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Optimal bounds for the first and second Seiffert means in terms of geometric, arithmetic and contraharmonic means

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Abstract
In this paper, we find the greatest values α, λ and the least values β, μ such that the double inequalities $\alpha[G(a,b)/3+2A(a,b)/3]+(1-\alpha)G^{1/3}(a,b)A^{2/3}(a,b)<P(a,b)<\beta[G(a,b)/3+2A(a,b)/3]+(1-\beta)G^{1/3}(a,b)A^{2/3}(a,b)$ and $\lambda[C(a,b)/3+2A(a,b)/3]+(1-\lambda)C^{1/3}(a,b)A^{2/3}(a,b)<T(a,b)<\mu[C(a,b)/3+2A(a,b)/3]+(1-\mu)C^{1/3}(a,b)A^{2/3}(a,b)$ hold for all $a,b>0$ with $a\neq b$. Here $G(a,b)$, $A(a,b)$, $C(a,b)$, $P(a,b)$ and $T(a,b)$ denote the geometric, arithmetic, contraharmonic, first Seiffert and second Seiffert means of two positive numbers a and b, respectively.
MSC: 26E60

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1 Introduction

For $a,b>0$ with $a\neq b$, the first and second Seiffert means $P(a,b)$ [1] and $T(a,b)$ [2] are defined by

$$P(a,b)=\frac{a-b}{4\arctan(\sqrt{a/b})-\pi}$$

and

$$T(a,b)=\frac{a-b}{2\arctan[(a-b)/(a+b)]}, \tag{1.1}$$

respectively.

Recently, both means P and T have been the subject of intensive research. In particular, many remarkable inequalities for P and T can be found in the literature [3–9]. The first Seiffert mean $P(a,b)$ can be rewritten as (see [10, Eq. (2.4)])

$$P(a,b)=\frac{a-b}{2\arcsin[(a-b)/(a+b)]}. \tag{1.2}$$

Let $H(a,b)=2ab/(a+b)$, $G(a,b)=\sqrt{ab}$, $L(a,b)=(b-a)/(\log b-\log a)$, $I(a,b)=1/e(b^b/a^a)^{1/(b-a)}$, $A(a,b)=(a+b)/2$, $Q(a,b)=\sqrt{(a^2+b^2)/2}$, $C(a,b)=(a^2+b^2)/(a+b)$,

$L_r(a,b) = (a^{r+1} + b^{r+1})/(a^r + b^r)$, and $M_r(a,b) = [(a^r + b^r)/2]^{1/r}$ $(r \neq 0)$ and $M_0(a,b) = G(a,b)$ be the harmonic, geometric, logarithmic, identric, arithmetic, quadratic, contraharmonic, rth Lehmer and rth power means of two distinct positive real numbers a and b, respectively. Then both $L_r(a,b)$ and $M_r(a,b)$ are strictly increasing with respect to $r \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$, and the inequalities

$$\begin{aligned} H(a,b) &= L_{-1}(a,b) = M_{-1}(a,b) < G(a,b) = L_{-1/2}(a,b) \\ &= M_0(a,b) < L(a,b) < P(a,b) < I(a,b) < A(a,b) = L_0(a,b) \\ &= M_1(a,b) < T(a,b) < Q(a,b) = M_2(a,b) < C(a,b) = L_1(a,b) \end{aligned}$$

hold for all $a, b > 0$ with $a \neq b$.

Jagers [11] and Seiffert [2] proved that the inequalities

$$M_{1/2}(a,b) < P(a,b) < M_{2/3}(a,b), \qquad M_1(a,b) < T(a,b) < M_2(a,b)$$

hold for $a, b > 0$ with $a \neq b$.

Costin and Toader [12] proved that the double inequality

$$M_{4/3}(a,b) < T(a,b) < M_{5/3}(a,b)$$

holds for $a, b > 0$ with $a \neq b$.

