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Spectrum of the Sturm-Liouville operators with boundary conditions polynomially dependent on the spectral parameter

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Abstract

In this paper, we consider the operator L generated in $L_2(\mathbb{R}_+)$ by the Sturm-Liouville equation $-y'' + q(x)y = \lambda^2 y$, $x \in \mathbb{R}_+ = [0, \infty)$, and the boundary condition $(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2)y'(0) - (\beta_0 + \beta_1\lambda + \beta_2\lambda^2)y(0) = 0$, where q is a complex-valued function, $\alpha_i, \beta_i \in \mathbb{C}$, $i = 0, 1, 2$, and λ is an eigenparameter. Under the conditions $q, q' \in AC(\mathbb{R}_+)$, $\lim_{x\to\infty} |q(x)| + |q'(x)| = 0$, $\sup_{x\in\mathbb{R}_+} [e^{\varepsilon\sqrt{x}}|q''(x)|] < \infty$, $\varepsilon > 0$, using the uniqueness theorems of analytic functions, we prove that L has a finite number of eigenvalues and spectral singularities with finite multiplicities.

MSC: 34B08; 34B09; 34B24

Keywords: Sturm-Liouville equations; eigenparameter; eigenvalues; spectral singularities

1 Introduction

Let us consider the non-selfadjoint Sturm-Liouville operator L_0 generated in $L_2(\mathbb{R}_+)$ by the differential expression

$$l_0(y) := -y'' + q(x)y, \quad x \in \mathbb{R}_+, \tag{1.1}$$

and the boundary condition $y'(0) - hy(0) = 0$, where q is a complex-valued function and $h \in \mathbb{C}$. The spectrum and eigenfunction expansion of L_0 were investigated by Naimark [1]. In this study, the spectrum of L_0 is investigated and it is shown that it is composed of the eigenvalues, a continuous spectrum, and spectral singularities. The spectral singularities are poles of the resolvent which are embedded in the continuous spectrum and are not eigenvalues.

The effect of the spectral singularities in the spectral expansion of L_0 in terms of the principal functions has been investigated in [2–4].

The spectral analysis of the non-selfadjoint operator, generated in $L_2(\mathbb{R}_+)$ by (1.1) and the integral boundary condition

$$\int_0^\infty A(x)y(x)\,dx + \alpha y'(0) - \beta y(0) = 0,$$

where $A \in L_2(\mathbb{R}_+)$ is a complex-valued function, and $\alpha, \beta \in \mathbb{C}$, was investigated in detail by Krall [5, 6].

Some problems of spectral theory of differential and other types of operators with spectral singularities were also studied in [7–12].

Note that in all the above articles, the boundary conditions are independent of the spectral parameter.

In 1977, Fulton [13], considered the Sturm-Liouville equation with one boundary condition dependent on the spectral parameter and obtained asymptotic estimates of eigenvalues or eigenfunctions. Since 1977, one of such Sturm-Liouville equations with boundary condition dependent on the spectral parameter was discussed by a number of authors (see [14–21]).

Let L denote the operator generated in $L_2(\mathbb{R}_+)$ by

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \mathbb{R}_+, \tag{1.2}$$

$$\bigl(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\bigr)y'(0) - \bigl(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\bigr)y(0) = 0, \tag{1.3}$$

where q is a complex-valued function, $\alpha_i, \beta_i \in \mathbb{C}$, $i = 0,1,2$, with $|\alpha_2| + |\beta_2| \neq 0$.

Differently from other studies in the literature, the specific feature of this paper, which is one of the articles having applicability in study areas such as physics, engineering, and mathematics, is the presence of the spectral parameter not only in the Sturm-Liouville equation but also in the boundary condition for a quadratic form.

In this article, we intend to investigate eigenvalues and the spectral singularities of the L, which has a finite number of eigenvalues and spectral singularities with a finite multiplicities, if the conditions

$$q, q' \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} \bigl|q(x)\bigr| + \bigl|q'(x)\bigr| = 0,$$

$$\sup_{x\in\mathbb{R}_+} \bigl[e^{\varepsilon\sqrt{x}}\bigl|q''(x)\bigr|\bigr] < \infty, \quad \varepsilon > 0,$$

hold, where $\mathrm{AC}(\mathbb{R}_+)$ denotes the class of complex-valued absolutely continuous functions on $\mathbb{R}_+$.

