Two new regularization methods for solving sideways heat equation

Huy Tuan Nguyen[1*] and Vu Cam Hoan Luu[2]

*Correspondence:
nguyenhuytuan@tdt.edu.vn
[1] Applied Analysis Research Group,
Faculty of Mathematics and
Statistics, Ton Duc Thang University,
Ho Chi Minh City, Vietnam
Full list of author information is
available at the end of the article

Abstract

We consider a non-standard inverse heat conduction problem in a bounded domain which appears in some applied subjects. We want to know the surface temperature in a body from a measured temperature history at a fixed location inside the body. This is an exponentially ill-posed problem in the sense that the solution (if it exists) does not depend continuously on the data. In this paper, we introduce the two new classes of quasi-type methods and iteration methods to solve the problem and prove that our methods are stable under both *a priori* and *a posteriori* parameter choice rules. An appropriate selection of a parameter in the scheme will get a satisfactory approximate solution. Furthermore, if we use the discrepancy principle we can avoid the selection of the *a priori* bound.

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1 Introduction

In this paper, we want to determine the surface temperature $u(x,t)$ for $0 < x < L$ from well-known temperature measurements $u(L,t) = g(t)$ when $u(x,t)$ satisfies the following system:

$$\begin{cases} u_t = u_{xx}, & 0 < x < L, 0 \le t \le 2\pi, \\ u(L,t) = g(t), & 0 \le t \le 2\pi, \\ u(x,0) = 0, & 0 \le t \le 2\pi. \end{cases} \qquad (1)$$

In order to guarantee the uniqueness of the solution, here we assume that u be bounded as $x \to +\infty$. The functions g, h given in $L^2(0, 2\pi)$ are to be noised. Physically, we only obtain the measured Cauchy data with measurement errors. This is a severely ill-posed problem: any small perturbation in the observation data can cause large errors in the solution $u(x,t)$ for $x \in [0, L]$. Therefore, most classical numerical methods often fail to give an acceptable approximation of the solution. Thus it requires regularization techniques to stabilize the numerical computations [1].

The problem has a long history and has attracted to many authors. In several engineering contexts, it is sometimes necessary to determine the surface temperature and heat flux in a body from a measured temperature history at a fixed location inside the body [2, 3]. The physical situation at the surface may be unsuitable for attaching a sensor, or the accuracy

of a surface measurement may be seriously impaired by the presence of the sensor. The special case of estimating a surface condition from interior measurements has come to be known as the inverse heat conduction problem (IHCP).

In recent years, IHCP have been researched by many authors and some valuable method are proposed, such as the difference regularization method [4], the Fourier method [5], quasi-reversibility method [2, 6], wavelet and wavelet-Galerkin method [7], a spectral regularization method [7, 8], *etc*.

Most of the papers involving regularization methods for sideways heat are devoted to giving a convergence analysis with the regularization parameter chosen by an *a priori* choice rule, under which some *a priori* information on the unknown exact solution which is to be simulated is often used. However, in general, the *a priori* bound M cannot be known exactly in practice, and working with a wrong constant M may lead to the bad regularized solution. In the present paper, we proposed two new regularization method: a modified quasi-boundary value method and an iteration method for solving Problem (1). This method has been used by Deng and Liu to deal with the sideways parabolic equation [9], but here we give another way to choose the regularization parameter by an *a posteriori* rule. Moreover, we shall give a criterion for making an *a posteriori* choice of the regularization parameter. To the authors' knowledge, there are very few papers for choosing the regularization parameter by the *a posteriori* rule for this sideways heat.

This paper is constructed as follows: in Section 2, we propose two new regularization method. Convergence estimates under priori and posteriori assumptions are given in Section 3 and Section 4. Finally, we draw a conclusion to our method.

2 Formulation of the problem and a quasi-boundary value regularization method

Suppose g^ϵ is for measured data in $L^2(0, 2\pi)$ and satisfies

$$\left\| g^\epsilon - g \right\|_{L^2(0,2\pi)} \leq \epsilon, \tag{2}$$

where $\epsilon > 0$ represents a noisy level. For $f \in L^2(0, 2\pi)$, we have the Fourier series

$$f(t) = \sum_{k \in \mathbb{Z}} f_k e^{ikt}, \tag{3}$$

where the Fourier coefficient is

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \, dt. \tag{4}$$

The norm of f in L^2 is

$$\|f\|_{L^2(0,2\pi)}^2 = 2\pi \sum_{k \in \mathbb{Z}} |f_k|^2. \tag{5}$$

The principal value of $\sqrt{ik}$ is

$$\sqrt{ik} = \begin{cases} (1+i)\sqrt{\frac{|k|}{2}}, & k \geq 0, \\ (1-i)\sqrt{\frac{|k|}{2}}, & k < 0. \end{cases} \tag{6}$$

Suppose that the solution of the problem is represented as a Fourier series

$$u(x,t) = \sum_{k \in \mathbb{Z}} u_k(x) e^{ikt}, \tag{7}$$

with

$$u_k(x) = \frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-ikt} \, dt. \tag{8}$$

Let $F(x,t) = \sum_{k \in \mathbb{Z}} F_k(x) e^{ikt}$ with

$$F_k(x) = \frac{1}{2\pi} \int_0^{2\pi} F(x,t) e^{-ikt} \, dt. \tag{9}$$

Then by taking the partial derivative with respect to t and x, we obtain

$$u_t(x,t) = \sum_{k \in \mathbb{Z}} u_k(x) ik e^{ikt},$$

$$u_{xx}(x,t) = \sum_{k \in \mathbb{Z}} \frac{d^2}{dx^2} u_k(x) e^{ikt}. \tag{10}$$

The main equation $u_t = u_{xx}$ leads to

$$\sum_{k \in \mathbb{Z}} u_k(x) ik e^{ikt} = \sum_{k \in \mathbb{Z}} \frac{d^2}{dx^2} u_k(x) e^{ikt}.$$

Combining the Cauchy data g, h, we have the following systems of second ordinary equations:

$$\frac{d^2}{dx^2} u_k(x) - ik u_k(x) = 0, \qquad u_k(L) = g_k, \tag{11}$$

where

$$g_k = \langle g(t), e^{-ikt} \rangle = \frac{1}{2\pi} \int_0^{2\pi} g(t) e^{-ikt} \, dt.$$

