 + (1 - \tau) g_2 \subset B(u, \tau v_1 + (1 - \tau) v_2) - \mathbb{R}_+.$$

Thus, there is a $g_\tau \in B(x, \tau v_1 + (1-\tau)v_2)$ such that $\tau g_1 + (1-\tau)g_2 \le g_\tau$. Hence, $\tau v_1 + (1-\tau)v_2 \in V \setminus M(u)$ and the set $V \setminus M(u)$ is convex in V for each $u \in U$.

Since all conditions of Lemma 1 hold, by Lemma 1, either there is an $v_0 \in V$ such that $L^{-1}(v_0)$ is an empty set, or the whole intersection $\bigcap_{v\in V} D^{-1}(v)$ is nonempty. That is, for each $t \in \mathbb{R}$, either there is $v_0 \in V$ such that

$$C(u, v_0) \cap (t + \mathbb{R}_+) \neq \emptyset$$

for all $u \in U$, or there is $u_0 \in U$ such that

$$A(u_0, v) \subset t - \operatorname{int} \mathbb{R}_+$$

for all $v \in V$. □

Theorem 2 *We work under the framework of Theorem* 1, *in addition*, U *is compact*, *for each* $(u, v) \in U \times V$, *the union* $\bigcup_{u\in U} C(u, v)$ *is compact*, *and the mappings* $u \mapsto A(u, v)$ *and* $v \mapsto C(u, v)$ *are lower semicontinuous on* U *and* V, *respectively*. *If the following condition holds*: *for each* $v \in V$, *there is an* $u_v \in U$ *such that*

$$\max C(u_v, v) \le \max \bigcup_{v\in V} \min \bigcup_{u\in U} C(u, v), \tag{L}$$

then (sH) *is valid*.

Proof For any $t > \max \bigcup_{v\in V} \min \bigcup_{u\in U} C(u, v)$. From (L), we see that, for each $v \in V$ there is an $u_v \in U$ such that

$$C(u_v, v) \cap (t + \mathbb{R}_+) = \emptyset.$$

Hence, by Theorem 1, there is $u_0 \in U$ such that

$$A(u_0, v) \subset t - \operatorname{int} \mathbb{R}_+$$

for all $v \in V$. This will suffice to show that (sH) holds. □

We note that Theorems 1 and 2 include some special cases as follows.

Corollary 1 *If we replace* (iii) *of Theorem* 2 *by any one of the following conditions*:

(i) *for all* $(u, v) \in U \times V$, $A(u, v) = B(u, v) = C(u, v)$;

(ii) *for all* $(u, v) \in U \times V$, $A(u, v) \subset B(u, v) = C(u, v)$;

(iii) *for all* $(u, v) \in U \times V$, $A(u, v) = B(u, v) \subset C(u, v)$;

(iv) *for all* $(u, v) \in U \times V$, $A(u, v) \subset B(u, v) \subset C(u, v)$;

(v) *for all* $(u, v) \in U \times V$, $\max A(u, v) \le \max B(u, v) \le \max C(u, v)$, *but* $A(u, v) \neq B(u, v) \neq C(u, v)$;

(vi) *for all* $(u, v) \in U \times V$, $\max A(u, v) \le \max B(u, v) \le \max C(u, v)$, *but* $A(u, v) \subset B(u, v) \neq C(u, v)$;

(vii) *for all* $(u, v) \in U \times V$, $\max A(u, v) \le \max B(u, v) \le \max C(u, v)$, *but* $A(u, v) \neq B(u, v) \subset C(u, v)$,

then (sH) *is valid*.

We state the first one of Corollary 1 as follows.

Corollary 2 *Let U, V be two nonempty compact convex subset of real Hausdorff topological spaces, respectively. Suppose that the set-valued mappings $A : U \times V \rightrightarrows \mathbb{R}$ come with nonempty compact values and satisfy the following conditions*:

(i) *the mapping $u \mapsto A(u,v)$ is above-$\mathbb{R}_+$-quasi-convex on U for each $v \in V$, and the mapping $v \mapsto A(u,v)$ is continuous on V for each $u \in U$;*

(ii) *the mapping $u \mapsto A(u,v)$ is continuous on U for each $v \in V$, and the mapping $v \mapsto A(u,v)$ is above-$\mathbb{R}_+$-concave for each $u \in U$.*

If the following condition holds: *for each $v \in V$, there is an $u_v \in U$ such that*

$$\max A(u_v, v) \leq \max \bigcup_{v\in V} \min \bigcup_{u\in U} A(u,v),$$

then (sH) *with $A = B = C$ is valid.*

From Proposition 3.12 [3], every above-naturally $\mathbb{R}_+$-quasi-convex is an above-$\mathbb{R}_+$-quasi-convex. We can see that Corollary 1 slightly generalizes Theorem 2.1 [4].