2 Jost solutions and Jost functions of *L*

Let us suppose that

$$\int_0^\infty x\bigl|q(x)\bigr|\,dx < \infty. \tag{2.1}$$

By $e(x,\lambda)$, we will denote the bounded solution of (1.2) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x} = 1, \quad \text{for } \lambda \in \overline{\mathbb{C}}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \geq 0\}. \tag{2.2}$$

The solution $e(x,\lambda)$ is called the Jost solution of (1.2). Under the condition (2.1), the solution $e(x,\lambda)$ has the integral representation [22, Chapter 3]

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}\,dt, \tag{2.3}$$

where the function $K(x,t)$ is the solution of the integral equation

$$\begin{aligned} K(x,t) &= \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} q(s)\,ds + \frac{1}{2}\int_{x}^{\frac{x+t}{2}}\int_{t+x-s}^{t+s-x} q(s)K(s,u)\,du\,ds \\ &\quad + \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}\int_{s}^{t+s-x} q(s)K(s,u)\,du\,ds, \end{aligned} \tag{2.4}$$

and $K(x,t)$ is continuously differentiable with respect to its arguments. We also have

$$\begin{aligned} &\bigl|K(x,t)\bigr| < cw\left(\frac{x+t}{2}\right), \\ &\bigl|K_x(x,t)\bigr|, \bigl|K_t(x,t)\bigr| \le \frac{1}{4}\left|q\left(\frac{x+t}{2}\right)\right| + cw\left(\frac{x+t}{2}\right), \end{aligned} \tag{2.5}$$

where $w(x) = \int_x^{\infty} |q(s)|\,ds$ and $c > 0$ is a constant.

Let

$$\begin{aligned} N^+(\lambda) &:= \bigl(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\bigr)e'(0,\lambda) - \bigl(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\bigr)e(0,\lambda), \quad \lambda \in \overline{\mathbb{C}}_+, \\ N^-(\lambda) &:= \bigl(\alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2\bigr)e'(0,-\lambda) - \bigl(\beta_0 + \beta_1\lambda + \beta_2\lambda^2\bigr)e(0,-\lambda), \quad \lambda \in \overline{\mathbb{C}}_-, \end{aligned} \tag{2.6}$$

where $\overline{\mathbb{C}}_- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda \le 0\}$.

Therefore, N^+ and N^- are analytic in $\mathbb{C}_+ = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and $\mathbb{C}_- = \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda < 0\}$, respectively, and are continuous up to the real axis. The functions N^+ and N^- are called Jost functions of L.

3 Eigenvalues and spectral singularities of *L*

We will denote the set of all eigenvalues and spectral singularities of L by $\sigma_d(L)$ and $\sigma_{ss}(L)$, respectively. It is evident that

$$\begin{aligned} &\sigma_d(L) = \bigl\{\lambda : \lambda \in \mathbb{C}_+, N^+(\lambda) = 0\bigr\} \cup \bigl\{\lambda : \lambda \in \mathbb{C}_-, N^-(\lambda) = 0\bigr\}, \\ &\sigma_{ss}(L) = \bigl\{\lambda : \lambda \in \mathbb{R}^*, N^+(\lambda) = 0\bigr\} \cup \bigl\{\lambda : \lambda \in \mathbb{R}^*, N^-(\lambda) = 0\bigr\}, \\ &\bigl\{\lambda : \lambda \in \mathbb{R}^*, N^+(\lambda) = 0\bigr\} \cap \bigl\{\lambda : \lambda \in \mathbb{R}^*, N^-(\lambda) = 0\bigr\} = \emptyset, \end{aligned} \tag{3.1}$$

where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

Definition 1 The multiplicity of a zero of N^+ (or N^-) in $\overline{\mathbb{C}}_+$ (or $\overline{\mathbb{C}}_-$) is called the multiplicity of the corresponding eigenvalue or spectral singularity of L.

From (3.1) we find that, in order to investigate the quantitative properties of the eigenvalues and the spectral singularities of L, we need to discuss the quantitative properties of the zeros of N^+ and N^- in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively.

Let

$$M_1^{\pm} := \bigl\{\lambda : \lambda \in \mathbb{C}_{\pm}, N^{\pm}(\lambda) = 0\bigr\}, \qquad M_2^{\pm} := \bigl\{\lambda : \lambda \in \mathbb{R}^*, N^{\pm}(\lambda) = 0\bigr\}. \tag{3.2}$$

Let us denote the set of all limit points of M_1^+ and M_1^- by M_3^+ and M_3^- and the set of all zeros of N^+ and N^- with infinite multiplicity in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, by M_4^+ and M_4^-, respectively.

It follows from the boundary uniqueness theorem of analytic functions that

$$M_3^{\pm} \subset M_2^{\pm}, \qquad M_4^{\pm} \subset M_2^{\pm}, \qquad M_3^{\pm} \subset M_4^{\pm}, \tag{3.3}$$

and the linear Lebesgue measures of $M_3^{\pm}$ and $M_4^{\pm}$ are zero.

