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Some relationships among the constraint qualifications for Lagrangian dualities in DC infinite optimization problems

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Abstract
In this paper, we establish some relationships among several constraint qualifications, which characterize strong Lagrangian dualities and total Lagrangian dualities for DC infinite optimization problems.
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1 Introduction

Consider the following DC infinite optimization problem:

$$
(\mathrm{P})\quad \begin{array}{ll} \text{Min} & f(x) - g(x), \\ \text{s. t.} & f_t(x) - g_t(x) \le 0, \quad t \in T, \\ & x \in C, \end{array} \tag{1.1}
$$

where T is an arbitrary (possibly infinite) index set, C is a nonempty convex subset of a locally convex Hausdorff topological vector space X and $f, g, f_t, g_t : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $t \in T$, are proper convex functions. This problem has been studied extensively by many researchers. For example, the authors in [1–11] studied Lagrange dualities, Farkas lemmas, and optimality condition in the case when $g = g_t = 0$, $t \in T$ and the authors in [12] established the Fenchel-Lagrange duality in the case when $X = \mathbb{R}^n$ and T is finite, and Sun *et al.* gave some dualities and Farkas-type results in [13, 14]. In particular, the authors in [15] defined the dual problem of (1.1) by

$$
(\mathrm{D})\quad \sup_{\lambda \in \mathbb{R}_+^{(T)}} \inf_{w^* \in H^*} L(w^*, \lambda), \tag{1.2}
$$

where $H^* = \operatorname{dom} g^* \times \prod_{t \in T} \operatorname{dom} g_t^*$, and the Lagrange function $L : H^* \times \mathbb{R}_+^{(T)} \to \overline{\mathbb{R}}$ for (1.1) is defined by

$$
L(w^*, \lambda) := g^*(u^*) + \sum_{t \in T} \lambda_t g_t^*(v_t^*) - \Big(f + \delta_C + \sum_{t \in T} \lambda_t f_t\Big)^* \Big(u^* + \sum_{t \in T} \lambda_t v_t^*\Big) \tag{1.3}
$$

for any $(w^*,\lambda) \in H^* \times \mathbb{R}_+^{(T)}$ with $w^* = (u^*,(v_t^*)) \in H^*$ and $\lambda = (\lambda_t) \in \mathbb{R}_+^{(T)}$, and they established some Lagrangian dualities between (P) and (D).

Usually, the main interest for the above optimization problems is focused on two aspects: one is about strong Lagrangian duality and the other is about total Lagrangian duality. For the strong Lagrangian duality for problem (1.1), one seeks conditions ensuring

$$\inf_{x\in A}\bigl\{f(x)-g(x)\bigr\} = \max_{\lambda\in\mathbb{R}_+^{(T)}}\inf_{w^*\in H^*} L\bigl(w^*,\lambda\bigr); \tag{1.4}$$

and, for the problem of total Lagrangian duality, one seeks conditions ensuring the following equality holds:

$$\min_{x\in A}\bigl\{f(x)-g(x)\bigr\} = \max_{\lambda\in\mathbb{R}_+^{(T)}}\inf_{w^*\in H^*} L\bigl(w^*,\lambda\bigr), \tag{1.5}$$

where $A := \{x \in C : f_t(x) - g_t(x) \le 0, \text{for each } t \in T\}$. To establish the strong Lagrangian duality, the authors in [15] introduced the following constraint qualification (the conical $(WEHP)$):

$$\begin{aligned}\operatorname{epi}(f-g+\delta_A)^* = \bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\Biggl(\bigcap_{(u^*,v^*)\in H^*}\Biggl(\operatorname{epi}\Biggl(f+\delta_C+\sum_{t\in T}\lambda_t f_t\Biggr)^* - \bigl(u^*,g^*\bigl(u^*\bigr)\bigr)\\ -\sum_{t\in T}\lambda_t\bigl(v_t^*,g_t^*\bigl(v_t^*\bigr)\bigr)\Biggr)\Biggr),\end{aligned}$$

and to consider the total Lagrangian duality, the authors in [16] introduced two constraint qualifications: the quasi-$(WBCQ)$

$$\partial(f-g+\delta_A)(x) \subseteq \bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\Biggl(\bigcap_{(u^*,v^*)\in\partial H(x)}\Biggl(\partial\Biggl(f+\delta_C+\sum_{t\in T(x)}\lambda_t f_t\Biggr)(x) - u^* - \sum_{t\in T(x)}\lambda_t v_t^*\Biggr)\Biggr),$$

and the $(WBCQ)$

$$\partial(f-g+\delta_A)(x) \subseteq \bigcup_{\lambda\in\mathbb{R}_+^{(T)}}\Biggl(\bigcap_{(u^*,v^*)\in H^*}\Biggl(\partial\Biggl(f+\delta_C+\sum_{t\in T(x)}\lambda_t f_t\Biggr)(x) - u^* - \sum_{t\in T(x)}\lambda_t v_t^*\Biggr)\Biggr),$$

where $\partial H(x) := \partial g(x) \times \prod_{t\in T}\partial g_t(x)$, for each $x\in X$ and $T(x) := \{t\in T : f_t(x) - g_t(x) = 0\}$.

