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A sharp two-sided inequality for bounding the Wallis ratio

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Abstract
In this article we establish a sharp two-sided inequality for bounding the Wallis ratio. Some best constants for the estimation of the Wallis ratio are obtained. An asymptotic formula for the Wallis ratio is also presented.

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1 Introduction and main results

For $n \in \mathbb{N}$ (the set of all positive integers), the double factorial $n!!$ is defined by

$$n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} (n-2i), \tag{1}$$

where in (1) the floor function $\lfloor t \rfloor$ denotes the largest integer less than or equal to t. For our own convenience, in what follows, we denote the ratio of two neighboring double factorials by

$$W_n = \frac{(2n-1)!!}{(2n)!!}, \tag{2}$$

which is called the Wallis ratio in the literature.

The Wallis ratio W_n can be represented as follows (see [1, p.258]):

$$W_n = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{2^{2n}}\binom{2n}{n} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)}, \tag{3}$$

where in (3) $\Gamma(x)$ is the classical Euler's gamma function defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt. \tag{4}$$

In [2] the author proved, for all $n \in \mathbb{N}$,

$$\left[n\pi\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n+\frac{15}{4n}}}\right)\right]^{-1/2} < W_n < \left[n\pi\left(1+\frac{1}{4n-\frac{1}{2}+\frac{3}{16n}}\right)\right]^{-1/2}. \tag{5}$$

In [3], the following result was established:
For $n \in \mathbb{N}$ and $\varepsilon \in (0, \frac{1}{2})$,

$$\left[n\pi\left(1+\frac{1}{4n-\frac{1}{2}}\right)\right]^{-1/2} < W_n < \left[n\pi\left(1+\frac{1}{4n-\frac{1}{2}+\varepsilon}\right)\right]^{-1/2}. \tag{6}$$

The right-hand inequality in (6) holds for $n > n^*$, where n^* is the maximal root on

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

We also note that in [4] the authors proved the result below.
For all $n \in \mathbb{N}$

$$\frac{\sqrt{\pi}}{2\sqrt{n+\frac{9\pi}{16}-1}} \le \frac{(2n)!!}{(2n+1)!!} < \frac{\sqrt{\pi}}{2\sqrt{n+\frac{3}{4}}}, \tag{7}$$

which is equivalent to the following:

$$\frac{2\sqrt{n+\frac{3}{4}}}{(2n+1)\sqrt{\pi}} < W_n \le \frac{2\sqrt{n+\frac{9\pi}{16}-1}}{(2n+1)\sqrt{\pi}}. \tag{8}$$

In this article we shall establish a sharp two-sided inequality for bounding the Wallis ratio in the form

$$C_1 P_n < W_n < C_2 P_n, \tag{9}$$

where in (9) the constants $C_1 > 0$ and $C_2 > 0$ are best possible. This means that the constant C_1 in (9) cannot be replaced by a number which is greater than C_1 and the constant C_2 in (9) cannot be replaced by a number which is less than C_2. An asymptotic formula for the Wallis ratio is also given.

Our main result may be stated as the following theorem.

Theorem 1 *For all* $n \ge 2$,

$$\left(\frac{2}{3}\right)^{3/2}\left(1-\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}} \le W_n < \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}}. \tag{10}$$

The constants $(\frac{2}{3})^{3/2}$ *and* $\sqrt{\frac{e}{\pi}}$ *in* (10) *are best possible. Furthermore, the asymptotic formula*

$$W_n \sim \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+\frac{1}{2}}\left(n-\frac{3}{2}\right)^{-\frac{1}{2}}, \quad n \to \infty, \tag{11}$$

is valid.

2 Proof of main result

We are now in a position to prove our main result stated in Theorem 1.

Proof of Theorem 1 Define

$$f(x) := \frac{x^{x+1/2}}{e^x\Gamma(x+1)}. \tag{12}$$

Taking the logarithm of $f(x)$ and then differentiating yield

$$\ln f(x) = \left(x - \frac{1}{2}\right)\ln x - x - \ln\Gamma(x), \tag{13}$$

$$\bigl[\ln f(x)\bigr]' = \ln x - \frac{1}{2x} - \psi(x). \tag{14}$$

In (14)

$$\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It is well known that (see [5, p.892])

$$\psi(x) = \ln x - \frac{1}{2x} - 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}, \quad x > 0. \tag{15}$$

From (14) and (15) we get

$$\bigl[\ln f(x)\bigr]' = 2\int_0^\infty \frac{t\,dt}{(t^2+x^2)(e^{2\pi t}-1)}, \quad x > 0. \tag{16}$$

Hence,

$$\bigl[\ln f(x)\bigr]' > 0, \quad x \in (0,\infty),$$

which means that $\ln f(x)$, and thus $f(x)$, is strictly increasing on $(0,\infty)$.

