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Barnes-Godunova-Levin type inequality of the Sugeno integral for an (α, m)-concave function

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Abstract

In this paper, a Barnes-Godunova-Levin type inequality for the Sugeno integral based on an (α, m)-concave function is proved. Some examples are given to illustrate the validity of these inequalities. Finally, several important results, as special cases of an (α, m)-concave function, are also obtained.

MSC: 03E72; 28B15; 28E10; 26D10

Keywords: Barnes-Godunova-Levin type inequality; Sugeno integral; (α, m)-concave function

1 Introduction

As a tool for modeling non-deterministic problems, the theory of fuzzy measures and fuzzy integrals was introduced by Sugeno in [1]. Many authors generalized the Sugeno integral by using some other operators to replace the special operator(s) $\vee$ and/or $\wedge$ and introduced Choquet-like integral [2], Shilkret integral [3], $\perp$-integral [4], and pseudo-integral [5]. Suárez and Gil [6] presented two families of fuzzy integrals, the so-called seminormed fuzzy integral and semiconormed fuzzy integral. Wang and Klir [7] provided a general overview on fuzzy measurement and fuzzy integration.

Recently, Flores-Franulič *et al.* [8–21] generalized several classical integral inequalities of the Sugeno integral. Agahi *et al.* [22] proved a general Barnes-Godunova-Levin type inequality of the Sugeno integral for a concave function. In [23], Mihesan introduced the concept of (α, m)-convex function. For recent results and generalizations concerning m-convex and (α, m)-convex functions, we refer to [24–26]. The purpose of this paper is to prove a Barnes-Godunova-Levin type inequality for the Sugeno integral based on an (α, m)-concave function. Some examples are given to illustrate the results.

After some preliminaries and summarization of previous known results in Section 2, Section 3 deals with a Barnes-Godunova-Levin type inequality for the Sugeno integral, and some examples are given to illustrate the results. Finally, as special cases, some remarks are obtained.

2 Preliminaries

In this section, we recall some basic definitions or properties of a fuzzy integral and an (α, m)-concave function. For details, we refer the reader to Refs. [1, 7, 23].

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Suppose that $\wp$ is a σ-algebra of the subsets of X, and let $\mu : \wp \to [0,\infty)$ be a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if it satisfies:

(1) $\mu(\emptyset) = 0$;

(2) $E, F \in \wp$ and $E \subset F$ imply $\mu(E) \le \mu(F)$;

(3) $\{E_n\} \subset \wp$, $E_1 \subset E_2 \subset \cdots$, imply $\lim_{n\to\infty}\mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n)$;

(4) $\{E_n\} \subset \wp$, $E_1 \supset E_2 \supset \cdots$, $\mu(E_1) < \infty$, imply $\lim_{n\to\infty}\mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n)$.

Definition 2.1 (Mihesan [23]) The function $f : [0,b] \to R$ is said to be (α,m)-concave, where $(\alpha,m) \in [0,1]^2$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, it satisfies

$$f\bigl(tx + m(1-t)y\bigr) \ge t^{\alpha} f(x) + m\bigl(1 - t^{\alpha}\bigr) f(y). \tag{2.1}$$

Note that for $(\alpha,m) \in \{(0,0), (\alpha,0), (1,0), (1,m), (1,1), (\alpha,1)\}$ one obtains the following classes of functions: decreasing, α-starshaped, starshaped, m-concave, concave and α-concave.

If f is a non-negative real-valued function defined on X, we denote the set $\{x \in X : f(x) \ge \alpha\} = \{x \in X : f \ge \alpha\}$ by F_α for $\alpha \ge 0$. Note that if $\alpha \le \beta$ then $F_\beta \subset F_\alpha$.

Let $(X, \wp, \mu)$ be a fuzzy measure space, we denote by M^+ the set of all non-negative measurable functions with respect to $\wp$.