In this paper, we continuous to study the general case, that is, C is not necessarily closed and $f, g, f_t, g_t, t\in T$, are not necessarily lsc. Our main aim in the present paper is focused on the relationships among the conical $(WEHP)$, the quasi-$(WBCQ)$, and the $(WBCQ)$. The paper is organized as follows. The next section contains some necessary notations and preliminary results. In Section 3, some relationships among the conical $(WEHP)$, the quasi-$(WBCQ)$, and the $(WBCQ)$ are obtained and some examples illustrating the relationships are given.

2 Notations and preliminaries

The notations used in this paper are standard (*cf.* [17]). In particular, we assume throughout the whole paper that X is a real locally convex space and let X^* denote the dual space of X. For $x \in X$ and $x^* \in X^*$, we write $\langle x^*, x\rangle$ for the value of x^* at x, that is, $\langle x^*, x\rangle := x^*(x)$. Let Z be a set in X. The closure of Z is denoted by $\operatorname{cl} Z$. If $W \subseteq X^*$, then $\operatorname{cl} W$ denotes the weak*-closure of W. For the whole paper, we endow $X^* \times \mathbb{R}$ with the product topology of $w^*(X^*, X)$ and the usual Euclidean topology.

The normal cone of Z at $z_0 \in Z$ is denoted by $N_Z(z_0)$ and is defined by

$$N_Z(z_0) = \{x^* \in X^* : \langle x^*, z - z_0\rangle \le 0 \text{ for all } z \in Z\}.$$

The indicator function δ_Z of Z is defined by

$$\delta_Z(x) := \begin{cases} 0, & x \in Z, \\ +\infty, & \text{otherwise.} \end{cases}$$

Let f be a proper function defined on X. The effective domain, the conjugate function, and the epigraph of f are denoted by $\operatorname{dom} f$, f^*, and $\operatorname{epi} f$, respectively; they are defined by

$$\operatorname{dom} f := \{x \in X : f(x) < +\infty\},$$
$$f^*(x^*) := \sup\{\langle x^*, x\rangle - f(x) : x \in X\}, \quad \text{for each } x^* \in X^*,$$

and

$$\operatorname{epi} f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}.$$

It is well known and easy to verify that $\operatorname{epi} f^*$ is weak*-closed. The closure of f is denoted by $\operatorname{cl} f$, which is defined by

$$\operatorname{epi}(\operatorname{cl} f) = \operatorname{cl}(\operatorname{epi} f).$$

Then (*cf.* [17, Theorems 2.3.1]),

$$f^* = (\operatorname{cl} f)^*. \tag{2.1}$$

By [17, Theorem 2.3.4], if $\operatorname{cl} f$ is proper and convex, then the following equality holds:

$$f^{**} = \operatorname{cl} f. \tag{2.2}$$

Let $x \in X$. The subdifferential of f at x is defined by

$$\partial f(x) := \{x^* \in X^* : f(x) + \langle x^*, y - x\rangle \le f(y), \text{for each } y \in X\} \tag{2.3}$$

if $x \in \operatorname{dom} f$, and $\partial f(x) := \emptyset$ otherwise. We also define

$$\operatorname{dom} \partial f = \{x \in X : \partial f(x) \ne \emptyset\},$$

and

$$\operatorname{Im} \partial f = \{x^* \in X^* : x^* \in \partial f(x) \text{ for some } x \in X\}.$$

By [17, Theorems 2.3.1 and 2.4.2(iii)], the Young-Fenchel inequality below holds:

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle, \quad \text{for each pair } (x, x^*) \in X \times X^*, \tag{2.4}$$

and the Young equality holds:

$$f(x) + f^*(x^*) = \langle x^*, x \rangle \quad \text{if and only if} \quad x^* \in \partial f(x). \tag{2.5}$$

Furthermore, if g, h are proper functions, then

$$\operatorname{epi} g^* + \operatorname{epi} h^* \subseteq \operatorname{epi}(g+h)^*, \tag{2.6}$$

$$g \leq h \quad \Rightarrow \quad g^* \geq h^* \quad \Leftrightarrow \quad \operatorname{epi} g^* \subseteq \operatorname{epi} h^*, \tag{2.7}$$

and

$$\partial g(a) + \partial h(a) \subseteq \partial(g+h)(a), \quad \text{for each } a \in \operatorname{dom} g \cap \operatorname{dom} h. \tag{2.8}$$