In [13–17], the authors proved that the inequalities

$$\begin{aligned} &M_p(a,b) < P(a,b) < M_q(a,b), \qquad M_r(a,b) < T(a,b) < M_s(a,b), \\ &L_\alpha(a,b) < P(a,b) < L_\beta(a,b), \qquad L_\sigma(a,b) < T(a,b) < L_\tau(a,b), \\ &P(a,b) > \left[\frac{b^\lambda - a^\lambda}{\lambda(\log b - \log a)}\right]^{1/\lambda}, \qquad T(a,b) > \left[\frac{b^\mu - a^\mu}{\mu(\log b - \log a)}\right]^{1/\mu} \end{aligned}$$

hold for $a, b > 0$ with $a \neq b$ if and only if $p \leq \log\pi/\log 2$, $q \geq 2/3$, $r \leq \log 2/(\log\pi - \log 2)$, $s \geq 5/3$, $\alpha \leq -1/6$, $\beta \geq 0$, $\sigma \leq 0$, $\tau \geq 1/3$, $\lambda \geq 2$ and $\mu \geq 5$.

Gao [18] proved that $\alpha = e/\pi$, $\beta = 1$, $\lambda = 1$ and $\mu = 2e/\pi$ are the best possible constants such that the double inequalities

$$\alpha I(a,b) < P(a,b) < \beta I(a,b), \qquad \lambda I(a,b) < T(a,b) < \mu I(a,b)$$

hold for $a, b > 0$ with $a \neq b$.

In [19, 20], the authors proved that the double inequalities

$$\begin{aligned} &\alpha_1 C(a,b) + (1-\alpha_1)G(a,b) < P(a,b) < \beta_1 C(a,b) + (1-\beta_1)G(a,b), \\ &\alpha_2 A(a,b) + (1-\alpha_2)G(a,b) < P(a,b) < \beta_2 A(a,b) + (1-\beta_2)G(a,b), \\ &\alpha_3 C(a,b) + (1-\alpha_3)H(a,b) < T(a,b) < \beta_3 C(a,b) + (1-\beta_3)H(a,b) \end{aligned}$$

hold for $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 2/9$, $\beta_1 \geq 1/\pi$, $\alpha_2 \leq 2/\pi$, $\beta_2 \geq 2/3$, $\alpha_3 \leq 2/\pi$ and $\beta_3 \geq 2/3$.

Let $p \geq 1/2$, $q \geq 1$, $t_1, t_2 \in (1/2, 1)$ and $t_3, t_4 \in (0, 1/2)$. Then the authors in [21, 22] proved that the double inequalities

$$\begin{aligned} &C^p\bigl[t_1 a + (1-t_1)b, t_1 b + (1-t_1)a\bigr]A^{1-p}(a,b) \\ &\quad < T(a,b) < C^p\bigl[t_2 a + (1-t_2)b, t_2 b + (1-t_2)a\bigr]A^{1-p}(a,b), \\ &G^q\bigl[t_3 a + (1-t_3)b, t_3 b + (1-t_3)a\bigr]A^{1-q}(a,b) \\ &\quad < P(a,b) < G^q\bigl[t_4 a + (1-t_4)b, t_4 b + (1-t_4)a\bigr]A^{1-q}(a,b) \end{aligned}$$

hold for $a, b > 0$ with $a \neq b$ if and only if $t_1 \leq [1 + \sqrt{(4/\pi)^{1/p} - 1}]/2$, $t_2 \geq 1/2 + \sqrt{3p}/(6p)$, $t_3 \leq [1 - \sqrt{1 - (2/\pi)^{2/q}}]/2$ and $t_4 \geq (1 - 1/\sqrt{3q})/2$.

Yang *et al.* [23] proved that the double inequality

$$\frac{Q^2(a,b)}{L_{p-1}(a,b)} < T(a,b) < \frac{Q^2(a,b)}{L_{q-1}(a,b)}$$

holds for $a, b > 0$ with $a \neq b$ if and only if $p \geq 5/3$ and $q \leq 1$.

Sándor [24] and Jiang *et al.* [25] proved that the inequalities

$$G^{1/3}(a,b)A^{2/3}(a,b) < P(a,b) < \frac{1}{3}G(a,b) + \frac{2}{3}A(a,b), \tag{1.3}$$

$$T(a,b) < \frac{1}{3}C(a,b) + \frac{2}{3}A(a,b) \tag{1.4}$$

hold for $a, b > 0$ with $a \neq b$.