The solution of (11) is

$$u_k(x) = C_k \exp(\sqrt{ik}x) + D_k \exp(-\sqrt{ik}x). \tag{12}$$

Since $u(x,t)$ is bounded when $x \to +\infty$, we conclude that $C_k = 0$. It follows from $u_k(L) = D_k \exp(-\sqrt{ik}L) = g_k$ that

$$D_k = g_k \exp(\sqrt{ik}L).$$

This leads to the formula of u_k:

$$u_k(x) = \exp\big(\sqrt{ik}(L-x)\big) g_k. \tag{13}$$

The exact form of u is

$$u(x,t) = \sum_{k \in \mathbb{Z}} \exp\left(\sqrt{ik}(L-x)\right)\langle g(t), e^{-ikt}\rangle e^{ikt}, \quad 0 \le x \le L. \tag{14}$$

Lemma 1 *For $y \in \mathbb{R}$, we have*

$$\left|e^{\sqrt{ik}y}\right| = e^{\sqrt{\frac{|k|}{2}}y}. \tag{15}$$

Proof We can rewrite the term $|e^{\sqrt{ik}y}|$ by

$$\left|e^{\sqrt{ik}y}\right| = \left|e^{(1+i)\sqrt{\frac{|k|}{2}}y}\right| = \left|e^{\sqrt{\frac{|k|}{2}}y}\right|\sqrt{\cos^2\left(\frac{|k|y}{2}\right) + \sin^2\left(\frac{|k|y}{2}\right)} = e^{\sqrt{\frac{|k|}{2}}y}. \tag{16}$$
$\square$

3 Quasi-boundary value method for a sideways heat equation

Define the Sobolev space $H^s(0, 2\pi)$ as

$$\left\|u(0,t)\right\|_{H^s(0,2\pi)}^2 = \sum_{k \in \mathbb{Z}}\left(1 + k^2\right)^s\left|\langle u(0,t), e^{-ikt}\rangle\right|^2.$$

In this section, we present a QBV regularization problem as follows:

$$\begin{cases} u_t^{\beta(\epsilon)} = u_{xx}^{\beta(\epsilon)}, & 0 < x < L, 0 \le t \le 2\pi, \\ u^{\beta(\epsilon)}(L,t) = B_\epsilon g^\epsilon(t), & 0 \le t \le 2\pi, \\ u^{\beta(\epsilon)}(x,0) = 0, & 0 \le t \le 2\pi, \end{cases} \tag{17}$$

with the additive information that $u^\epsilon(x,t)$ bounded when $x \to +\infty$. And B_ϵ is defined as

$$B_\epsilon g^\epsilon(t) = \sum_{k \in \mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)}\langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt}. \tag{18}$$

3.1 A priori parameter choice rule

Theorem 1 *Let u be an exact solution to Problem* (1). *Let g_ϵ be as in* (2).

(1) *If $\|u(0,t)\|_{L^2(0,2\pi)} \le M$ then by $\beta(\epsilon) = \frac{\epsilon}{M}$, we have*

$$\left\|u^{\beta(\epsilon)}(x,\cdot) - u(x,\cdot)\right\|_{L^2(0,2\pi)} \le 2M^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}}.$$

(2) *If $\|u(0,t)\|_{H^s(0,2\pi)} \le M_1$ then by $\beta(\epsilon) = \frac{\epsilon^k}{M_1}$, $0 < k < 1$, we have*

$$\left\|u^{\beta(\epsilon)}(x,\cdot) - u(x,\cdot)\right\|_{L^2(0,2\pi)}\left(\frac{1}{M_1}\right)^{\frac{x}{L}}\epsilon^{\frac{kx}{L}}\epsilon^{1-k} + M_1 A(s)\left(\frac{L}{\ln(M_1/\epsilon)}\right)^{2s},$$

where

$$A(s) = 2^s(2s)^{2s}e^{1-2s}\left(1 + L^{-2s}\right).$$

Proof

Proof of Part 1. The solution of this regularized problem is

$$u^{\beta(\epsilon)}(x,t) = \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)\exp(\sqrt{ik}(L-x))}{\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)}\langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt}. \tag{19}$$

Let $v^{\beta(\epsilon)}$ obey

$$v^{\beta(\epsilon)}(x,t) = \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)\exp(\sqrt{ik}(L-x))}{\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)}\langle g(t), e^{-ikt}\rangle e^{ikt}.$$

Step 1. Estimate $\|u^{\beta(\epsilon)} - v^{\beta(\epsilon)}\|_{L^2(0,2\pi)}$.
We have

$$\begin{aligned}
\|u^{\beta(\epsilon)} - v^{\beta(\epsilon)}\|^2_{L^2(0,2\pi)} &= 2\pi \sum_{k\in\mathbb{Z}} \left| \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)}\exp(\sqrt{ik}(L-x))\right|^2 \\
&\quad\times \left|\langle g^\epsilon(t) - g(t), e^{-ikt}\rangle\right|^2 \\
&= 2\pi \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{2|k|}x)}{[\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)]^2}\left|\langle g^\epsilon(t) - g(t), e^{-ikt}\rangle\right|^2 \\
&\leq \beta(\epsilon)^{\frac{2x}{L}-2}\|g^\epsilon(t) - g(t)\|^2_{L^2(0,2\pi)} \leq \beta(\epsilon)^{\frac{2x}{L}-2}\epsilon^2. \tag{20}
\end{aligned}$$

This implies that

$$\|u^{\beta(\epsilon)} - v^{\beta(\epsilon)}\|_{L^2(0,2\pi)} \leq \beta(\epsilon)^{\frac{x}{L}-1}\epsilon = M^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}}. \tag{21}$$

Step 2. Estimate $\|v^{\beta(\epsilon)} - u\|_{L^2(0,2\pi)}$. Using $\langle u(0,t), e^{-ikt}\rangle = \exp(\sqrt{ik}L)\langle g(t), e^{-ikt}\rangle$, we have

$$\begin{aligned}
\|v^{\beta(\epsilon)} - u\|^2_{L^2(0,2\pi)} &= 2\pi \sum_{k\in\mathbb{Z}} \left|\frac{\beta(\epsilon)\exp(\sqrt{\frac{|k|}{2}}L)}{1+\beta\exp(\sqrt{\frac{|k|}{2}}L)}\right|^2 \exp(2\sqrt{ik}(L-x))\left|\langle g(t), e^{-ikt}\rangle\right|^2 \\
&= 2\pi \sum_{k\in\mathbb{Z}} \frac{\beta^2|\exp(-2\sqrt{ik}x)|}{[\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)]^2}\left|\langle u(0,t), e^{-ikt}\rangle\right|^2 \\
&= \beta(\epsilon)^2 2\pi \sum_{k\in\mathbb{Z}} \left[\frac{\exp(-\sqrt{\frac{|k|}{2}}x)}{\beta(\epsilon)+\exp(-\sqrt{\frac{|k|}{2}}L)}\right]^2 \left|\langle u(0,t), e^{-ikt}\rangle\right|^2 \\
&\leq \beta(\epsilon)^2 \beta(\epsilon)^{\frac{2x}{L}-2}\|u(0,t)\|^2_{L^2(0,2\pi)} \leq \beta(\epsilon)^{\frac{2x}{L}}M^2. \tag{22}
\end{aligned}$$