We end this section with the remark that an element $p \in X^*$ can be naturally regarded as a function on X in such way that

$$p(x) := \langle p, x \rangle, \quad \text{for each } x \in X. \tag{2.9}$$

Thus the following fact is clear for any $a \in \mathbb{R}$ and real-valued proper function f:

$$\operatorname{epi}(f + p + a)^* = \operatorname{epi} f^* + (p, -a). \tag{2.10}$$

3 Relationships among constraint qualifications

Let X be a real locally convex Hausdorff vector space, and $C \subseteq X$ be a convex set. Let T be an index set and let f, g, f_t, g_t, $t \in T$ be proper convex functions such that $f - g$ and $f_t - g_t$, $t \in T$, are proper functions. Here and throughout the whole paper, following [17, p.39], we adapt the convention that $(+\infty) + (-\infty) = (+\infty) - (+\infty) = +\infty$, $0 \cdot (+\infty) = +\infty$, and $0 \cdot (-\infty) = 0$. Then

$$\emptyset \neq \operatorname{dom} f \subseteq \operatorname{dom} g \quad \text{and} \quad \emptyset \neq \operatorname{dom} f_t \subseteq \operatorname{dom} g_t. \tag{3.1}$$

Let $A \neq \emptyset$ be the solution set of the following system with the assumption that $A \cap \operatorname{dom}(f - g)$ is nonempty:

$$x \in C; \quad f_t(x) - g_t(x) \leq 0, \quad \text{for each } t \in T, \tag{3.2}$$

and let A^{cl} be the solution set of the following system:

$$x \in C; \quad f_t(x) - \operatorname{cl} g_t(x) \leq 0, \quad \text{for each } t \in T. \tag{3.3}$$

Then $A^{\mathrm{cl}} \subseteq A$. Following [18], we use $\mathbb{R}^{(T)}$ to denote the space of real tuples $\lambda = (\lambda_t)$ with only finitely many $\lambda_t \neq 0$, and let $\mathbb{R}_+^{(T)}$ denote the nonnegative cone in $\mathbb{R}^{(T)}$, that is,

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t) \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \text{for each } t \in T\}.$$

For simplicity, we denote

$$H^* := \operatorname{dom} g^* \times \prod_{t\in T} \operatorname{dom} g_t^*$$

and

$$\partial H(x) := \partial g(x) \times \prod_{t\in T} \partial g_t(x), \quad \text{for each } x \in X.$$

To make the dual problem considered here well defined, we further assume that $\operatorname{cl} g$ and $\operatorname{cl} g_t$, $t \in T$, are proper. Then $H^* \neq \emptyset$. For the whole paper, any elements $\lambda \in \mathbb{R}^{(T)}$ and $v^* \in \prod_{t\in T} \operatorname{dom} g_t^*$ are understood as $\lambda = (\lambda_t) \in \mathbb{R}^{(T)}$ and $v^* = (v_t^*) \in \prod_{t\in T} \operatorname{dom} g_t^*$, respectively. Following [15], we define the characteristic set K for the DC optimization problem (1.1) by

$$K := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in H^*} \left(\operatorname{epi}\left(f + \delta_C + \sum_{t\in T} \lambda_t f_t\right)^* - \left(u^*, g^*(u^*)\right) - \sum_{t\in T} \lambda_t\left(v_t^*, g_t^*(v_t^*)\right)\right)\right), \tag{3.4}$$

where we adopt the convention that $\bigcap_{t\in\emptyset} S_t = X$ (see [17, p.2]). Below we will make use of the subdifferential $\partial h(x)$ for a general proper function (not necessarily convex) $h : X \to \overline{\mathbb{R}}$; see (2.3). Clearly, the following equivalence holds:

$$x_0 \text{ is a minimizer of } h \text{ if and only if } 0 \in \partial h(x_0). \tag{3.5}$$

For each $x \in X$, let $T(x)$ be the active index set of system (3.2), that is,

$$T(x) := \{t \in T : f_t(x) - g_t(x) = 0\}.$$

Define $N'(x)$ by

$$N'(x) := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in H^*} \left(\partial\left(f + \delta_C + \sum_{t\in T(x)} \lambda_t f_t\right)(x) - u^* - \sum_{t\in T(x)} \lambda_t v_t^*\right)\right) \tag{3.6}$$

and define $N_0'(x)$ by

$$N_0'(x) := \bigcup_{\lambda\in\mathbb{R}_+^{(T)}} \left(\bigcap_{(u^*,v^*)\in \partial H(x)} \left(\partial\left(f + \delta_C + \sum_{t\in T(x)} \lambda_t f_t\right)(x) - u^* - \sum_{t\in T(x)} \lambda_t v_t^*\right)\right). \tag{3.7}$$