3 Hierarchical minimax theorems

In this section, we will discuss three versions of hierarchical minimax theorems. The first one is as follows.

Theorem 3 *Let U, V be nonempty compact convex subsets of real Hausdorff topological spaces, respectively, W be a complete locally convex Hausdorff topological vector space. Suppose that the set-valued mappings $A, B, C : U \times V \rightrightarrows W$ come with nonempty compact values and satisfy the following conditions*:

(i) *$(u,v) \mapsto A(u,v)$ is upper semicontinuous on $U \times V$, and $u \mapsto A(u,v)$ is above-naturally D-quasi-convex and lower semicontinuous on U for each $v \in V$;*

(ii) *$u \mapsto B(u,v)$ is upper semicontinuous on U for each $v \in V$, and $v \mapsto B(u,v)$ is above-D-concave on V for each $u \in U$;*

(iii) *$(u,v) \mapsto C(u,v)$ is upper semicontinuous on $U \times V$, $u \mapsto C(u,v)$ is above-naturally D-quasi-convex on U for each $v \in V$, and $v \mapsto C(u,v)$ is continuous on V for each $u \in U$;*

(iv) *for any $\varphi \in C^\star$ and for each $v \in V$, there is an $u_v \in U$ such that*

$$\max \varphi C(u_v, v) \leq \max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi C(u,v);$$

(v) *for each $v \in V$,*

$$\mathrm{Max} \bigcup_{v\in V} \mathrm{Min}_w \bigcup_{u\in U} C(u,v) \subset \mathrm{Min}_w \bigcup_{u\in U} C(u,v) + D; \quad \textit{and}$$

(vi) *for all $(u,v) \in U \times V$, $\mathrm{Max}_w A(u,v) \subset \mathrm{Max}_w B(u,v) - D$, and $\mathrm{Max}_w B(u,v) \subset \mathrm{Max}_w C(u,v) - D$.*

Then (H_1) *is valid.*

Proof We omit some parts of the proof since the techniques of the proof are similar to Theorem 3.1 [1]. Suppose that $v \notin \mathrm{co}(\bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v)) + D$. There is a nonzero continuous linear functional $\varphi : Z \mapsto \mathbb{R}$ such that

$$\varphi(v) < \min \bigcup_{u\in U} \max \bigcup_{v\in V} \varphi A(u,v).$$

Since $u \mapsto A(u,v)$ and $u \mapsto C(u,v)$ are above-naturally D-quasi-convex for each $v \in V$, by Proposition 3.13 [3], $u \mapsto \varphi A(u,v)$ and $u \mapsto \varphi C(u,v)$ are above-naturally $\mathbb{R}_+$-quasi-convex for each $v \in V$ and $\varphi \in C^\star$. Since $v \mapsto B(u,v)$ is above-D-concave on V for each $u \in U$, by Proposition 3.9 [3], $v \mapsto \varphi B(u,v)$ is above-$\mathbb{R}_+$-concave on V for each $u \in U$ and $\varphi \in C^\star$. Since every $\varphi \in C^\star$ is continuous, all continuities of Theorem 2 are satisfied for the mappings φA, φB, φC. By Proposition 2 and (vi), $\varphi A(u,v) \leq \varphi B(u,v) \leq \varphi C(u,v)$ for all $(u,v) \in U \times V$. Thus, all conditions of Theorem 2 hold for φA, φB, φC. Hence,

$$\varphi(v) < \max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi C(u,v).$$

Since V is compact, there is a $v' \in V$ such that

$$\varphi(v) < \min \bigcup_{u\in U} \varphi C(u,v').$$

Thus,

$$v \notin \bigcup_{u\in U} C(u,v') + D,$$

and hence,

$$v \notin \mathrm{Min}_w \bigcup_{u\in U} C(u,v') + D. \tag{2}$$

If $v \in \mathrm{Max} \bigcup_{v\in V} \mathrm{Min}_w \bigcup_{u\in U} C(u,v)$, then, by (v),

$$v \in \mathrm{Min}_w \bigcup_{u\in U} C(u,v') + D,$$

which contradicts (2). Hence, for every $v \in \mathrm{Max} \bigcup_{v\in V} \mathrm{Min}_w \bigcup_{u\in U} C(u,v)$,

$$v \in \mathrm{co}\left(\bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v)\right) + D.$$