From $\beta(\epsilon) = \frac{\epsilon}{M}$, we obtain

$$\|v^{\beta(\epsilon)} - u\|^2_{L^2(0,2\pi)} \leq \epsilon^{\frac{2x}{L}}M^{2-\frac{2x}{L}}.$$

This implies that

$$\left\|v^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)} \leq M^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}}. \tag{23}$$

Combining (22) and (23), we obtain

$$\left\|u^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)} \leq \left\|u^{\beta(\epsilon)} - v^{\beta(\epsilon)}\right\|_{L^2(0,2\pi)} + \left\|v^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)} \leq 2M^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}}.$$

Proof of Part 2.

Lemma 2 *Let $s > 0$, $X \geq 0$. Then for $0 < \beta < 1$, we have*

$$\frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))} \leq s^s e^{1-s}(1 + L^{-s})\left(\frac{L}{\ln(1/\beta)}\right)^s.$$

Proof

Case 1. $\sqrt{\frac{|k|}{2}} \in [0, \frac{1}{L}]$. It is clear to see that

$$\frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))} \leq \frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s \exp(-\sqrt{\frac{|k|}{2}}L)} \leq \beta \exp\left(\sqrt{\frac{|k|}{2}}L\right) \leq e\beta.$$

From the inequality $\beta \leq (\frac{s}{e})^s(\frac{1}{\ln(1/\beta)})^s$, we get

$$\frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))} \leq s^s e^{1-s}(1 + L^{-s})\left(\frac{L}{\ln(1/\epsilon)}\right)^s. \tag{24}$$

Case 2. $\sqrt{\frac{|k|}{2}} > \frac{1}{L}$. Set $Y = \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta}$. Then we obtain

$$\frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))} = \frac{\beta}{\beta + \beta Y}\left(\frac{L}{L - \ln(\beta Y)}\right)^s$$

$$= \frac{1}{1 + Y}\left(\frac{L}{L - \ln(\beta Y)}\right)^s$$

$$= \frac{1}{1 + Y}\left(\frac{L}{\ln(1/\beta)}\right)^s\left(\frac{-\ln(\beta)}{L - \ln(\beta Y)}\right)^s$$

$$= \left(\frac{L}{\ln(1/\beta)}\right)^s\frac{1}{1 + Y}\left(\frac{-\ln(\beta)}{L - \ln(\beta Y)}\right)^s. \tag{25}$$

We continue to estimate the term $\frac{1}{1+Y}(\frac{-\ln(\beta)}{L-\ln(\beta Y)})^s$.

If $0 < Y \leq 1$ then $0 < -\ln(\beta) < -\ln(\beta Y)$, thus

$$\frac{1}{1 + Y}\left(\frac{-\ln(\beta)}{L - \ln(\beta Y)}\right)^s < 1, \tag{26}$$

else if $Y > 1$ then $\ln Y > 0$ and $\ln(\beta Y) = -L\sqrt{\frac{|k|}{2}} < -1$ due to the assumption $\sqrt{\frac{|k|}{2}} \in (\frac{1}{L}, \infty)$. Therefore $\ln Y(1 + \ln(\beta Y)) \leq 0$. This implies that

$$0 < \frac{-\ln\beta}{L - \ln(\beta Y)} < \frac{-\ln\beta}{-\ln(\beta Y)} < 1 + \ln Y.$$

Hence, in this case, we get

$$\frac{1}{1+Y}\left(\frac{-\ln(\beta)}{L - \ln(\beta Y)}\right)^s < \frac{(1 + \ln Y)^s}{Y} = (1 + \ln Y)^s Y^{-1}.$$

Set $g(Y) = (1 + \ln Y)^s Y^{-1}$ for $Y > e^{-1}$. Taking the derivative of this function, we get

$$g'(Y) = (1 + \ln Y)^{s-1} Y^{-2}(s - 1 - \ln Y).$$

The function g has a maximum at the point Y_0 such that $g'(Y_0) = 0$. This implies that $Y_0 = e^{s-1}$. And therefore

$$\sup_{Y \geq 1}(1 + \ln Y)^s Y^{-1} \leq g(Y_0) = s^s e^{1-s}. \tag{27}$$

Since (26) and (27) hold, we have

$$\frac{1}{1+Y}\left(\frac{-\ln(\beta)}{L - \ln(\beta Y)}\right)^s \leq s^s e^{1-s}.$$

From (25), we get

$$\frac{\beta}{(1 + \sqrt{\frac{|k|}{2}})^s(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))} \leq s^s e^{1-s}\left(\frac{T}{\ln(1/\beta)}\right)^s \leq s^s e^{1-s}(1 + T^{-s})\left(\frac{T}{\ln(1/\beta)}\right)^s. \tag{28}$$

We have

$$\|v^\epsilon - u\|^2_{L^2(0,2\pi)}$$

$$= 2\pi \sum_{k \in \mathbb{Z}}\left[\frac{\beta \exp(-\sqrt{\frac{|k|}{2}}x)}{\beta + \exp(-\sqrt{\frac{|k|}{2}}L)}\right]^2 \left|\langle u(0,t), e^{-ikt}\rangle\right|^2$$

$$\leq 2\pi \sum_{k \in \mathbb{Z}}\frac{\beta^2}{(1 + \sqrt{\frac{|k|}{2}})^{4s}(\beta + \exp(-\sqrt{\frac{|k|}{2}}L))^2}\left(1 + \sqrt{\frac{|k|}{2}}\right)^{4s}\left|\langle u(0,t), e^{-ikt}\rangle\right|^2$$