Then, for each $x \in X$,

$$N'(x) \subseteq N_0'(x).$$

Definition 3.1 The family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ is said to satisfy

(a) the lower semi-continuity closure ((LSC)) if

$$\operatorname{epi}(f - g + \delta_A)^* = \operatorname{epi}(f - \operatorname{cl} g + \delta_{A^{\mathrm{cl}}})^*; \tag{3.8}$$

(b) the conical weak epigraph hull property ((*WEHP*)) if

$$\operatorname{epi}(f - g + \delta_A)^* = K; \tag{3.9}$$

(c) the quasi-weakly basic constraint qualification (the quasi-(*WBCQ*)) at $x \in A$ if

$$\partial(f - g + \delta_A)(x) \subseteq N_0'(x); \tag{3.10}$$

(d) the weakly basic constraint qualification (the (*WBCQ*)) at $x \in A$ if

$$\partial(f - g + \delta_A)(x) \subseteq N'(x). \tag{3.11}$$

It is said that the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ satisfies the quasi-(*WBCQ*) (resp. the (*WBCQ*)) if it satisfies the quasi-(*WBCQ*) (resp. the (*WBCQ*)) at each point $x \in A$.

Remark 3.1

(a) The notions of (*LSC*) and the conical (*WEHP*) were introduced in [15] and the quasi-(*WBCQ*) and the (*WBCQ*) were taken from [16].

(b) Recall from [3, 4] that the family $\{\delta_C; f_t : t \in T\}$ has the conical $(WEHP)_f$ if

$$\operatorname{epi}(f + \delta_A)^* = \bigcup_{\lambda \in R_+^{(T)}} \operatorname{epi}\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)^* \tag{3.12}$$

and has the $(WBCQ)_f$ at $x \in \operatorname{dom} f \cap A$ if

$$\partial(f + \delta_A)(x) = \bigcup_{\substack{\lambda \in R_+^{(T)} \\ \sum_{t \in T} \lambda_t f_t(x) = 0}} \partial\left(f + \delta_C + \sum_{t \in T} \lambda_t f_t\right)(x). \tag{3.13}$$

Thus, in the special case when $g = g_t = 0$, $t \in T$, the conical (*WEHP*) coincides with the conical $(WEHP)_f$ for the family $\{\delta_C; f_t : t \in T\}$ and the quasi-(*WBCQ*) and (*WBCQ*) are reduced to the $(WBCQ)_f$ for the family $\{\delta_C; f_t : t \in T\}$.

Theorems 3.1 and 3.2 characterize the relationships among the quasi-(*WBCQ*), the (*WBCQ*), and the conical (*WEHP*).

Theorem 3.1 *The following implication holds*:

$$\left[\operatorname{epi}(f - g + \delta_A)^* \subseteq K\right] \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ). \tag{3.14}$$

Consequently,

$$\textit{the conical } (WEHP) \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ). \tag{3.15}$$

Proof Suppose that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. To show the quasi-(*WBCQ*), let $x_0 \in A$ and let $x^* \in \partial(f - g + \delta_A)(x_0)$. Then, by (2.5),

$$\langle x^*, x_0 \rangle - (f - g + \delta_A)(x_0) = (f - g + \delta_A)^*(x^*).$$

This implies that

$$\big(x^*,\langle x^*,x_0\rangle-(f-g+\delta_A)(x_0)\big)\in\operatorname{epi}(f-g+\delta_A)^*\subseteq K.$$

Hence, there exists $\lambda\in\mathbb{R}_+^{(T)}$ such that, for each $(u^*,v^*)\in\partial H(x_0)$,

$$\big(x^*,\langle x^*,x_0\rangle-(f-g+\delta_A)(x_0)\big)\in\operatorname{epi}\Big(f+\delta_C+\sum_{t\in T}\lambda_t f_t\Big)^*-\big(u^*,g^*(u^*)\big)-\sum_{t\in T}\lambda_t\big(v_t^*,g_t^*(v_t^*)\big).$$