Using (3.1) and (3.2), we get

$$\sigma_d(L) = M_1^+ \cup M_1^-, \qquad \sigma_{ss}(L) = M_2^+ \cup M_2^-. \tag{3.4}$$

Now, let us suppose that

$$q, q' \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} |q(x)| + |q'(x)| = 0, \qquad \int_0^\infty x^5 |q''(x)|\, dx < \infty. \tag{3.5}$$

Theorem 1 *Under condition* (3.5) *the functions N^+ and N^- have the following representations*:

$$N^+(\lambda) = i\alpha_2\lambda^3 + \beta^+\lambda^2 + \delta^+\lambda + \varphi^+ + \int_0^\infty f^+(t)e^{i\lambda t}\, dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{3.6}$$

$$N^-(\lambda) = i\alpha_2\lambda^3 + \beta^-\lambda^2 + \delta^-\lambda + \varphi^- + \int_0^\infty f^-(t)e^{-i\lambda t}\, dt, \quad \lambda \in \overline{\mathbb{C}}_-, \tag{3.7}$$

where $\beta^{\pm}, \delta^{\pm}, \varphi^{\pm} \in \mathbb{C}$, and $f^{\pm} \in L_1(\mathbb{R}_+)$.

Proof Using (2.3), (2.4), and (2.6) we have (3.6), where

$$\begin{aligned}
\beta^+ &= i\alpha_1 - \alpha_2 K(0,0) - \beta_2, \\
\delta^+ &= i\alpha_0 + i\alpha_2 K_x(0,0) - \alpha_1 K(0,0) - \beta_1 - i\beta_2 K(0,0), \\
\varphi^+ &= -\alpha_2 K_{xt}(0,0) + i\alpha_1 K_x(0,0) - \alpha_0 K(0,0) - \beta_0 + \beta_2 K_t(0,0), \\
f^+(t) &= -\alpha_2 K_{xtt}(0,t) + i\alpha_1 K_{xt}(0,t) + \alpha_0 K_x(0,t) + \beta_2 K_{tt}(0,t) - i\beta_1 K_t(0,t) - \beta_0 K(0,t).
\end{aligned} \tag{3.8}$$

The following result is obtained in [23]:

$$\left|K_{tt}(0,t)\right| \le c\left[t\left|q\left(\frac{t}{2}\right)\right| + \left|q'\left(\frac{t}{2}\right)\right| + tw\left(\frac{t}{2}\right) + w_1\left(\frac{t}{2}\right)\right]. \tag{3.9}$$

Then from (2.4), (2.5), and (3.9), we get

$$\begin{aligned}
\left|K_{xtt}(0,t)\right| \le c\Bigg[&\left|q''\left(\frac{t}{2}\right)\right| + t\left|q'\left(\frac{t}{2}\right)\right| + t^2\left|q\left(\frac{t}{2}\right)\right| \\
&+ t^2\sigma\left(\frac{t}{2}\right) + t\sigma_1\left(\frac{t}{2}\right) + \delta\left(\frac{t}{2}\right) + \delta_1\left(\frac{t}{2}\right)\Bigg],
\end{aligned} \tag{3.10}$$

where

$$\delta(x) = \int_x^\infty |q'(s)|\, ds, \qquad \delta_1(x) = \int_x^\infty \delta(t)\, dt$$

and $c > 0$ is a constant.

It follows from (2.5), (3.8), (3.9), and (3.10) that $f^+ \in L_1(\mathbb{R}_+)$. In a similar way we obtain (3.7). □

Theorem 2 *Under the condition* (3.5), *we have*:

(i) *The set of eigenvalues of* L *is bounded, has at most a countable number of elements, and its limit points can lie only in a bounded subinterval of the real axis.*

(ii) *The set of spectral singularities of* L *is bounded and* $\mu(\sigma_{ss}(L)) = 0$.

Proof From (3.8), (3.9), and (3.10), we have

$$\begin{aligned} N^+(\lambda) &= i\alpha_2\lambda^3 + \beta^+\lambda^2 + \delta^+\lambda + \varphi^+ + o(1), \quad \lambda \in \overline{\mathbb{C}}_+, |\lambda| \to \infty, \\ N^-(\lambda) &= i\alpha_2\lambda^3 + \beta^-\lambda^2 + \delta^-\lambda + \varphi^- + o(1), \quad \lambda \in \overline{\mathbb{C}}_-, |\lambda| \to \infty. \end{aligned} \tag{3.11}$$

Using (3.4), (3.11), and the uniqueness theorems of analytic functions [24], we obtain (i) and (ii). □