That is, (H_1) is valid. □

Corollary 3 *If we replace* (vi) *of Theorem* 3 *by any one of the following conditions*:

(i) *for all* $(u,v) \in U \times V$, $A(u,v) = B(u,v) = C(u,v)$;
(ii) *for all* $(u,v) \in U \times V$, $A(u,v) \subset B(u,v) = C(u,v)$;
(iii) *for all* $(u,v) \in U \times V$, $A(u,v) = B(u,v) \subset C(u,v)$;
(iv) *for all* $(u,v) \in U \times V$, $A(u,v) \subset B(u,v) \subset C(u,v)$;

(v) *for all* $(u,v) \in U \times V$, *for all* $(u,v) \in U \times V$, $\mathrm{Max}_w A(u,v) \subset \mathrm{Max}_w B(u,v) - D$, *and* $\mathrm{Max}_w B(u,v) \subset \mathrm{Max}_w C(u,v) - D$, *but* $A(u,v) \not\subset B(u,v) \not\subset C(u,v)$;

(vi) *for all* $(u,v) \in U \times V$, *for all* $(u,v) \in U \times V$, $\mathrm{Max}_w A(u,v) \subset \mathrm{Max}_w B(u,v) - D$, *and* $\mathrm{Max}_w B(u,v) \subset \mathrm{Max}_w C(u,v) - D$, *but* $A(u,v) \subset B(u,v) \not\subset C(u,v)$;

(vii) *for all* $(u,v) \in U \times V$, $\mathrm{Max}_w A(u,v) \subset \mathrm{Max}_w B(u,v) - D$, *and* $\mathrm{Max}_w B(u,v) \subset \mathrm{Max}_w C(u,v) - D$, *but* $A(u,v) \not\subset B(u,v) \subset C(u,v)$,

then (H_1) *is valid*.

The following example illustrates that Theorem 3 is true.

Example 1 Let $U = V = [0,1]$, $D = \mathbb{R}^2_+$, and $f : U \rightrightarrows \mathbb{R}$ be defined by

$$f(v) = \begin{cases} [-1,0], & v = 0, \\ \{0\}, & v \neq 0. \end{cases}$$

Define $A, B, C : U \times V \rightrightarrows \mathbb{R}^2$ by

$$\begin{aligned} A(u,v) &= \{1 - \cos(u\pi/2)\} \times f(v), \\ B(u,v) &= \{1 + \cos(u\pi/2)\} \times [v-1,1], \\ C(u,v) &= \{2 + u^2\} \times [v^2+1,2], \end{aligned}$$

for all $(u,v) \in U \times V$.

We can easily see that the mappings A, B, C satisfy (vi) and all continuities in Theorem 3. For each $v \in V$, the mapping $u \mapsto A(u,v)$ is above-naturally D-quasi-convex on U for each $v \in V$ since, for any $\alpha \in [0,1]$ and $u_1, u_2 \in U$,

$$\begin{aligned} &A(\alpha u_1 + (1-\alpha)u_2, v) \\ &\quad = \{1 - \cos((\alpha u_1 + (1-\alpha)u_2)\pi/2)\} \times f(v) \\ &\quad \subset \alpha\{1 - \cos(u_1\pi/2)\} \times f(v) + (1-\alpha)\{1 - \cos(u_2\pi/2)\} \times f(v) - D \\ &\quad = \mathrm{co}\{A(u_1,v) \cup A(u_2,v)\} - D. \end{aligned}$$

We see that the mapping $v \mapsto B(u,v)$ is above-D-concave on V for each $u \in U$ since, for any $\alpha \in [0,1]$ and $v_1, v_2 \in V$,

$$\begin{aligned} &\alpha B(u,v_1) + (1-\alpha)B(v_2) \\ &\quad = \alpha\{1 + \cos(u\pi/2)\} \times [v_1 - 1, 1] + (1-\alpha)\{1 + \cos(u\pi/2)\} \times [v_2 - 1, 1] \\ &\quad = \{1 + \cos(u\pi/2)\} \times [\alpha v_1 + (1-\alpha)v_2 - 1, 1] \\ &\quad \subset B(u, \alpha v_1) + (1-\alpha)v_2 - D. \end{aligned}$$