$$\leq (2s)^{4s}e^{2-4s}\left(1 + L^{-2s}\right)^2\left(\frac{L}{\ln(1/\beta)}\right)^{4s}2\pi \sum_{k \in \mathbb{Z}}\left(1 + \sqrt{\frac{|k|}{2}}\right)^{4s}\left|\langle u(0,t), e^{-ikt}\rangle\right|^2$$

$$\leq 4^s(2s)^{4s}e^{2-4s}\left(1 + L^{-2s}\right)^2\left(\frac{L}{\ln(1/\beta)}\right)^{4s}2\pi \sum_{k \in \mathbb{Z}}\left(1 + k^2\right)^s\left|\langle u(0,t), e^{-ikt}\rangle\right|^2$$

$$\leq A(s)^2\left(\frac{L}{\ln(1/\beta)}\right)^{4s}\|u(0,t)\|^2_{H^s(0,2\pi)}, \tag{29}$$

where

$$A(s) = 2^s(2s)^{2s}e^{1-2s}\left(1 + L^{-2s}\right).$$

This leads to

$$\left\|v^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)} \leq A(s)\left(\frac{L}{\ln(1/\beta)}\right)^{2s}\left\|u(0,t)\right\|_{H^s(0,2\pi)}$$

$$\leq M_1 A(s)\left(\frac{L}{\ln(1/\beta)}\right)^{2s}. \tag{30}$$

Combining (20) and (30), we obtain

$$\left\|u^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)} \leq \left\|u^{\beta(\epsilon)} - v^{\beta(\epsilon)}\right\|_{L^2(0,2\pi)} + \left\|v^{\beta(\epsilon)} - u\right\|_{L^2(0,2\pi)}$$

$$\leq \beta(\epsilon)^{\frac{x}{L}-1}\epsilon + M_1 A(s)\left(\frac{L}{\ln(1/\beta)}\right)^{2s}. \qquad \square$$

3.2 A posteriori parameter choice rule

Theorem 2 *Suppose that* $0 < \epsilon < \|g^\epsilon\|_{L^2(0,2\pi)}$. *Choose* $m > 1$ *such that*

$$0 < m\epsilon < \left\|g^\epsilon\right\|_{L^2(0,2\pi)}.$$

Then there exists a unique number $\beta(\epsilon)$ *such that*

$$\left\|u^{\beta(\epsilon)}(L,t) - g^\epsilon\right\|_{L^2(0,2\pi)} = m\epsilon. \tag{31}$$

Furthermore, if $\|u(0,t)\|_{L^2(0,2\pi)} \leq M$ *then by* $\beta(\epsilon) = \frac{\epsilon}{M}$, *we have the following estimate*:

$$\left\|u^{\beta(\epsilon)}(x,\cdot) - u(x,\cdot)\right\|_{L^2(0,2\pi)} \leq \left[\frac{2m}{m-1}M\right]^{1-\frac{x}{L}}\left[(m+1)\epsilon\right]^{\frac{x}{L}}.$$

Proof The following results are straightforward.

Lemma 3 *Set* $\rho(\beta) = \|u^{\beta(\epsilon)}(L,t) - g^\epsilon\|_{L^2(0,2\pi)}$. *If* $0 < \epsilon < \|g^\epsilon\|_{L^2(0,2\pi)}$ *then* ρ *is a continuous strictly increasing function and satisfies*

(a) $\lim_{\beta \to 0} \rho(\beta) = 0$,
(b) $\lim_{\beta \to +\infty} \rho(\beta) = \|g^\epsilon\|_{L^2(0,2\pi)}$.

From this lemma, it follows that there exists a unique number $\beta(\epsilon)$ satisfying (31).
Set $z^{\beta(\epsilon)}(x,t) = u(x,t) - u^{\beta(\epsilon)}(x,t)$. Then $z^{\beta(\epsilon)}$ satisfies the heat equation

$$z_t^{\beta(\epsilon)} = z_{xx}^{\beta(\epsilon)}. \tag{32}$$

From the formula of u and $u^{\beta(\epsilon)}$, we get

$$z^{\beta(\epsilon)}(x,t) = u^{\beta(\epsilon)}(x,t) - u(x,t) = \sum_{k\in\mathbb{Z}}\exp\left(\sqrt{ik}(L-x)\right)R_k\left(\beta,g,g^\epsilon\right)e^{ikt},$$

where

$$R_k(\beta, g, g^\epsilon) = \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)} \langle g^\epsilon(t), e^{-ikt} \rangle - \langle g(t), e^{-ikt} \rangle.$$

Then using the Hölder inequality, we get

$$
\begin{aligned}
\left\| z^{\beta(\epsilon)}(x,t) \right\|_{L^2(0,2\pi)}^2 &= \sum_{k \in \mathbb{Z}} \exp\left(\sqrt{2|k|}L\left(1 - \frac{x}{L}\right) \right) \left| R_k(\beta, g, g^\epsilon) \right|^{2 - \frac{2x}{L}} \left| R_k(\beta, g, g^\epsilon) \right|^{\frac{2x}{L}} \\
&\leq \left[\sum_{k \in \mathbb{Z}} \left(\exp\left(\sqrt{2|k|}L\left(1 - \frac{x}{L}\right) \right) \left| R_k(\beta, g, g^\epsilon) \right|^{2 - \frac{2x}{L}} \right)^{\frac{1}{1 - \frac{x}{L}}} \right]^{1 - \frac{x}{L}} \\
&\quad \times \left[\sum_{k \in \mathbb{Z}} \left(\left| R_k(\beta, g, g^\epsilon) \right|^{\frac{2x}{L}} \right)^{\frac{1}{\frac{x}{L}}} \right]^{\frac{x}{L}} \\
&= \left[\sum_{k \in \mathbb{Z}} \exp(\sqrt{2|k|}L) \left| R_k(\beta, g, g^\epsilon) \right|^2 \right]^{1 - \frac{x}{L}} \left[\sum_{k \in \mathbb{Z}} \left| R_k(\beta, g, g^\epsilon) \right|^2 \right]^{\frac{x}{L}} \\
&= \left\| z^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)}^{2 - \frac{2x}{L}} \left\| z^{\beta(\epsilon)}(L,t) \right\|_{L^2(0,2\pi)}^{\frac{2x}{L}}.
\end{aligned}
$$

This implies that

$$\left\| z^{\beta(\epsilon)}(x,t) \right\|_{L^2(0,2\pi)} \leq \left\| z^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)}^{1 - \frac{x}{L}} \left\| z^{\beta(\epsilon)}(L,t) \right\|_{L^2(0,2\pi)}^{\frac{x}{L}}. \tag{33}$$