In [26], Sándor found that $T(a,b)$ is the common limit of the sequences $\{u_n\}$ and $\{v_n\}$ given by

$$u_0 = A(a,b), \qquad v_0 = Q(a,b), \qquad u_{n+1} = \frac{u_n + v_n}{2}, \qquad v_{n+1} = \sqrt{u_{n+1}v_n} \quad (n \geq 0)$$

and established a more general inequality

$$\sqrt[3]{u_n v_n^2} < T(a,b) < \frac{u_n + 2v_n}{3} \tag{1.5}$$

for all $n \geq 0$ and $a, b > 0$ with $a \neq b$. In particular, let $n = 0$, then (1.4) and (1.5) together with the identity $Q^{2/3}(a,b)A^{1/3}(a,b) = C^{1/3}(a,b)A^{2/3}(a,b)$ lead to

$$C^{1/3}(a,b)A^{2/3}(a,b) < T(a,b) < \frac{1}{3}C(a,b) + \frac{2}{3}A(a,b) \tag{1.6}$$

for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (1.3) and (1.6), we naturally ask: what are the best possible parameters α, β, λ and μ such that the double inequalities

$$\begin{aligned} &\alpha\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + (1-\alpha)G^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad < P(a,b) < \beta\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + (1-\beta)G^{1/3}(a,b)A^{2/3}(a,b), \end{aligned}$$

$$\begin{aligned}&\lambda\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\lambda)C^{1/3}(a,b)A^{2/3}(a,b)\\&\quad<T(a,b)<\mu\left[\frac{1}{3}C(a,b)+\frac{2}{3}A(a,b)\right]+(1-\mu)C^{1/3}(a,b)A^{2/3}(a,b)\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$? The purpose of this paper is to answer this question.

2 Lemmas

In order to establish our main results, we need two lemmas, which we present in this section.

Lemma 2.1 *Let* $g(t)=-p^2t^6-2p^2t^5+3(p^2-4p+2)t^4+2(2p^2-9p+6)t^3-(4p^2+6p-9)t^2+6(1-p)t+3(1-p)$. *Then the following statements are true*:

(1) *If* $p=4/5$, *then* $g(t)>0$ *for all* $t\in(0,1)$.
(2) *If* $p=3/\pi$, *then there exists* $\lambda_0\in(0,1)$ *such that* $g(t)>0$ *for* $t\in(0,\lambda_0)$ *and* $g(t)<0$ *for* $t\in(\lambda_0,1)$.

Proof Part (1) follows easily from

$$g(t)=\frac{1}{25}(1-t)\bigl(16t^5+48t^4+90t^3+86t^2+45t+15\bigr)>0$$

for all $t\in(0,1)$ if $p=4/5$.

For part (2), if $p=3/\pi$, then numerical computations lead to

$$p^2-4p+2=\frac{2\pi^2-12\pi+9}{\pi^2}<0, \tag{2.1}$$

$$2p^2-9p+6=\frac{6\pi^2-27\pi+18}{\pi^2}<0, \tag{2.2}$$

$$4p^2+6p-9=\frac{-9\pi^2+18\pi+36}{\pi^2}>0, \tag{2.3}$$

$$g(0)=\frac{3(\pi-3)}{\pi}>0, \tag{2.4}$$

$$g(1)=\frac{9(4\pi-15)}{\pi}<0, \tag{2.5}$$

$$\begin{aligned}g'(t)&=-6p^2t^5-10p^2t^4+12\bigl(p^2-4p+2\bigr)t^3\\&\quad+6\bigl(2p^2-9p+6\bigr)t^2-2\bigl(4p^2+6p-9\bigr)t+6(1-p),\end{aligned}$$

$$g'(0)=\frac{6(\pi-3)}{\pi}>0, \tag{2.6}$$

$$g'(1)=\frac{84\pi-360}{\pi}<0, \tag{2.7}$$

$$\begin{aligned}g''(t)&=-30p^2t^4-40p^2t^3+36\bigl(p^2-4p+2\bigr)t^2\\&\quad+12\bigl(2p^2-9p+6\bigr)t-2\bigl(4p^2+6p-9\bigr).\end{aligned} \tag{2.8}$$

It follows from (2.1)-(2.3) and (2.8) that g' is strictly decreasing on $(0,1)$. Then (2.6) and (2.7) lead to the conclusion that there exists $\lambda_1\in(0,1)$ such that g is strictly increasing on $(0,\lambda_1]$ and strictly decreasing on $[\lambda_1,1)$.