We note that the mapping $u \mapsto C(u,v)$ is above-D-convex on U for each $v \in V$. Hence, by definition, $u \mapsto C(u,v)$ is above-naturally D-quasi-convex on U for each $v \in V$. Thus, conditions (i)-(iii) of Theorem 3 are valid. Now we claim that condition (iv) holds. Indeed, for each $v \in V$ and $\varphi = (\varphi_1, \varphi_2) \in D^\star$, we need to find an $u_v \in U$ such that

$$\begin{aligned}\max \varphi C(u_v, v) &= \max\{\varphi_1(2 + u^2) + \varphi_2 t : v^2 + 1 \le t \le 2\} \\ &= \varphi_1(2 + u^2) + 2\varphi_2 \\ &\le 2\varphi_1 + 2\varphi_2 \\ &= \max \bigcup_{v \in V} \min \bigcup_{u \in U} \varphi C(u, v).\end{aligned}$$

Hence, we choose u_v by the following rule:

$$u_v = \begin{cases} \text{any point in } [0,1], & \varphi_1 = 0, \\ 0, & \varphi_1 \neq 0, \end{cases}$$

then (iv) of Theorem 3 holds. Next, we claim (v) of Theorem 3 is valid. Indeed, by a simple calculation, we get

$$\begin{aligned}&\operatorname{Max} \bigcup_{v \in V} \operatorname{Min}_w \bigcup_{u \in U} C(u, v) \\ &\quad = \{(3,2)\} \\ &\quad \subset (\{2\} \times [y^2 + 1, 2]) \bigcup ([2,3] \times \{y^2 + 1\}) + D \\ &\quad = \operatorname{Min}_w \bigcup_{u \in U} C(u, v) + D\end{aligned}$$

for each $v \in V$. Thus, condition (v) of Theorem 3 holds. By Theorem 3, (H_1) is valid. Indeed,

$$\begin{aligned}&\operatorname{Max} \bigcup_{v \in V} \operatorname{Min}_w \bigcup_{u \in U} C(u, v) \\ &\quad = \{(3,2)\} \\ &\quad \subset \{(0,-1)\} + D \\ &\quad = \operatorname{Min}\left(\operatorname{co} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in V} A(u, v)\right) + D,\end{aligned}$$

and hence the conclusion of Theorem 3 is valid.

In the following result, we apply the Gerstewitz function $\varphi_{kw} : W \mapsto \mathbb{R}$ to introduce the second version of the hierarchical minimax theorems, where $k \in \operatorname{int} D$ and $w \in W$.

Theorem 4 *Let U, V be nonempty compact convex subsets of real Hausdorff topological spaces, respectively, W be a real Hausdorff topological vector space. We work under the framework of Theorem* 3 *except* (iv) *and the concavity of B. If, in addition, the mapping $v \mapsto \varphi_{kw} B(u, v)$ is above-$\mathbb{R}_+$-concave on V for each $u \in U$, and for any Gerstewitz function*

φ_{kw} *and for each* $v \in V$, *there is an* $u_v \in U$ *such that*

$$\text{(iv)}' \quad \max \varphi_{kw} C(u_v, v) \le \max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi_{kw} C(u,v);$$

then (H_2) *is valid.*

Proof Using the same steps as in the proof of Theorem 3, we see that the set $\bigcup_{u\in U} \mathrm{Max}_w \times \bigcup_{v\in V} A(u,v)$ is nonempty and compact. Suppose that $v \notin \bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v) + D$. For any $k \in \operatorname{int} D$, there is a Gerstewitz function $\varphi_{kw} : W \mapsto \mathbb{R}$ such that

$$\varphi_{kw}(u) > 0 \tag{3}$$

for all $u \in \bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v)$. Then, for each $u \in U$, there is $v_u^\star \in Y$ and $f(u, v_u^\star) \in F(u, v_x^\star)$ with $f(u, v_u^\star) \in \mathrm{Max}_w \bigcup_{v\in V} A(u,v)$ such that

$$\varphi_{kw}\bigl(f\bigl(u, v_u^\star\bigr)\bigr) = \max \bigcup_{v\in V} \varphi_{kw} A(u,v).$$