Proof of Part (a). It is easy to see that

$$
\begin{aligned}
\left\| z^{\beta(\epsilon)}(L,t) \right\|_{L^2(0,2\pi)} &= \left\| u^{\beta(\epsilon)}(L,t) - u(L,t) \right\|_{L^2(0,2\pi)} \\
&\leq \left\| u^{\beta(\epsilon)}(L,t) - g^\epsilon \right\|_{L^2(0,2\pi)} + \left\| g - g^\epsilon \right\|_{L^2(0,2\pi)} \leq (m+1)\epsilon \tag{34}
\end{aligned}
$$

and

$$\left\| z^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)} \leq \left\| u(0,t) \right\|_{L^2(0,2\pi)} + \left\| u^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)}. \tag{35}$$

We have

$$
\begin{aligned}
m\epsilon &= \left\| u^{\beta(\epsilon)}(L,t) - g^\epsilon(t) \right\|_{L^2(0,2\pi)} \\
&= \left\| \sum_{k \in \mathbb{Z}} \frac{\beta(\epsilon) \exp(-\sqrt{\frac{|k|}{2}}L)}{1 + \beta(\epsilon) \exp(-\sqrt{\frac{|k|}{2}}L)} \langle g^\epsilon(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)} \\
&\leq \left\| \sum_{k \in \mathbb{Z}} \frac{\beta(\epsilon) \exp(-\sqrt{\frac{|k|}{2}}L)}{1 + \beta(\epsilon) \exp(-\sqrt{\frac{|k|}{2}}L)} \langle g^\epsilon(t) - g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)} \\
&\quad + \left\| \sum_{k \in \mathbb{Z}} \frac{\beta(\epsilon)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)} \langle g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}
\end{aligned}
$$

$$\leq \left\| \sum_{k\in\mathbb{Z}} \langle g^{\epsilon}(t) - g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)} + \left\| \sum_{k\in\mathbb{Z}} \beta(\epsilon) \exp\left(\sqrt{\frac{|k|}{2}}L\right) \langle g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$= \left\| g^{\epsilon} - g \right\|_{L^2(0,2\pi)} + \beta(\epsilon) \left\| u(0,t) \right\|_{L^2(0,2\pi)}$$

$$\leq \epsilon + \beta(\epsilon) \left\| u(0,t) \right\|_{L^2(0,2\pi)}.$$

This implies that

$$\frac{\epsilon}{\beta(\epsilon)} \leq \frac{1}{m-1} \left\| u(0,t) \right\|_{L^2(0,2\pi)}. \tag{36}$$

It follows from (19) that

$$\left\| u^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)} = \left\| \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)} \exp(\sqrt{ik}L) \langle g^{\epsilon}(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)} \exp(\sqrt{ik}L) \langle g^{\epsilon}(t) - g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \left\| \sum_{k\in\mathbb{Z}} \frac{\exp(-\sqrt{\frac{|k|}{2}}L)}{\beta(\epsilon) + \exp(-\sqrt{\frac{|k|}{2}}L)} \exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k\in\mathbb{Z}} \frac{1}{\beta(\epsilon)} \langle g^{\epsilon}(t) - g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \left\| \sum_{k\in\mathbb{Z}} \exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \frac{1}{\beta(\epsilon)} \left\| g^{\epsilon} - g \right\|_{L^2(0,2\pi)} + \left\| u(0,t) \right\|_{L^2(0,2\pi)}$$

$$\leq \frac{\epsilon}{\beta(\epsilon)} + \left\| u(0,t) \right\|_{L^2(0,2\pi)}. \tag{37}$$

Combining (36) and (37), we obtain

$$\left\| u^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)} \leq \left(\frac{1}{m-1} + 1 \right) \left\| u(0,t) \right\|_{L^2(0,2\pi)} = \frac{m}{m-1} \left\| u(0,t) \right\|_{L^2(0,2\pi)}. \tag{38}$$

From (35) and (38), we obtain

$$\left\| z^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)} \leq \left(1 + \frac{m}{m-1} \right) \left\| u(0,t) \right\|_{L^2(0,2\pi)} = \frac{2m-1}{m-1} \left\| u(0,t) \right\|_{L^2(0,2\pi)}. \tag{39}$$

This together with (33) and (34) leads to

$$\left\| z^{\beta(\epsilon)}(x,t) \right\|_{L^2(0,2\pi)} \leq \left\| z^{\beta(\epsilon)}(0,t) \right\|_{L^2(0,2\pi)}^{1-\frac{x}{L}} \left\| z^{\beta(\epsilon)}(L,t) \right\|_{L^2(0,2\pi)}^{\frac{x}{L}}$$

$$\leq \left[\frac{2m-1}{m-1} \left\| u(0,t) \right\|_{L^2(0,2\pi)} \right]^{1-\frac{x}{L}} \left[(m+1)\epsilon \right]^{\frac{x}{L}}.$$

Hence

$$\left\| u(x,t) - u^{\beta(\epsilon)}(x,t) \right\|_{L^2(0,2\pi)} \le \left[\frac{(2m-1)M}{m-1} \right]^{1-\frac{x}{L}} (m+1)^{\frac{x}{L}} \epsilon^{\frac{x}{L}}. \qquad \square$$

4 An iteration regularization method

4.1 A priori parameter choice

To the authors' knowledge, there are few papers for choosing the regularization parameter by *a posteriori* rule for the sideways heat equation. In this section, a new regularization method of iteration type for solving this problem will be given. We will use this method solving Problem (1), an *a priori* and *a posteriori* rule for choosing regularization parameters and the Hölder-type error estimates are given. To solve the problem, we introduce the following iteration scheme:

$$\langle u_n^\epsilon(x,t), e^{-ikt} \rangle = (1-h)\langle u_{n-1}^\epsilon(x,t), e^{-ikt} \rangle + h \exp\left(\sqrt{ik}(L-x)\right)\langle g^\epsilon(t), e^{-ikt} \rangle \qquad (40)$$

with initial guess $\langle u_0^\epsilon(x,t), e^{-ikt} \rangle$ and $0 < h = e^{-L\sqrt{\frac{|k|}{2}}} < 1$, which plays an important role in the convergence proof. From the formula, it is easy to see that

$$\langle u_n^\epsilon(x,t), e^{-ikt} \rangle = (1-h)^n\langle u_0^\epsilon(x,t), e^{-ikt} \rangle + \sum_{i=0}^{n-1}(1-h)^i h \exp\left(\sqrt{ik}(L-x)\right)\langle g^\epsilon(t), e^{-ikt} \rangle$$

$$= (1-h)^n\langle u_0^\epsilon(x,t), e^{-ikt} \rangle + \left[1-(1-h)^n\right] \exp\left(\sqrt{ik}(L-x)\right)\langle g^\epsilon(t), e^{-ikt} \rangle. \quad (41)$$

Therefore, the approximate solution of Problem (1) has the following form:

$$u_n^\epsilon(x,t) = \sum_{k \in \mathbb{Z}} \langle u_n^\epsilon(x,t), e^{-ikt} \rangle e^{ikt}. \qquad (42)$$

We have the main theorem as follows.