Therefore, part (2) follows from (2.4) and (2.5) together with the piecewise monotonicity of g'. □

Lemma 2.2 *Let* $h(t) = q(q+3)t^4 + 2q(q+3)t^3 - 3(q^2-6q+1)t^2 - 2(2q^2-9q+3)t + 4q^2$. *Then the following statements are true*:

(1) *If* $q = 1/5$, *then* $h(t) > 0$ *for* $t \in (1, \sqrt[3]{2})$.
(2) *If* $q = [3(\sqrt[3]{2}\pi - 4)]/[(3\sqrt[3]{2} - 4)\pi] = 0.1814\ldots$, *then there exists* $\mu_0 \in (1, \sqrt[3]{2})$ *such that* $h(t) < 0$ *for* $t \in (1, \mu_0)$ *and* $h(t) > 0$ *for* $t \in (\mu_0, \sqrt[3]{2})$.

Proof Part (1) follows easily from

$$h(t) = \frac{4(t-1)}{25}\left(4t^3 + 12t^2 + 15t - 1\right) > 0$$

for all $t \in (1, \sqrt[3]{2})$ if $q = 1/5$.

For part (2), if $q = [3(\sqrt[3]{2}\pi - 4)]/[(3\sqrt[3]{2} - 4)\pi]$, then numerical computations lead to

$$q^2 - 6q + 1 = -0.0556\ldots < 0, \tag{2.9}$$

$$h(1) = 9(5q-1) = -0.836\ldots < 0, \tag{2.10}$$

$$h\left(\sqrt[3]{2}\right) = 0.548\ldots > 0, \tag{2.11}$$

$$h'(t) = 4q(q+3)t^3 + 6q(q+3)t^2 - 6\left(q^2 - 6q + 1\right)t - 2\left(2q^2 - 9q + 3\right). \tag{2.12}$$

It follows from (2.9) and (2.12) that

$$\begin{aligned} h'(t) &> 4q(q+3) + 6q(q+3) - 6\left(q^2 - 6q + 1\right) - 2\left(2q^2 - 9q + 3\right) \\ &= 12(7q-1) = 3.239\ldots > 0 \end{aligned} \tag{2.13}$$

for $t \in (1, \sqrt[3]{2})$.

Therefore, part (2) follows easily from (2.10) and (2.11) together with (2.13). □

3 Main results

Theorem 3.1 *The double inequality*

$$\begin{aligned} &\alpha\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + (1-\alpha)G^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad < P(a,b) < \beta\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + (1-\beta)G^{1/3}(a,b)A^{2/3}(a,b) \end{aligned}$$

holds for all $a, b > 0$ *with* $a \neq b$ *if and only if* $\alpha \leq 4/5$ *and* $\beta \geq 3/\pi$.

Proof Firstly, we prove that the inequalities

$$P(a,b) > \frac{4}{5}\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + \frac{1}{5}G^{1/3}(a,b)A^{2/3}(a,b), \tag{3.1}$$

$$P(a,b) < \frac{3}{\pi}\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] + \left(1 - \frac{3}{\pi}\right)G^{1/3}(a,b)A^{2/3}(a,b) \tag{3.2}$$

hold for all $a, b > 0$ with $a \neq b$.

Since $P(a,b)$, $A(a,b)$ and $G(a,b)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b) \in (0,1)$ and $p \in (0,1)$. Then (1.2) leads to

$$\frac{P(a,b)}{A(a,b)} = \frac{x}{\arcsin(x)}, \qquad \frac{G(a,b)}{A(a,b)} = \sqrt{1-x^2},$$

$$\frac{P(a,b) - G^{1/3}(a,b)A^{2/3}(a,b)}{G(a,b)/3 + 2A(a,b)/3 - G^{1/3}(a,b)A^{2/3}(a,b)} = \frac{x/\arcsin(x) - \sqrt[6]{1-x^2}}{2/3 + \sqrt{1-x^2}/3 - \sqrt[6]{1-x^2}}, \tag{3.3}$$