Let $(u^*,v^*)\in\partial H(x_0)$. There exists $(x_1^*,r_1)\in\operatorname{epi}(f+\delta_C+\sum_{t\in J}\lambda_t f_t)^*$ such that

$$x^*=x_1^*-u^*-\sum_{t\in J}\lambda_t v_t^*\tag{3.16}$$

and

$$\langle x^*,x_0\rangle-(f-g+\delta_A)(x_0)=r_1-g^*(u^*)-\sum_{t\in J}\lambda_t g_t^*(v_t^*),\tag{3.17}$$

where $J:=\{t\in T:\lambda_t\neq 0\}$ is a finite subset of T. Below we only need to show that $x_1^*\in\partial(f+\delta_C+\sum_{t\in J}\lambda_t f_t)(x_0)$ and $J\subseteq T(x_0)$. To do this, note by the definition of epigraph, one has

$$\Big(f+\delta_C+\sum_{t\in J}\lambda_t f_t\Big)^*(x_1^*)\le r_1.\tag{3.18}$$

Note that $(u^*,v^*)\in\partial H(x_0)$, it follows from (2.5) that

$$g(x_0)+g^*(u^*)=\langle u^*,x_0\rangle\quad\text{and}\quad g_t(x_0)+g_t^*(v_t^*)=\langle v_t^*,x_0\rangle,\quad\text{for each } t\in T.\tag{3.19}$$

This together with (3.16), (3.17), and (3.18) implies that

$$\begin{aligned}
&\Big(f+\delta_C+\sum_{t\in J}\lambda_t f_t\Big)^*(x_1^*)\\
&\quad\le\langle x^*,x_0\rangle-(f-g+\delta_A)(x_0)+g^*(u^*)+\sum_{t\in J}\lambda_t g_t^*(v_t^*)\\
&\quad\le\Big\langle x_1^*-u^*-\sum_{t\in J}\lambda_t v_t^*,x_0\Big\rangle-\Big(f-g+\delta_C+\sum_{t\in J}\lambda_t(f_t-g_t)\Big)(x_0)\\
&\qquad+g^*(u^*)+\sum_{t\in J}\lambda_t g_t^*(v_t^*)\\
&\quad\le\langle x_1^*,x_0\rangle-\Big(f+\delta_C+\sum_{t\in J}\lambda_t f_t\Big)(x_0)+\big\{g(x_0)-\langle u^*,x_0\rangle+g^*(u^*)\big\}\\
&\qquad+\sum_{t\in J}\lambda_t\big\{g_t(x_0)-\langle v_t^*,x_0\rangle+g_t^*(v_t^*)\big\}\\
&\quad=\langle x_1^*,x_0\rangle-\Big(f+\delta_C+\sum_{t\in J}\lambda_t f_t\Big)(x_0),
\end{aligned}$$

where the second inequality holds because $x_0 \in A$. Hence,

$$\Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)^*(x_1^*) + \Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)(x_0) = \langle x_1^*, x_0\rangle$$

since

$$\Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)^*(x_1^*) \geq \langle x_1^*, x_0\rangle - \Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)(x_0)$$

holds automatically by the Fenchel-Young inequality (2.4). Therefore, by (2.5), $x^* \in \partial(f + \delta_C + \sum_{t\in J} \lambda_t f_t)(x_0)$. To show $J \subseteq T(x_0)$, note that $x_0 \in A$, then

$$\Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)^*(x_1^*) \leq \langle x^*, x_0\rangle - f(x_0) + g(x_0) + g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*)$$

and

$$\Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)^*(x_1^*) \geq \langle x_1^*, x_0\rangle - f(x_0) - \sum_{t\in J} \lambda_t f_t(x_0).$$

Thus, by (3.16) and (3.19), we have

$$\begin{aligned} f(x_0) - g(x_0) - \langle x^*, x_0\rangle &\leq g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) - \Big(f + \delta_C + \sum_{t\in J} \lambda_t f_t\Big)^*(x_1^*) \\ &\leq g^*(u^*) + \sum_{t\in J} \lambda_t g_t^*(v_t^*) - \langle x_1^*, x_0\rangle + f(x_0) + \sum_{t\in J} \lambda_t f_t(x_0) \\ &= f(x_0) - g(x_0) - \langle x^*, x_0\rangle + \sum_{t\in J} \lambda_t\big(f_t(x_0) - g_t(x_0)\big) \\ &\leq f(x_0) - g(x_0) - \langle x^*, x_0\rangle. \end{aligned}$$

Since $\lambda_t > 0$ and $f_t(x_0) - g_t(x_0) \leq 0$, for each $t \in J$, it follows that $\lambda_t(f_t(x_0) - g_t(x_0)) = 0$, that is, $f_t(x_0) - g_t(x_0) = 0$, for each $t \in J$. Thus, $J \subseteq T(x_0)$ and hence the quasi-($WBCQ$) holds. □

Theorem 3.2 *If* $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im}\partial(f - g + \delta_A)$, *then*

$$\textit{the } (WBCQ) \quad \Longrightarrow \quad \big[\operatorname{epi}(f - g + \delta_A)^* \subseteq K\big]. \tag{3.20}$$