Definition 2.2 (Sugeno [1]) Let $(X, \wp, \mu)$ be a fuzzy measure space, $f \in M^+$ and $A \in \wp$. The Sugeno integral (or the fuzzy integral) of f on A, with respect to the fuzzy measure μ, is defined as

$$(S)\int_A f\,d\mu = \bigvee_{\alpha\ge 0}\bigl[\alpha \wedge \mu(A \cap F_\alpha)\bigr], \tag{2.2}$$

when $A = X$,

$$(S)\int_X f\,d\mu = \bigvee_{\alpha\ge 0}\bigl[\alpha \wedge \mu(F_\alpha)\bigr], \tag{2.3}$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0,\infty)$, respectively.

The properties of the Sugeno integral are well known and can be found in [7].

Proposition 2.3 *Let $(X, \wp, \mu)$ be a fuzzy measure space, $A, B \in \wp$ and $f, g \in M^+$ then*:

(1) $(S)\int_A f\,d\mu \le \mu(A)$;

(2) $(S)\int_A k\,d\mu = k \wedge \mu(A)$, *$k$ for a non-negative constant*;

(3) $(S)\int_A f\,d\mu \le (S)\int_A g\,d\mu$ *for $f \le g$*;

(4) $(S)\int_{A\cup B} f\,d\mu \ge (S)\int_A f\,d\mu \vee (S)\int_B f\,d\mu$;

(5) $\mu(A \cap \{f \ge \alpha\}) \ge \alpha \Rightarrow (S)\int_A f\,d\mu \ge \alpha$;

(6) $\mu(A \cap \{f \ge \alpha\}) \le \alpha \Rightarrow (S)\int_A f\,d\mu \le \alpha$;

(7) $(S)\int_A f\,d\mu > \alpha \Leftrightarrow$ *there exists $\gamma > \alpha$ such that $\mu(A \cap \{f \ge \gamma\}) > \alpha$*;

(8) $(S)\int_A f\,d\mu < \alpha \Leftrightarrow$ *there exists $\gamma < \alpha$ such that $\mu(A \cap \{f \ge \gamma\}) < \alpha$*.

Remark 2.4 Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \ge \alpha\})$. Then, due to (4) and (5) of Proposition 2.3, we have $F(\alpha) = \alpha \Rightarrow$

$(S)\int_A f\,d\mu = \alpha$. Thus, from a numerical point of view, the Sugeno integral can be calculated solving the equation $F(\alpha) = \alpha$.

Definition 2.5 Functions $f, g : X \to R$ are said to be co-monotone if for all $x, y \in X$,

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \geq 0, \tag{2.4}$$

and f and g are said to be counter-monotone if for all $x, y \in X$,

$$\big(f(x) - f(y)\big)\big(g(x) - g(y)\big) \leq 0. \tag{2.5}$$

It is clear that if f and g are co-monotone, then for any real numbers s, t either $F_s \subseteq G_t$ or $F_t \subseteq G_s$.

2.1 Barnes-Godunova-Levin type inequality for the Sugeno integral based on an (α, m)-concave function

The classical Barnes-Godunova-Levin type inequality provides the inequality

$$\left(\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}} \leq B(p,q) \int_a^b f(x)g(x)\,dx, \tag{2.6}$$

where $p, q > 1$, $B(p,q) = \frac{6(b-a)^{\frac{1}{p}+\frac{1}{q}-1}}{(1+p)^{\frac{1}{p}}(1+q)^{\frac{1}{q}}}$ and f, g are non-negative concave functions on $[a, b]$.

Unfortunately, the following example shows that the Barnes-Godunova-Levin type inequality for the Sugeno integral is not valid.