$$\lim_{x\to 0^+} \frac{x/\arcsin(x) - \sqrt[6]{1-x^2}}{2/3 + \sqrt{1-x^2}/3 - \sqrt[6]{1-x^2}} = \frac{4}{5}, \tag{3.4}$$

$$\lim_{x\to 1^-} \frac{x/\arcsin(x) - \sqrt[6]{1-x^2}}{2/3 + \sqrt{1-x^2}/3 - \sqrt[6]{1-x^2}} = \frac{3}{\pi}, \tag{3.5}$$

$$\begin{aligned} &P(a,b) - p\left[\frac{1}{3}G(a,b) + \frac{2}{3}A(a,b)\right] - (1-p)G^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad = A(a,b)\left[\frac{x}{\arcsin(x)} - p\left(\frac{1}{3}\sqrt{1-x^2} + \frac{2}{3}\right) - (1-p)\sqrt[6]{1-x^2}\right] \\ &\quad = \frac{A(a,b)[p(2+\sqrt{1-x^2}) + 3(1-p)\sqrt[6]{1-x^2}]}{3\arcsin(x)} G(x), \end{aligned} \tag{3.6}$$

where

$$G(x) = \frac{3x}{p(2+\sqrt{1-x^2}) + 3(1-p)\sqrt[6]{1-x^2}} - \arcsin(x),$$

$$G(0) = 0, \tag{3.7}$$

$$G(1) = \frac{3}{2p} - \frac{\pi}{2}, \tag{3.8}$$

$$G'(x) = \frac{(1-\sqrt[6]{1-x^2})^2}{\sqrt[6]{(1-x^2)^5}[p(2+\sqrt{1-x^2}) + 3(1-p)\sqrt[6]{1-x^2}]^2} g\big(\sqrt[6]{1-x^2}\big), \tag{3.9}$$

where the function $g(\cdot)$ is defined as in Lemma 2.1.

We divide the proof into two cases.

Case 1. $p = 4/5$. Then (3.1) follows easily from (3.6), (3.7), (3.9) and Lemma 2.1(1).

Case 2. $p = 3/\pi$. Then Lemma 2.1(2) and (3.9) lead to the conclusion that there exists $x_0 \in (0,1)$ such that G is strictly decreasing on $(0, x_0]$ and strictly increasing on $[x_0, 1)$.

Note that (3.8) becomes

$$G(1) = 0. \tag{3.10}$$

It follows from (3.7) and (3.10) together with the piecewise monotonicity of G that

$$G(x) < 0 \tag{3.11}$$

for all $x \in (0,1)$.

Therefore, (3.2) follows from (3.6) and (3.11), and Theorem 3.1 follows from (3.1) and (3.2) in conjunction with the following statements.

- If $\alpha > 4/5$, then equations (3.3) and (3.4) lead to the conclusion that there exists $0 < \delta_1 < 1$ such that $P(a,b) < \alpha[G(a,b)/3 + 2A(a,b)/3] + (1-\alpha)G^{1/3}(a,b)A^{2/3}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (0,\delta_1)$.
- If $\beta < 3/\pi$, then equations (3.3) and (3.5) imply that there exists $0 < \delta_2 < 1$ such that $P(a,b) > \beta[G(a,b)/3 + 2A(a,b)/3] + (1-\beta)G^{1/3}(a,b)A^{2/3}(a,b)$ for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\delta_2, 1)$. □

Theorem 3.2 *The double inequality*

$$\lambda\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\lambda)C^{1/3}(a,b)A^{2/3}(a,b) < T(a,b) < \mu\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + (1-\mu)C^{1/3}(a,b)A^{2/3}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ if and only if $\lambda \leq [3(\sqrt[3]{2}\pi - 4)]/[(3\sqrt[3]{2} - 4)\pi] = 0.1814\ldots$ and $\mu \geq 1/5$.

Proof Let $\lambda^* = [3(\sqrt[3]{2}\pi - 4)]/[(3\sqrt[3]{2} - 4)\pi]$. Firstly, we prove that the inequalities

$$T(a,b) < \frac{1}{5}\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + \frac{4}{5}C^{1/3}(a,b)A^{2/3}(a,b), \tag{3.12}$$

$$T(a,b) > \lambda^*\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] + \left(1-\lambda^*\right)C^{1/3}(a,b)A^{2/3}(a,b) \tag{3.13}$$

hold for all $a,b > 0$ with $a \neq b$.