Theorem 3 *If*

$$q, q' \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} |q(x)| + |q'(x)| = 0, \qquad \int_0^\infty e^{\varepsilon x} |q''(x)| < \infty, \quad \varepsilon > 0, \tag{3.12}$$

then the operator L *has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

Proof Using (2.5), (3.8), (3.9), (3.10), and (3.12) we find that

$$|f^+(t)| \le ce^{-(\frac{\varepsilon}{2})t}, \tag{3.13}$$

where $c > 0$ is a constant. By (3.6) and (3.13) we observe that the function N^+ has an analytic continuation to the half-plane $\operatorname{Im}\lambda > -\frac{\varepsilon}{2}$. So we get $M_4^+ = \emptyset$. It follows from (3.3) that $M_3^+ = \emptyset$. Therefore the sets M_1^+ and M_2^+ have a finite number of elements with a finite multiplicity. We obtain similar results for the sets M_1^- and M_2^-. From (3.4) we have the proof of the theorem. □

It is seen that the condition (3.12) guarantees the analytic continuation of the functions N^+ and N^- from the real axis to the lower and upper half-planes, respectively. So the finiteness of eigenvalues and spectral singularities of L are achieved as a result of this analytic continuation.

Now let us suppose that

$$q, q' \in \mathrm{AC}(\mathbb{R}_+), \quad \lim_{x\to\infty} |q(x)| + |q'(x)| = 0, \qquad \sup_{x\in\mathbb{R}_+} \left[e^{\varepsilon\sqrt{x}} |q''(x)|\right] < \infty, \quad \varepsilon > 0, \tag{3.14}$$

which is weaker than (3.12).

It is evident that under the condition (3.14) the function N^+ is analytic in $\mathbb{C}_+$ and infinitely differentiable on the real axis. But N^+ does not have an analytic continuation from the real axis to the lower half-plane. Similarly, N^- does not have an analytic continuation from the real axis to the upper half-plane, either. Therefore, under the condition (3.14) the finiteness of eigenvalues and spectral singularities of L cannot be proved in a way similar to Theorem 3.

Lemma 4 *If* (3.14) *holds, then* $M_4^+ = M_4^- = \emptyset$.

Proof It follows from (3.6) and (3.14) that the function N^+ is analytic in $\mathbb{C}_+$, and all of its derivatives are continuous up to the real axis. Moreover, by Theorem 2 for sufficiently large $T > 0$, we have

$$\left|\int_{-\infty}^{-T} \frac{In|N^+(\lambda)|}{1+\lambda^2}\,d\lambda\right| < \infty, \qquad \left|\int_{T}^{\infty} \frac{In|N^+(\lambda)|}{1+\lambda^2}\,d\lambda\right| < \infty.$$

From (3.6), we obtain

$$\left|\frac{d^n}{d\lambda^n}N^+(\lambda)\right| \le A_n^+, \quad \lambda \in \overline{\mathbb{C}}_+, |\lambda| \le 2T, n \in \mathbb{N} \cup \{0\},$$

where

$$A_n^+ = 2^n c \int_0^\infty t^n e^{-(\frac{\varepsilon}{2})\sqrt{t}}\,dt, \quad n \in \mathbb{N} \cup \{0\}, \tag{3.15}$$

and $c > 0$ is a constant. Since the function N^+ is not equal to zero identically, by Pavlov's theorem [25], M_4^+ satisfies

$$\int_0^h \ln T^+(s)\,d\mu\left(M_4^+, s\right) > -\infty, \tag{3.16}$$

where $T^+(s) = \inf_n \frac{A_n^+ s^n}{n!}$, $\mu(M_4^+, s)$ is the linear Lebesgue measure of an s-neighborhood of M_4^+, and the constant A_n^+ is defined by (3.15).

Now we obtain the following estimates for A_n^+:

$$A_n^+ \le A a^n n! n^n, \tag{3.17}$$

where A and a are constants depending on c and ε. Substituting (3.17) in the definition of $T^+(s)$, we arrive at

$$T^+(s) \le A \inf_n \left(a^n s^n n^n\right) \le A \exp\left(-a^{-1} e^{-1} s^{-1}\right).$$

Now by (3.16) we get

$$\int_0^h \frac{1}{s}\,d\mu\left(M_4^+, s\right) < \infty. \tag{3.18}$$

Inequality (3.18) holds for arbitrary s if and only if $\mu(M_4^+, s) = 0$ or $M_4^+ = \emptyset$. In a similar way we can prove that $M_4^- = \emptyset$. □

Theorem 5 *Under the condition* (3.14) *the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of a finite multiplicity.*

Proof To be able to prove the theorem, we have to show that the functions N^+ and N^- have a finite number of zeros with finite multiplicities in $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$, respectively. We give the proof for N^+.

It follows from (3.3) and Lemma 4 that $M_3^+ = \emptyset$. So the bounded sets M_1^+ and M_2^+ have no limit points, *i.e.*, the function N^+ has only a finite number of zeros in $\overline{\mathbb{C}}_+$. Since $M_4^+ = \emptyset$, these zeros are of finite multiplicity. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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