It is easy to see that

$$\lim_{x\to 0+} f(x) = 0.$$

Since (see [6, p.20])

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty, \tag{17}$$

from (13) and (17), we have

$$\ln f(x) = -\ln\sqrt{2\pi} + O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty, \tag{18}$$

which implies

$$\lim_{x\to\infty} f(x) = \frac{1}{\sqrt{2\pi}}. \tag{19}$$

Since the function $f(x)$ is strictly increasing from $(0,\infty)$ onto $(0,\frac{1}{\sqrt{2\pi}})$ and

$$\Gamma(n+1)=n!, \tag{20}$$

we obtain

$$\frac{2\sqrt{2}}{e^2}=f(2)\leq f(n)=\frac{n^{n+1/2}}{e^n n!}<\frac{1}{\sqrt{2\pi}},\quad n\geq 2, \tag{21}$$

and

$$\lim_{n\to\infty}\frac{n^{n+1/2}}{e^n n!}=\frac{1}{\sqrt{2\pi}}. \tag{22}$$

The lower and upper bounds in (21) are best possible.

Also we define

$$h(x):=\frac{e^x\sqrt{x-1}\,\Gamma(x+1)}{x^{x+1}}. \tag{23}$$

Since the function $h(x)$ is strictly increasing from $(1,\infty)$ onto $(0,\sqrt{2\pi})$ (see [7, Theorem 1.3]) and in view of (20), we obtain

$$\sqrt{\pi}\left(\frac{e}{3}\right)^{3/2}=h\left(\frac{3}{2}\right)\leq h\left(n-\frac{1}{2}\right)<\sqrt{2\pi},\quad n\geq 2, \tag{24}$$

and

$$\lim_{n\to\infty}h\left(n-\frac{1}{2}\right)=\sqrt{2\pi}. \tag{25}$$

It is well known that [1, p.258] for all $n\in\mathbb{N}$,

$$\Gamma\left(n+\frac{1}{2}\right)=n!\sqrt{\pi}\,W_n. \tag{26}$$

By using (26), after some algebra, (24) and (25) can be rewritten, respectively, as

$$\frac{e^2}{3\sqrt{3}}\leq\frac{e^n n!\sqrt{n-\frac{3}{2}}\,W_n}{(n-\frac{1}{2})^{n+1/2}}<\sqrt{2e},\quad n\geq 2, \tag{27}$$

and

$$\lim_{n\to\infty}\frac{e^n n!\sqrt{n-\frac{3}{2}}\,W_n}{(n-\frac{1}{2})^{n+1/2}}=\sqrt{2e}. \tag{28}$$

The constants $\frac{e^2}{3\sqrt{3}}$ and $\sqrt{2e}$ in (27) are best possible.

Combining (21) and (27) yields

$$\left(\frac{2}{3}\right)^{3/2}\leq\frac{n^{n+1/2}\sqrt{n-\frac{3}{2}}\,W_n}{(n-\frac{1}{2})^{n+1/2}}<\sqrt{\frac{e}{\pi}},\quad n\geq 2. \tag{29}$$

The constants $(\frac{2}{3})^{3/2}$ and $\sqrt{\frac{e}{\pi}}$ in (29) are best possible. From (29) the inequality (10) follows.

Combining (22) and (28) gives

$$\lim_{n\to\infty} \frac{n^{n+1/2}\sqrt{n-\frac{3}{2}}W_n}{(n-\frac{1}{2})^{n+1/2}} = \sqrt{\frac{e}{\pi}}, \tag{30}$$

which is equivalent to the asymptotic formula (11). The proof of Theorem 1 is thus completed. □

Remark 1 Some related functions associated with $f(x)$, defined by (12), were proved [8–12] to be logarithmically completely monotonic.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors contributed to the writing of the present article. They also read and approved the final manuscript.

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