Theorem 3 *Let $u(x,t)$ be the exact solution of Problem* (1), *and u_n^ϵ be its regularization approximation defined by* (42) *with $\langle u_0^\epsilon(x,t), e^{-ikt} \rangle = 0$. Assume that $\|u(0,t)\|_{L^2(0,2\pi)} \le M$ for $M > 0$ and take $n = [\frac{M}{\epsilon}]$, where $[k]$ denotes the largest integer not exceeding k, then we have the estimate*

$$\left\| u_n^\epsilon(x,\cdot) - u(x,\cdot) \right\|_{L^2(0,2\pi)} \le 2M^{1-\frac{x}{L}} \epsilon^{\frac{x}{L}}.$$

Lemma 4 *For $0 \le h \le 1$ and $n \ge 1$, the following inequalities hold*:

$$(1-h)^n h \le \frac{1}{n+1},$$
$$\frac{1-(1-h)^n}{h} \le n.$$

Lemma 5 *For $0 \le h \le 1$, $0 \le \alpha \le 1$, and $n \ge 1$, the following inequalities hold*:

$$(1-h)^n h^\alpha \le \frac{1}{(n+1)^\alpha},$$

$$\frac{1 - (1 - h)^n}{h^\alpha} \le n^\alpha.$$

From $\langle u_0^\epsilon(x, t), e^{-ikt} \rangle = 0$, we see that

$$u_n^\epsilon(x, t) = \sum_{k \in \mathbb{Z}} \left[1 - (1 - h)^n\right] \exp\left(\sqrt{ik}(L - x)\right) \langle g^\epsilon(t), e^{-ikt} \rangle e^{ikt}.$$

Let $v^\epsilon(x, t)$ be given by

$$v_n^\epsilon(x, t) = \sum_{k \in \mathbb{Z}} \left[1 - (1 - h)^n\right] \exp\left(\sqrt{ik}(L - x)\right) \langle g(t), e^{-ikt} \rangle e^{ikt}.$$

We divide the argument into two steps.

Step 1. Estimate $\|u_n^\epsilon(x, \cdot) - v_n^\epsilon(x, \cdot)\|_{L^2(0, 2\pi)}$. We have

$$\begin{aligned}
&\left\| u_n^\epsilon(x, \cdot) - v_n^\epsilon(x, \cdot) \right\|_{L^2(0, 2\pi)}^2 \\
&\quad = 2\pi \sum_{k \in \mathbb{Z}} \left[1 - (1 - h)^n\right]^2 \left| \exp\left(2\sqrt{ik}(L - x)\right) \right| \left| \langle g^\epsilon - g(t), e^{-ikt} \rangle \right|^2 \\
&\quad \le 2\pi \sup_{k \in \mathbb{Z}} \left[1 - (1 - h)^n\right]^2 \left| \exp\left(2\sqrt{ik}(L - x)\right) \right| \sum_{k \in \mathbb{Z}} \left| \langle g^\epsilon - g(t), e^{-ikt} \rangle \right|^2 \\
&\quad \le 2\pi \sup_{k \in \mathbb{Z}} \left[1 - (1 - h)^n\right]^2 \exp\left(\sqrt{2|k|}(L - x)\right) \left\| g^\epsilon - g \right\|_{L^2(0, 2\pi)}^2.
\end{aligned}$$

It follows from $\exp\left(\sqrt{2|k|}(L - x)\right) = h^{\frac{2x}{L} - 2}$ and $\|g^\epsilon - g\|_{L^2(0, 2\pi)} \le \epsilon$ that

$$\left\| u_n^\epsilon(x, \cdot) - v_n^\epsilon(x, \cdot) \right\|_{L^2(0, 2\pi)}^2 \le 2\pi \sup_{0 < h < 1} \left[1 - (1 - h)^n\right]^2 h^{\frac{2x}{L} - 2} \epsilon^2.$$

Because of Lemma 5, we have

$$\left\| u_n^\epsilon(x, \cdot) - v_n^\epsilon(x, \cdot) \right\|_{L^2(0, 2\pi)}^2 \le n^{2 - \frac{2x}{L}} \epsilon^2.$$

Therefore

$$\left\| u_n^\epsilon(x, \cdot) - v_n^\epsilon(x, \cdot) \right\|_{L^2(0, 2\pi)} \le n^{1 - \frac{x}{L}} \epsilon. \tag{43}$$

Step 2. Estimate $\|v_n^\epsilon(x, \cdot) - u(x, \cdot)\|_{L^2(0, 2\pi)}$. We have

$$\begin{aligned}
\left\| v_n^\epsilon(x, \cdot) - u(x, \cdot) \right\|_{L^2(0, 2\pi)}^2 &= 2\pi \sum_{k \in \mathbb{Z}} (1 - h)^{2n} \exp\left(\sqrt{2|k|}(L - x)\right) \left| \langle g(t), e^{-ikt} \rangle \right|^2 \\
&\le \left\| u(0, t) \right\|_{L^2(0, 2\pi)}^2 \sup_{0 < h < 1} (1 - h)^{2n} h^{\frac{2x}{L}} \\
&\le M^2 (n + 1)^{-\frac{2x}{L}}.
\end{aligned}$$

This implies that

$$\left\| v_n^\epsilon(x, \cdot) - u(x, \cdot) \right\|_{L^2(0, 2\pi)} \le M(n + 1)^{-\frac{x}{L}}. \tag{44}$$

Combining (43) and (44), we obtain

$$\left\| u_n^\epsilon(x,\cdot) - u(x,\cdot) \right\|_{L^2(0,2\pi)} \le \left\| u^\epsilon(x,\cdot) - v^\epsilon(x,\cdot) \right\|_{L^2(0,2\pi)} + \left\| v^\epsilon(x,\cdot) - u(x,\cdot) \right\|_{L^2(0,2\pi)}$$
$$\le n^{1-\frac{x}{L}}\epsilon + M(n+1)^{-\frac{x}{L}}.$$