Choosing $u = f(u, v_u^\star)$ in (3),

$$\max \bigcup_{v\in V} \varphi_{kw} A(u,v) > 0$$

for all $u \in U$. Therefore,

$$\min \bigcup_{u\in U} \max \bigcup_{v\in V} \varphi_{kw} A(u,v) > 0.$$

By conditions (i)-(iii) and (iv′), we see that all conditions of Theorem 2 hold for the mappings $\varphi_{kw}A$, $\varphi_{kw}B$, $\varphi_{kw}C$, and hence, by (sH),

$$\max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi_{kw} C(u,v) > 0.$$

Since V is compact, there is a $y' \in Y$ such that

$$\min \bigcup_{u\in U} \varphi_{kw} C\bigl(u, v'\bigr) > 0.$$

Thus,

$$v \notin \bigcup_{u\in U} C\bigl(u, v'\bigr) + D,$$

and hence

$$v \notin \mathrm{Min}_w \bigcup_{u\in U} C\bigl(u, v'\bigr) + D. \tag{4}$$

If $v \in \mathrm{Max} \bigcup_{v\in V} \mathrm{Min}_w \bigcup_{u\in U} A(u,v)$, then, by (v),

$$v \in \mathrm{Min}_w \bigcup_{u\in U} C(u,v') + D,$$

which contradicts (4). From this, we can deduce (H_2). □

The third version of hierarchical minimax theorems is as follows. We remove condition (v) in Theorem 4 to deduce (H_3).

Theorem 5 *We work under the framework of Theorem* 4 *except condition* (v). *Equation* (H_3) *is valid*.

Proof Following the proof of Theorem 4. Fix any $v \in \mathrm{Min} \bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v)$. Then

$$\left(\bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v)\right) \setminus \{v\} \cap (v - D) = \emptyset.$$

For any $k \in \operatorname{int} D$, there is a Gerstewitz function $\varphi_{kw} : W \mapsto \mathbb{R}$ such that

$$\varphi_{kw}(u) > 0$$

for all $u \in \bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in V} A(u,v) \setminus \{v\}$. For each $u \in U$,

$$\max \bigcup_{v\in V} \varphi_{kw} A(u,v) \geq 0,$$

or

$$\min \bigcup_{u\in U} \max \bigcup_{v\in V} \varphi_{kw} A(u,v) \geq 0.$$

Hence, by Theorem 2 for the mappings $\varphi_{kw}A$, $\varphi_{kw}B$, $\varphi_{kw}C$,

$$\max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi_{kw} C(u,v) \geq 0.$$

Since U and V are compact, there are $u_0 \in U$, $v_0 \in V$, and $h_0 \in C(u_0, v_0)$ such that

$$\varphi_{kw}(h_0) = \min \bigcup_{u\in U} \varphi_{kw} C(u, v_0) \geq 0.$$

Applying Proposition 3.14 [3], $h_0 \in \mathrm{Min}_w \bigcup_{u\in U} C(u,v_0)$. If $h_0 = v$, $v \notin h_0 + (D \setminus \{0\})$. If $h_0 \neq v$, $\varphi_{kw}(h_0) > 0$, and hence $h_0 \notin v - D$. Therefore, $v \notin h_0 + (D \setminus \{0\})$. Thus, in any case, $v \in h_0 + W \setminus (D \setminus \{0\})$. This implies ($\mathrm{H}_3$). □

4 Hierarchical minimax inequalities

As an application of scalar hierarchical minimax theorems, we discuss minimax inequalities which were investigated by Lin *et al.* [2]. The following result as regards (Hi_1) is different from [2] and holds under very different conditions.

Theorem 6 *Let U be a nonempty compact convex subset of a real Hausdorff topological vector space, W be a complete locally convex Hausdorff topological vector space. Let the set-valued mappings $A, B, C : U \times U \rightrightarrows W$ come with nonempty compact values and satisfy the following conditions*:

(i) *the mappings $u \mapsto A(u,v)$ and $u \mapsto C(u,v)$ are above-naturally D-quasi-convex on U for each $v \in U$, the mappings $(u,v) \mapsto A(u,v)$ and $(u,v) \mapsto C(u,v)$ are upper semicontinuous on U for each $u \in U$, and the mappings $u \mapsto A(u,v)$ and $v \mapsto C(u,v)$ are lower semicontinuous on U*;

(ii) *$u \mapsto B(u,v)$ is upper semicontinuous on U for each $v \in U$, and the mapping $v \mapsto B(u,v)$ is above-D-concave on U for each $u \in U$*;