Furthermore, if the (LSC) *holds, then*

$$\textit{the } (WBCQ) \Longrightarrow \textit{ the conical } (WEHP). \tag{3.21}$$

Proof Suppose that $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im}\partial(f - g + \delta_A)$ and that the ($WBCQ$) holds. To show $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$, let $(x^*, \alpha) \in \operatorname{epi}(f - g + \delta_A)^*$. Since $x^* \in \operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im}\partial(f - g + \delta_A)$, it follows that there exists $x_0 \in \operatorname{dom}(f - g) \cap A$ such that $x^* \in \partial(f - g + \delta_A)(x_0) \subseteq N'(x_0)$, thanks to the assumed (*WBCQ*). This means that there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that, for each $(u^*, v^*) \in H^*$,

$$x^* \in \partial\left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)(x_0) - u^* - \sum_{t\in J}\lambda_t v_t^*$$

for some finite subset $J \subseteq T(x_0)$ and $\{\lambda_t\} \subseteq \mathbb{R}$ with $\lambda_t \geq 0$, for each $t \in J$. Let $(u^*, v^*) \in H^*$. Then there exists $x_1^* \in \partial(f + \delta_C + \sum_{t\in J}\lambda_t f_t)(x_0)$ such that

$$x^* = x_1^* - u^* - \sum_{t\in J}\lambda_t v_t^*. \tag{3.22}$$

By the Young equality (2.5), we have

$$\langle x_1^*, x_0\rangle = \left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)^*(x_1^*) + \left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)(x_0) \tag{3.23}$$

and

$$\langle x^*, x_0\rangle = (f - g + \delta_A)^*(x^*) + (f - g + \delta_A)(x_0) \leq \alpha + f(x_0) - g(x_0), \tag{3.24}$$

where the last inequality holds because of $(x^*, \alpha) \in \operatorname{epi}(f - g + \delta_A)^*$ and $x_0 \in A$. This together with (3.22) and (3.23) implies that

$$\begin{aligned}\left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)^*(x_1^*) &\leq \langle u^*, x_0\rangle + \sum_{t\in J}\lambda_t\langle v_t^*, x_0\rangle + \alpha - g(x_0) - \sum_{t\in J}\lambda_t f_t(x_0)\\ &\leq \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*) - \sum_{t\in J}\lambda_t\big(f_t(x_0) - g_t(x_0)\big)\\ &= \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*),\end{aligned}$$

where the second inequality holds by the Fenchel-Young inequality and the last equality holds because $J \subseteq T(x_0)$. This means that

$$\left(x_1^*, \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*)\right) \in \operatorname{epi}\left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)^*.$$

Hence,

$$\begin{aligned}(x^*, \alpha) &= \left(x_1^*, \alpha + g^*(u^*) + \sum_{t\in J} g_t^*(v_t^*)\right) - \big(u^*, g^*(u^*)\big) - \sum_{t\in J}\lambda_t\big(v_t^*, g_t^*(v_t^*)\big)\\ &\in \operatorname{epi}\left(f + \delta_C + \sum_{t\in J}\lambda_t f_t\right)^* - \big(u^*, g^*(u^*)\big) - \sum_{t\in J}\lambda_t\big(v_t^*, g_t^*(v_t^*)\big)\end{aligned}$$

and so $(x^*, \alpha) \in K$ by the arbitrary of $(u^*, v^*) \in H^*$. Therefore,

$$\operatorname{epi}(f - g + \delta_A)^* \subseteq K. \tag{3.25}$$

Furthermore, we assume that the (LSC) holds. Then (3.8) holds. By [15, Lemma 3.1], we see that

$$K = \bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \left(f - \operatorname{cl} g + \delta_C + \sum_{t \in T} \lambda_t (f_t - \operatorname{cl} g_t) \right)^*; \tag{3.26}$$

while by [3, (3.5)],

$$\bigcup_{\lambda \in \mathbb{R}_+^{(T)}} \left(f - \operatorname{cl} g + \delta_C + \sum_{t \in T} \lambda_t (f_t - \operatorname{cl} g_t) \right)^* \subseteq \operatorname{epi}(f - \operatorname{cl} g + \delta_{A^{\mathrm{cl}}})^*. \tag{3.27}$$

Combining (3.26), (3.27) with (3.8), we have

$$K \subseteq \operatorname{epi}(f - g + \delta_A)^*. \tag{3.28}$$

Hence, by (3.25), the conical $(WEHP)$ holds and the proof is complete. □

Remark 3.2 By [16, Remark 3.2], we see that

$$\text{the } (WBCQ) \quad \Longrightarrow \quad \text{the quasi-}(WBCQ)$$

and by Theorems 3.1 and 3.2, we get

$$\begin{aligned} &\left[\text{the } (WBCQ) \ \& \ \operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im} \partial(f - g + \delta_A) \ \& \text{ the } (LSC)\right] \\ &\quad \Longrightarrow \quad \text{the conical } (WEHP) \quad \Longrightarrow \quad \text{the quasi-}(WBCQ). \end{aligned}$$

By Theorems 3.1 and 3.2, we get the following corollary directly, which was given in [4, Proposition 3.1]. Note that the conical $(WEHP)_f$ and the $(WBCQ)_f$ for the family $\{\delta_C; f_t : t \in T\}$ were introduced in [3, 4]; see also Remark 3.1(ii).