Due to $n = [\frac{M}{\epsilon}]$, we have $n \le \frac{M}{\epsilon}$ and $n+1 \ge \frac{M}{\epsilon}$, and therefore

$$\left\| u_n^\epsilon(x,\cdot) - u(x,\cdot) \right\|_{L^2(0,2\pi)} \le M\left(\frac{M}{\epsilon}\right)^{-\frac{x}{L}} + \epsilon\left(\frac{M}{\epsilon}\right)^{1-\frac{x}{L}} = 2M^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}}.$$

4.2 The discrepancy principle

In this section, we discuss the *a posteriori* stopping rule for iteration scheme (40) based on 'discrepancy principle' of Morozov in the following form:

$$\left\| g^\epsilon(t) - u_\beta^\epsilon(L,t) \right\|_{L^2(0,2\pi)} = m\epsilon, \tag{45}$$

where $m > 1$ is a constant and β denotes the regularization parameter.

If we take $\langle u_0^\epsilon(x,t), e^{-ikt}\rangle = 0$, then (45) can be simplified to the form

$$\left\| g^\epsilon(t) - u_\beta^\epsilon(L,t) \right\|_{L^2(0,2\pi)} = \left\| \sum_{k\in\mathbb{Z}} \langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt} - \sum_{k\in\mathbb{Z}} \left[1 - (1-h)^\beta\right]\langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$
$$= \left\| \sum_{k\in\mathbb{Z}} (1-h)^\beta \langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)} = m\epsilon. \tag{46}$$

We have the main theorem as follows.

Theorem 4 *Let $u(x,t)$ be the exact solution of Problem (1), and u_n^ϵ be its regularization approximation defined by (42) with $\langle u_0^\epsilon(x,t), e^{-ikt}\rangle = 0$. If the a priori bound $\|u(0,t)\|_{L^2(0,\pi)} \le M$ is valid and the iteration (40) is stopped by the discrepancy principle (45), then*

$$\left\| u_\beta^\epsilon(x,\cdot) - u(x,\cdot) \right\|_{L^2(0,2\pi)} \le EM^{1-\frac{x}{L}}\epsilon^{\frac{x}{L}},$$

where

$$E = (m+1)^{\frac{x}{L}}\left(\frac{m}{m-1}\right)^{1-\frac{x}{L}}.$$

The following results are obvious.

Lemma 6 *Set $P(\beta) = \|g^\epsilon(t) - u_\beta^\epsilon(L,t)\|_{L^2(0,2\pi)} = \|(1-h)^\beta g^\epsilon(t)\|_{L^2(0,2\pi)}$.*

Lemma 7 *Setting*

$$w_\beta^\epsilon(x,t) = \sum_{k\in\mathbb{Z}}\left[\exp\left(\sqrt{ik}(L-x)\right)\langle g(t), e^{-ikt}\rangle - \left[1 - (1-h)^n\right]\right.$$
$$\left. \times \exp\left(\sqrt{ik}(L-x)\right)\langle g^\epsilon(t), e^{-ikt}\rangle\right]e^{ikt},$$

the following inequality holds:

$$\left\| w_\beta^\epsilon(x,t) \right\|_{L^2(0,2\pi)} \leq \left\| w_\beta^\epsilon(0,t) \right\|_{L^2(0,2\pi)}^{1-\frac{x}{L}} \left\| w_\beta^\epsilon(L,t) \right\|_{L^2(0,2\pi)}^{\frac{x}{L}}.$$

Proof We have

$$w_\beta^\epsilon(0,t) = \sum_{k\in\mathbb{Z}} \left[\exp(\sqrt{ik}L)\langle g(t), e^{-ikt}\rangle - \left[1-(1-h)^n\right] \exp(\sqrt{ik}L)\langle g^\epsilon(t), e^{-ikt}\rangle \right] e^{ikt}$$

and

$$w_\beta^\epsilon(L,t) = \sum_{k\in\mathbb{Z}} \left[\langle g(t), e^{-ikt}\rangle - \left[1-(1-h)^n\right]\langle g^\epsilon(t), e^{-ikt}\rangle \right] e^{ikt}.$$

We have

$$\left\| w_\beta^\epsilon(x,t) \right\|_{L^2(0,2\pi)}^2 = \sum_{k\in\mathbb{Z}} \exp\left(\sqrt{2|k|}(L-x)\right)\left[\langle g(t), e^{-ikt}\rangle - \left[1-(1-h)^n\right]\langle g^\epsilon(t), e^{-ikt}\rangle \right]^2.$$

For brevity in this case, we set $Q_k(h,n) = \langle g(t), e^{-ikt}\rangle - [1-(1-h)^n]\langle g^\epsilon(t), e^{-ikt}\rangle$. Then using the Hölder inequality, we get

$$\left\| w_\beta^\epsilon(x,t) \right\|_{L^2(0,2\pi)}^2 = \sum_{k\in\mathbb{Z}} \exp\left(\sqrt{2|k|}L\left(1-\frac{x}{L}\right)\right)\left|Q_k(h,n)\right|^{2-\frac{2x}{L}}\left|Q_k(h,n)\right|^{\frac{2x}{L}}$$

$$\leq \left[\sum_{k\in\mathbb{Z}}\left(\exp\left(\sqrt{2|k|}L\left(1-\frac{x}{L}\right)\right)\left|Q_k(h,n)\right|^{2-\frac{2x}{L}}\right)^{\frac{1}{1-\frac{x}{L}}}\right]^{1-\frac{x}{L}}$$

$$\times \left[\sum_{k\in\mathbb{Z}}\left(\left|Q_k(h,n)\right|^{\frac{2x}{L}}\right)^{\frac{1}{\frac{x}{L}}}\right]^{\frac{x}{L}}$$

$$= \left[\sum_{k\in\mathbb{Z}}\exp\left(\sqrt{2|k|}L\right) * \left|Q_k(h,n)\right|^2\right]^{1-\frac{x}{L}}\left[\sum_{k\in\mathbb{Z}}\left|Q_k(h,n)\right|^2\right]^{\frac{x}{L}}$$

$$= \left\| w_\beta^\epsilon(0,t) \right\|_{L^2(0,2\pi)}^{2-\frac{2x}{L}}\left\| w_\beta^\epsilon(L,t) \right\|_{L^2(0,2\pi)}^{\frac{2x}{L}}.$$