Example Consider $X = [0, 100]$ and $p = q = 4$. Let m be the Lebesgue measure on X. If we take the functions $f(x) = g(x) = \sqrt[4]{x}$, then $f(x)$, $g(x)$ are two $(\frac{1}{2}, \frac{1}{3})$-concave functions. In fact,

$$\sqrt[4]{x} = f\left(\frac{x}{100} \cdot 100 + \frac{1}{3}\left(1 - \frac{x}{100}\right) \cdot 0\right) \geq \sqrt{\frac{x}{100}} f(100) + \frac{1}{3}\left(1 - \sqrt{\frac{x}{100}}\right) f(0) = \sqrt{\frac{x}{10}}.$$

A straightforward calculus shows that

$$(S)\int_0^{100} f^4(x)\,dm = (S)\int_0^{100} g^4(x)\,dm = 50,$$

$$(S)\int_0^{100} f(x)g(x)\,dm = 9.5125, \qquad B(4,4) = 0.26833.$$

However,

$$\begin{aligned} 7.0711 &= \left((S)\int_0^{100} f^4(x)\,dm\right)^{\frac{1}{4}} \left((S)\int_0^{100} g^4(x)\,dm\right)^{\frac{1}{4}} \\ &\geq B(4,4)(S)\int_0^{100} f(x)g(x)\,dm = 2.5525. \end{aligned}$$

This proves that the Barnes-Godunova-Levin type inequality for the Sugeno integral is not satisfied.

The aim of this work is to show a Barnes-Godunova-Levin type inequality for the Sugeno integral with respect to an (α, m)-concave function.

Theorem 2.6 *Let $X = [0,1]$, $\alpha, m \in (0,1)$ and f, g be (α, m)-concave functions for all $x \in X$. If m is a Lebesgue measure on X, then*

Case (i). *If $f(0) \le f(1)$ and $g(0) \le g(1)$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.7}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case (ii). *If $f(0) > f(1)$ and $g(0) > g(1)$, then*

Case (a). *If $\frac{f(1)}{f(0)} < \frac{g(1)}{g(0)}$, then*

Case 1. *If $m \in (0, \frac{f(1)}{f(0)})$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.8}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. *If $m = \frac{f(1)}{f(0)}$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge 1 \wedge f(1) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.9}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. *If $m \in (\frac{f(1)}{f(0)}, \frac{g(1)}{g(0)})$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.10}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case 4. *If $m = \frac{g(1)}{g(0)}$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}} \wedge 1 \wedge g(1), \tag{2.11}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case 5. *If $m \in (\frac{g(1)}{g(0)}, 1)$, then*

$$(S)\int_0^1 f(x)g(x)\,dx \ge t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}, \tag{2.12}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case (b). *If* $\frac{f(1)}{f(0)} = \frac{g(1)}{g(0)}$, *then*
Case 1. *If* $m \in (0, \frac{f(1)}{f(0)})$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.13}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case 2. *If* $m = \frac{f(1)}{f(0)}$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge f(1) \wedge g(1) \wedge 1, \tag{2.14}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case 3. *If* $m \in (\frac{f(1)}{f(0)}, 1)$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}, \tag{2.15}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case (c). *If* $\frac{f(1)}{f(0)} > \frac{g(1)}{g(0)}$, *then*
Case 1. *If* $m \in (0, \frac{g(1)}{g(0)})$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right), \tag{2.16}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case 2. *If* $m = \frac{g(1)}{g(0)}$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge g(1) \wedge 1, \tag{2.17}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case 3. *If* $m \in (\frac{g(1)}{g(0)}, \frac{f(1)}{f(0)})$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}, \tag{2.18}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.
Case 4. *If* $m = \frac{f(1)}{f(0)}$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge f(1) \wedge \left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}, \tag{2.19}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Case 5. *If* $m \in (\frac{f(1)}{f(0)}, 1)$, *then*

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}, \tag{2.20}$$

where $t_1 = ((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}}$.

Proof Let $p, q \in (0, \infty)$, $((S)\int_0^1 f^p(x)\,dx)^{\frac{1}{p}} = t_1$ and $((S)\int_0^1 g^q(x)\,dx)^{\frac{1}{q}} = t_2$. Since $f, g : [0,1] \to [0,\infty)$ are two (α, m)-concave functions for $x \in [0,1]$, we have

$$f(x) = f\bigl(m(1-x)\cdot 0 + x\cdot 1\bigr) \geq m\bigl(1 - x^\alpha\bigr)f(0) + x^\alpha f(1) = h_1(x), \tag{2.21}$$

$$g(x) = g\bigl(m(1-x)\cdot 0 + x\cdot 1\bigr) \geq m\bigl(1 - x^\alpha\bigr)g(0) + x^\alpha g(1) = h_2(x). \tag{2.22}$$