(iii) *for each $v \in U$, for each $\varphi \in D^\star$, there is an $u_v \in U$ such that*

$$\max \varphi C(u_v, u) \leq \max \bigcup_{v \in U} \min \bigcup_{u \in U} \varphi C(u,v);$$

(iv) *for each $v \in U$,*

$$\mathrm{Max} \bigcup_{u \in U} C(u,u) \subset \mathrm{Min}_w \bigcup_{u \in U} C(u,v) + D; \textit{and}$$

(v) *for all $(u,v) \in U \times U$,*

$$\mathrm{Max}_w A(u,v) \subset \mathrm{Max}_w B(u,v) - D$$

and

$$\mathrm{Max}_w B(u,v) \subset \mathrm{Max}_w C(u,v) - D.$$

Then (Hi_1) *is valid.*

Proof Suppose that $v \notin \mathrm{co}(\bigcup_{u \in U} \mathrm{Max}_w \bigcup_{v \in U} A(u,v)) + D$. With the help of technique in the proof of Theorems 2 and 3 for the mappings φA, φB, φC, we can see that

$$\varphi(v) < \max \bigcup_{y \in U} \min \bigcup_{u \in U} \varphi C(u,v).$$

In a similar way to Theorem 3, there is a $v' \in V$ such that

$$\varphi(v) < \min \bigcup_{u \in U} \varphi C\bigl(u, v'\bigr).$$

Hence, $v \notin \mathrm{Min}_w \bigcup_{u \in U} C(u,v') + D$. By condition (iv), we see that

$$v \notin \mathrm{Max} \bigcup_{u \in U} C(u,u).$$

Therefore, (Hi_1) is valid. □

In the following example we modify Example 1, which serves to illustrate Theorem 6.

Example 2 Let $X = [0,1]$, $D = \mathbb{R}^2_+$ and $f : U \rightrightarrows \mathbb{R}$ be defined by

$$f(v) = \begin{cases} [-1,0], & y = 0, \\ \{0\}, & y \neq 0. \end{cases}$$

Define $A, B, C : U \times U \rightrightarrows \mathbb{R}^2$ by

$$\begin{aligned} A(u,v) &= \{1 - \cos(u\pi/2)\} \times f(v), \\ B(u,v) &= \{1 + \cos(u\pi/2)\} \times [v-1,1], \\ C(u,v) &= \{2 + u^2\} \times [v^2 + 1, 2], \end{aligned}$$

for all $(u,v) \in U \times V$.

We can easily see that the mappings A, B, C satisfy (v) and all continuities in Theorem 6. From the illustrations in Example 1, we see that the mapping $u \mapsto A(u,v)$ is above-naturally D-quasi-convex on U for each $v \in V$, the mapping $v \mapsto B(u,v)$ is above-D-concave on V for each $u \in U$, the mapping $u \mapsto C(u,v)$ is above-naturally D-quasi-convex on U for each $v \in V$. Furthermore, for each $v \in V$ and $\varphi = (\varphi_1, \varphi_2) \in D^\star$, by using the same choice of u_v as in Example 1, (iii) of Theorem 6 holds. Next, we claim (iv) of Theorem 6 is valid. Indeed, by a simple calculation, we get

$$\begin{aligned} &\operatorname{Max} \bigcup_{u \in U} C(u,u) \\ &\quad = \operatorname{Max} \bigcup_{u \in U} \{2 + u^2\} \times [u^2 + 1, 2] \\ &\quad = \{(3,2)\} \\ &\quad \subset (\{2\} \times [v^2 + 1, 2]) \cup ([2,3] \times \{v^2 + 1\}) + D \\ &\quad = \operatorname{Min}_w \bigcup_{u \in U} C(u,v) + D \end{aligned}$$

for each $v \in V$. Thus, condition (iv) of Theorem 6 holds. By Theorem 6, (Hi_1) is valid. Indeed,

$$\begin{aligned} &\operatorname{Max} \bigcup_{u \in U} C(u,u) \\ &\quad = \{(3,2)\} \\ &\quad \subset \{(0,-1)\} + D \\ &\quad = \operatorname{Min}\left(\operatorname{co} \bigcup_{u \in U} \operatorname{Max}_w \bigcup_{v \in U} A(u,v)\right) + D, \end{aligned}$$

and hence the conclusion of Theorem 6 is valid.