This completes the proof of the lemma. □

Lemma 8 *The following inequality holds*:

$$\beta\epsilon \leq \frac{M}{m-1}.$$

Proof It follows from (46) that

$$m\epsilon = \left\|\sum_{k\in\mathbb{Z}}(1-h)^\beta\langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt}\right\|_{L^2(0,2\pi)} \leq \left\|\sum_{k\in\mathbb{Z}}(1-h)^\beta\langle g^\epsilon(t)-g(t), e^{-ikt}\rangle e^{ikt}\right\|_{L^2(0,2\pi)}$$

$$+ \left\|\sum_{k\in\mathbb{Z}}(1-h)^\beta\langle g(t), e^{-ikt}\rangle e^{ikt}\right\|_{L^2(0,2\pi)}$$

$$\leq \sup_{0<h<1} (1-h)^\beta \left\| \sum_{k\in\mathbb{Z}} \langle g^\epsilon(t) - g(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \sup_{0<h<1} (1-h)^\beta \left\| \langle g(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}.$$

Moreover, it is easy to see that

$$(1-h)^\beta \leq \frac{1}{\beta}$$

and

$$\left\| \langle g(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)} = \left\| \exp(-\sqrt{ik}L)\langle u(0,t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)} \leq \left\| u(0,t) \right\|_{L^2(0,2\pi)} \leq M.$$

This leads to

$$\beta\epsilon \leq \epsilon + \frac{M}{m}.$$

This completes the proof of the lemma. $\qquad\square$

Lemma 9 *The following inequality holds*:

$$\left\| w_\beta^\epsilon(L,\cdot) \right\|_{L^2(0,2\pi)} \leq (m+1)\epsilon.$$

Proof Due to the triangle inequality, we have

$$\left\| w_\beta^\epsilon(L,\cdot) \right\|_{L^2(0,2\pi)} = \left\| \sum_{k\in\mathbb{Z}} \left[\langle g(t), e^{-ikt}\rangle - [1-(1-h)^n]\langle g^\epsilon(t), e^{-ikt}\rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k\in\mathbb{Z}} \langle g(t) - g^\epsilon(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \left\| \sum_{k\in\mathbb{Z}} (1-h)^\beta \langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| g^\epsilon - g \right\|_{L^2(0,2\pi)} + \left\| \sum_{k\in\mathbb{Z}} (1-h)^\beta \langle g^\epsilon(t), e^{-ikt}\rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \epsilon + m\epsilon = (m+1)\epsilon. \qquad\square$$

Lemma 10 *The following inequality holds*:

$$\left\| w_\beta^\epsilon(0,\cdot) \right\|_{L^2(0,2\pi)} \leq \frac{mM}{m-1}.$$

Proof Due to the triangle inequality and $h = e^{-L\sqrt{\frac{|k|}{2}}}$, we have

$$\left\| w_\beta^\epsilon(0,\cdot) \right\|_{L^2(0,2\pi)}$$

$$= \left\| \sum_{k\in\mathbb{Z}} \left[\exp(\sqrt{ik}L)\langle g(t), e^{-ikt}\rangle - [1-(1-h)^\beta]\exp(\sqrt{ik}L)\langle g^\epsilon(t), e^{-ikt}\rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k \in \mathbb{Z}} \left[\exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle - \left[1 - (1-h)^n \right] \exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \left\| \sum_{k \in \mathbb{Z}} \left[\left[1 - (1-h)^\beta \right] \exp(\sqrt{ik}L) \langle g(t) - g^\epsilon(t), e^{-ikt} \rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k \in \mathbb{Z}} \left[(1-h)^\beta \exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$+ \left\| \sum_{k \in \mathbb{Z}} \frac{1 - (1-h)^\beta}{h} \langle g(t) - g^\epsilon(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}. \tag{47}$$

Moreover, we have

$$\left\| \sum_{k \in \mathbb{Z}} \left[(1-h)^\beta \exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| \sum_{k \in \mathbb{Z}} \left[\exp(\sqrt{ik}L) \langle g(t), e^{-ikt} \rangle \right] e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$= \left\| u(0,t) \right\|_{L^2(0,2\pi)} \leq M, \tag{48}$$

and using $\frac{1-(1-h)^\beta}{h} \leq \beta$, we have the estimate

$$\left\| \sum_{k \in \mathbb{Z}} \frac{1 - (1-h)^\beta}{h} \langle g(t) - g^\epsilon(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)} \leq \beta \left\| \sum_{k \in \mathbb{Z}} \langle g(t) - g^\epsilon(t), e^{-ikt} \rangle e^{ikt} \right\|_{L^2(0,2\pi)}$$

$$\leq \beta \left\| g^\epsilon - g \right\|_{L^2(0,2\pi)} \leq \beta\epsilon. \tag{49}$$

Combining (47), (48), and (49), we obtain

$$\left\| w_\beta^\epsilon(0, \cdot) \right\|_{L^2(0,2\pi)} \leq M + \beta\epsilon.$$

Lemma 8 leads to

$$\left\| w_\beta^\epsilon(0, \cdot) \right\|_{L^2(0,2\pi)} \leq M + \frac{M}{m-1} = \frac{Mm}{m-1}. \qquad \square$$

Now, we return to the proof of the theorem.

Combining Lemma 7, Lemma 8, and Lemma 9, we obtain

$$\left\| u^\epsilon(x, \cdot) - u(x, \cdot) \right\|_{L^2(0,2\pi)} = \left\| w_\beta^\epsilon(x, \cdot) \right\|_{L^2(0,2\pi)}$$

$$\leq \left\| w_\beta^\epsilon(0, t) \right\|_{L^2(0,2\pi)}^{1-\frac{x}{L}} \left\| w_\beta^\epsilon(L, t) \right\|_{L^2(0,2\pi)}^{\frac{x}{L}}$$

$$\leq \left(\frac{Mm}{m-1} \right)^{1-\frac{x}{L}} \left((m+1)\epsilon \right)^{\frac{x}{L}}$$

$$= EM^{1-\frac{x}{L}} \epsilon^{\frac{x}{L}},$$

where

$$E = (m+1)^{\frac{x}{L}} \left(\frac{m}{m-1} \right)^{1-\frac{x}{L}}.$$

Author details

^{1}Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. ^{2}Faculty of Basic Science, Posts and Telecommunications Institute of Technology, Ho Chi Minh City, Vietnam.

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