Corollary 3.1 *For the family* $\{\delta_C; f_t : t \in T\}$, *the following implication holds*:

$$\textit{the conical } (WEHP)_f \quad \Longrightarrow \quad \textit{the quasi-}(WBCQ)_f$$

and

$$\textit{the conical } (WEHP)_f \quad \Longleftrightarrow \quad \textit{the quasi-}(WBCQ)_f$$

if $\operatorname{dom}(f + \delta_A)^* \subseteq \operatorname{im} \partial(f + \delta_A)$.

The following example illustrates (3.14) and shows that the quasi-$(WBCQ)$ in (3.14) cannot be replaced by the $(WBCQ)$.

Example 3.1 Let $X = C := \mathbb{R}$ and let $T = \{1\}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$, respectively, by

$$f(x) := \begin{cases} x, & x \geq 0, \\ +\infty, & x < 0, \end{cases} \qquad g(x) := \begin{cases} 0, & x > 0, \\ 1, & x = 0, \\ +\infty, & x < 0, \end{cases} \quad \text{for each } x \in \mathbb{R},$$

$f_1 := \delta_{[0,+\infty)}$ and $g_1 := 0$. Then f, g, f_1, and g_1 are proper convex functions and $A = [0,+\infty)$. Note that, for each $x \in \mathbb{R}$,

$$(f - g + \delta_A)(x) = \begin{cases} x, & x > 0, \\ -1, & x = 0, \\ +\infty, & x < 0, \end{cases}$$

and $f + \delta_C + \lambda f_1 = f$ holds, for each $\lambda \geq 0$. Then, for each $x^* \in \mathbb{R}$, $g^* = \delta_{(-\infty,0]}$,

$$(f - g + \delta_A)^*(x^*) = \begin{cases} 1, & x^* \leq 1, \\ +\infty, & x^* > 1, \end{cases}$$

and, for each $\lambda \geq 0$,

$$(f + \delta_C + \lambda f_1)^*(x^*) = \begin{cases} 0, & x^* \leq 1, \\ +\infty, & x^* > 1. \end{cases}$$

This means that $\operatorname{dom} g^* = (-\infty, 0]$,

$$\operatorname{epi}(f - g + \delta_A)^* = (-\infty, 1] \times [1, +\infty)$$

and

$$\operatorname{epi}(f + \delta_C + \lambda f_1)^* = (-\infty, 1] \times [0, +\infty), \quad \text{for each } \lambda \geq 0.$$

Hence

$$K = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in (-\infty,0]} \left(\operatorname{epi}(f + \delta_C + \lambda f_1)^* - \left(u^*, g^*(u^*)\right)\right) \right) = (-\infty, 1] \times [0, +\infty).$$

This implies that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. Moreover, it is easy to see that, for each $x \in A$,

$$\partial g(x) = \begin{cases} \{0\}, & x > 0, \\ \emptyset, & x = 0, \end{cases}$$

and, for each $\lambda \geq 0$,

$$\partial(f - g + \delta_A)(x) = \partial(f + \delta_C + \lambda f_1)(x) = \begin{cases} 1, & x > 0, \\ (-\infty, 1], & x = 0. \end{cases}$$

Hence, for each $x \in A$,

$$N_0'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in \partial g(x)} \left(\partial(f + \delta_C + \lambda_1 f_1)(x) - u^*\right) \right) = \begin{cases} 1, & x > 0, \\ \mathbb{R}, & x = 0, \end{cases}$$

and

$$N'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in \operatorname{dom} g^*} \left(\partial(f + \delta_C + \lambda_1 f_1)(x) - u^*\right) \right) = \begin{cases} \emptyset, & x > 0, \\ (-\infty, 1], & x = 0. \end{cases}$$

This means that $\partial(f-g+\delta_A)(x) \subseteq N_0'(x)$ but $\partial(f-g+\delta_A)(x) \not\subseteq N'(x)$, for each $x \in A$. Thus, the quasi-($WBCQ$) holds but not the ($WBCQ$).