Since $T(a,b)$, $A(a,b)$ and $C(a,b)$ are symmetric and homogenous of degree 1, without loss of generality, we assume that $a > b$. Let $x = (a-b)/(a+b) \in (0,1)$ and $q \in (0,1)$. Then (1.1) leads to

$$\frac{T(a,b)}{A(a,b)} = \frac{x}{\arctan(x)}, \qquad \frac{C(a,b)}{A(a,b)} = 1 + x^2,$$

$$\frac{T(a,b) - C^{1/3}(a,b)A^{2/3}(a,b)}{C(a,b)/3 + 2A(a,b)/3 - C^{1/3}(a,b)A^{2/3}(a,b)} = \frac{x/\arctan x - \sqrt[3]{1+x^2}}{2/3 + (1+x^2)/3 - \sqrt[3]{1+x^2}}, \tag{3.14}$$

$$\lim_{x\to 0^+} \frac{x/\arctan x - \sqrt[3]{1+x^2}}{2/3 + (1+x^2)/3 - \sqrt[3]{1+x^2}} = \frac{1}{5}, \tag{3.15}$$

$$\lim_{x\to 1^-} \frac{x/\arctan x - \sqrt[3]{1+x^2}}{2/3 + (1+x^2)/3 - \sqrt[3]{1+x^2}} = \lambda^*, \tag{3.16}$$

$$\begin{aligned} &T(a,b) - q\left[\frac{1}{3}C(a,b) + \frac{2}{3}A(a,b)\right] - (1-q)C^{1/3}(a,b)A^{2/3}(a,b) \\ &\quad = A(a,b)\left[\frac{x}{\arctan(x)} - \frac{q}{3}\left(3 + x^2\right) - (1-q)\sqrt[3]{1+x^2}\right] \\ &\quad = \frac{A(a,b)[q(3+x^2) + 3(1-q)\sqrt[3]{1+x^2}]}{3\arctan(x)}H(x), \end{aligned} \tag{3.17}$$

where

$$H(x) = \frac{3x}{q(3+x^2)+3(1-q)\sqrt[3]{1+x^2}} - \arctan(x),$$

$$H(0) = 0, \tag{3.18}$$

$$H(1) = \frac{3}{4q+3\sqrt[3]{2}(1-q)} - \frac{\pi}{4}, \tag{3.19}$$

$$H'(x) = -\frac{(\sqrt[3]{1+x^2}-1)^2}{(1+x^2)[q(3+x^2)+3(1-q)\sqrt[3]{1+x^2}]^2} h\bigl(\sqrt[3]{1+x^2}\bigr), \tag{3.20}$$

where the function $h(\cdot)$ is defined as in Lemma 2.2.

We divide the proof into two cases.

Case 1. $q = 1/5$. Then (3.12) follows easily from Lemma 2.2(1), (3.17), (3.18) and (3.20).

Case 2. $q = \lambda^*$. Then Lemma 2.2(2) and (3.20) lead to the conclusion that there exists $x^* \in (0,1)$ such that H is strictly increasing on $(0, x^*]$ and strictly decreasing on $[x^*, 1)$.

Note that (3.19) becomes

$$H(1) = 0. \tag{3.21}$$

Therefore, (3.13) follows from (3.17), (3.18), (3.21) and the piecewise monotonicity of H, and Theorem 3.2 follows from (3.12) and (3.13) in conjunction with the following statements.

- If $\mu < 1/5$, then equations (3.14) and (3.15) lead to the conclusion that there exists $0 < \delta_3 < 1$ such that $T(a,b) > \mu[C(a,b)/3 + 2A(a,b)/3] + (1-\mu)C^{1/3}(a,b)A^{2/3}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (0, \delta_3)$.
- If $\lambda > \lambda^*$, then equations (3.14) and (3.16) imply that there exists $0 < \delta_4 < 1$ such that $T(a,b) < \lambda[C(a,b)/3 + 2A(a,b)/3] + (1-\lambda)C^{1/3}(a,b)A^{2/3}(a,b)$ for all $a, b > 0$ with $(a-b)/(a+b) \in (1-\delta_4, 1)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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