Theorem 7 *Let U be a nonempty compact convex subset of real Hausdorff topological vector space, W be a real Hausdorff topological vector space. We work under the framework of Theorem* 6 *except* (iii) *and the convexities of B. If, in addition, the mapping $v \mapsto \varphi_{kw} B(u,v)$*

is above-$\mathbb{R}_+$-concave on U for each $u \in U$, and for each $v \in U$, there is an $x_v \in U$ such that for any Gerstewitz function φ_{kw},

$$\text{(iii)}' \quad \max \varphi_{kw} C(u_v, v) \le \max \bigcup_{v\in U} \min \bigcup_{u\in U} \varphi_{kw} C(u,v),$$

then (Hi_2) *is valid.*

Proof Suppose that $v \notin \bigcup_{u\in U} \mathrm{Max}_w \bigcup_{v\in U} A(u,v) + D$. Using a similar technique to the proofs of Theorems 2 and 4 for the mappings $\varphi_{kw}A$, $\varphi_{kw}B$, $\varphi_{kw}C$, we can see that

$$\max \bigcup_{v\in U} \min \bigcup_{u\in U} \varphi_{kw} C(u,v) > 0.$$

By the same technique as in Theorem 6 and condition (iv), we see that

$$v \notin \mathrm{Max} \bigcup_{u\in U} C(u,u).$$

Hence, (Hi_2) is valid. □

5 Saddle points

In this section, we list the existence of saddle points for set-valued mappings as applications of scalar hierarchical minimax theorems. The proofs of the following results can be deduced directly from Corollary 2, so we omit them. We refer the reader to [2, 3] for more details. Nevertheless, the conditions used in Theorems 8-10 are quite different from the ones used in the literature [2, 3].

Theorem 8 *Under the framework of Corollary* 2.2, *we have*

$$\max \bigcup_{v\in V} A(\bar{u}, v) = \min \bigcup_{u\in U} A(x, \bar{v}) = A(\bar{u}, \bar{v}),$$

which means: *A has $\mathbb{R}_+$-saddle point $(\bar{u}, \bar{v})$.*

Theorem 9 *Let U, V be nonempty compact convex subsets of real Hausdorff topological spaces, respectively. W is a complete locally convex Hausdorff topological vector space. Suppose that the set-valued mappings $F: U \times V \rightrightarrows W$ have nonempty compact values and satisfy the following conditions*:

(i) *$(u,v) \mapsto A(u,v)$ is upper semicontinuous on $U \times V$, and $u \mapsto A(u,v)$ is above-naturally D-quasi-convex and lower semicontinuous on U for each $v \in V$;*
(ii) *$v \mapsto A(u,v)$ is above-D-concave on V for each $u \in U$;*
(iii) *$v \mapsto A(u,v)$ is continuous on V for each $u \in U$; and*
(iv) *for any $\varphi \in D^\star$ and for each $v \in V$, there is an $u_v \in U$ such that*

$$\max \varphi A(u_v, v) \le \max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi A(u,v).$$

Then

$$A(\bar{u},\bar{v}) \cap \left(\mathrm{Max}_w \bigcup_{v\in V} A(\bar{u},v)\right) \cap \left(\mathrm{Min}_w \bigcup_{u\in U} A(x,\bar{v})\right) \neq \emptyset,$$

which means: *A has a weakly D-saddle point* $(\bar{u},\bar{v})$.

Theorem 10 *Let* U, V *be nonempty compact convex subsets of real Hausdorff topological spaces, respectively.* W *is a real Hausdorff topological vector space. We work under the framework of Theorem* 9 *except* (iv) *and the convexities of* A. *If, in addition, the mapping* $v \mapsto \varphi_{kw}A(u,v)$ *is above-*$\mathbb{R}_+$*-concave on* V *for each* $u \in U$, *and for any Gerstewitz function* φ_{kw} *and for each* $v \in V$, *there is an* $u_v \in U$ *such that*

$$\text{(iv}'\text{)} \quad \max \varphi_{kw}A(u_v,v) \le \max \bigcup_{v\in V} \min \bigcup_{u\in U} \varphi_{kw}A(u,v);$$

then A *has a weakly D-saddle point* $(\bar{u},\bar{v})$.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This work was supported by grant MOST103-2115-M-039-001 of the Ministry of Science and Technology of Taiwan (Republic of China).

Received: 4 October 2014 Accepted: 26 January 2015 Published online: 19 February 2015

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