Example 3.2 illustrates Theorem 3.2 and Example 3.3 shows that the condition (LSC) is essential for (3.21) to hold.

Example 3.2 Let $X = C := \mathbb{R}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$, respectively, by $f = f_1 = g := \delta_{(-\infty,0]}$, $g_1 := 0$. Then f, g, f_1, and g_1 are proper convex functions. Consider the system (3.2) with $T := \{1\}$. Then one sees that

$$A = \{x \in \mathbb{R} : f_1(x) - g_1(x) \leq 0\} = (-\infty, 0].$$

It is easy to see that

$$f - g + \delta_A = \delta_A \quad \text{and} \quad (f - g + \delta_A)^* = \delta_{[0,+\infty)}.$$

Hence,

$$\operatorname{dom}(f - g + \delta_A)^* = [0, +\infty),$$

and, for each $x \in A$,

$$\partial(f - g + \delta_A)(x) = N_A(x) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

This implies that $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im} \partial(f - g + \delta_A)$. Note that $g_1^* = \delta_{\{0\}}$, $g^* = \delta_{[0,+\infty)}$, and $(f + \lambda f_1)^* = \delta_{[0,+\infty)}$, for each $\lambda \geq 0$. It follows that, for each $x \in A$,

$$N'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(N_A(x) - u^*\right) \right) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

Thus, $\partial(f - g + \delta_A)(x) = N'(x)$ and the ($WBCQ$) holds. Therefore, by Theorem 3.1, we see that $\operatorname{epi}(f - g + \delta_A)^* \subseteq K$. Moreover, since g is lsc, it follows that the (LSC) holds. Therefore, by (3.21), one sees that the conical ($WEHP$) holds. In fact, it is easy to see that

$$\operatorname{epi}(f - g + \delta_A)^* = [0, +\infty) \times [0, +\infty)$$

and

$$K = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(\operatorname{epi}(f + \lambda f_1)^* - \left(u^*, g^*\left(u^*\right)\right)\right) \right) = [0, +\infty) \times [0, +\infty).$$

Example 3.3 Let $X = C := \mathbb{R}$. Define $f, g, f_1, g_1 : \mathbb{R} \to \overline{\mathbb{R}}$ as in [15, Example 3.1], that is, $f = f_1 := \delta_{(-\infty,0]}$, $g_1 := 0$ and, for each $x \in \mathbb{R}$,

$$g(x) := \begin{cases} 0, & x < 0, \\ 1, & x = 0, \\ +\infty, & x > 0. \end{cases}$$

Then f, g, f_1, and g_1 are proper convex functions. Consider the system (3.2) with $T := \{1\}$. Then one sees that

$$A = \{x \in \mathbb{R} : f_1(x) - g_1(x) \leq 0\} = (-\infty, 0].$$

It is easy to see that, for each $x \in \mathbb{R}$,

$$(f - g + \delta_A)(x) = \begin{cases} 0, & x < 0, \\ -1, & x = 0, \\ +\infty, & x > 0, \end{cases}$$

and, for each $x^* \in \mathbb{R}$,

$$(f - g + \delta_A)^*(x^*) = \begin{cases} 1, & x^* \geq 0, \\ +\infty, & x^* < 0. \end{cases}$$

Moreover, for each $x \in A$, we see that

$$\partial(f - g + \delta_A)(x) = \begin{cases} \emptyset, & x < 0, \\ [0, +\infty), & x = 0. \end{cases}$$

Thus, $\operatorname{dom}(f - g + \delta_A)^* \subseteq \operatorname{im} \partial(f - g + \delta_A)$. Note that $g_1^* = \delta_{\{0\}}$, $g^* = \delta_{[0,+\infty)}$, and $(f + \lambda f_1)^* = \delta_{[0,+\infty)}$, for each $\lambda \geq 0$. It follows that, for each $x \in A$,

$$N'(x) = \bigcup_{\lambda \geq 0} \left(\bigcap_{u^* \in [0,+\infty)} \left(N_A(x) - u^*\right) \right) = \begin{cases} \{0\}, & x < 0, \\ [0, +\infty) & x = 0. \end{cases}$$

Therefore, the (*WBCQ*) holds. However, the conical (*WEHP*) does not hold as shown in Example 3.1 in [15]. Actually, the family $\{f, g, \delta_C; f_t, g_t : t \in T\}$ does not satisfy the (*LSC*), since

$$\operatorname{epi}(f - g + \delta_A)^* = [0, +\infty) \times [1, +\infty);$$

but

$$\operatorname{epi}(f - \operatorname{cl} g + \delta_A)^* = [0, +\infty) \times [0, +\infty).$$

Competing interests
The author declares that they have no competing interests.

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