Case (i). If $f(0) \leq f(1)$ and $g(0) \leq g(1)$, then by (3) of Proposition 2.3 and the co-monotonicity of $h_1(x)$ and $h_2(x)$, we have

$$\begin{aligned}
&(S)\int_0^1 f(x)g(x)\,dx \\
&\quad\geq (S)\int_0^1 h_1(x)h_2(x)\,dx \\
&\quad= \bigvee_{\beta\geq 0}\bigl(\beta \wedge \mu\bigl([0,1] \cap \bigl\{h_1(x)h_2(x) \geq \beta\bigr\}\bigr)\bigr) \\
&\quad\geq t_1t_2 \wedge \mu\bigl([0,1] \cap \bigl\{h_1(x)h_2(x) \geq t_1t_2\bigr\}\bigr) \\
&\quad\geq t_1t_2 \wedge \mu\bigl([0,1] \cap \bigl\{h_1(x) \geq t_1\bigr\} \cap \bigl\{h_2(x) \geq t_2\bigr\}\bigr) \\
&\quad= t_1t_2 \wedge \mu\bigl([0,1] \cap \bigl\{h_1(x) \geq t_1\bigr\}\bigr) \wedge \mu\bigl([0,1] \cap \bigl\{h_2(x) \geq t_2\bigr\}\bigr) \\
&\quad= t_1t_2 \wedge \mu\bigl([0,1] \cap \bigl\{h_1(x) \geq t_1\bigr\}\bigr) \wedge \mu\bigl([0,1] \cap \bigl\{h_2(x) \geq t_2\bigr\}\bigr) \\
&\quad= t_1t_2 \wedge \mu\bigl([0,1] \cap \bigl\{m\bigl(1 - x^\alpha\bigr)f(0) + x^\alpha f(1) \geq t_1\bigr\}\bigr) \\
&\qquad\wedge \mu\bigl([0,1] \cap \bigl\{m\bigl(1 - x^\alpha\bigr)g(0) + x^\alpha g(1) \geq t_2\bigr\}\bigr) \\
&\quad= t_1t_2 \wedge \mu\left([0,1] \cap \left\{x \geq \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right\}\right) \\
&\qquad\wedge \mu\left([0,1] \cap \left\{x \geq \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}\right\}\right) \\
&\quad= t_1t_2 \wedge \left(1 - \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(1 - \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}\right).
\end{aligned} \tag{2.23}$$

Case (ii). If $f(0) > f(1)$ and $g(0) > g(1)$, then by (3) of Proposition 2.3 and the co-monotonicity of $h_1(x)$ and $h_2(x)$, we have

$$\begin{aligned}
&(S)\int_0^1 f(x)g(x)\,dx \\
&\quad\geq (S)\int_0^1 h_1(x)h_2(x)\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\beta\geq 0}\bigl(\beta \wedge \mu\bigl([0,1]\cap\{h_1(x)h_2(x)\geq\beta\}\bigr)\bigr)\\
&\geq t_1t_2 \wedge \mu\bigl([0,1]\cap\{h_1(x)h_2(x)\geq t_1t_2\}\bigr)\\
&\geq t_1t_2 \wedge \mu\bigl([0,1]\cap\{h_1(x)\geq t_1\}\cap\{h_2(x)\geq t_2\}\bigr)\\
&= t_1t_2 \wedge \mu\bigl([0,1]\cap\{h_1(x)\geq t_1\}\bigr)\wedge\mu\bigl([0,1]\cap\{h_2(x)\geq t_2\}\bigr)\\
&= t_1t_2 \wedge \mu\bigl([0,1]\cap\{h_1(x)\geq t_1\}\bigr)\wedge\mu\bigl([0,1]\cap\{h_2(x)\geq t_2\}\bigr)\\
&= t_1t_2 \wedge \mu\bigl([0,1]\cap\{m(1-x^\alpha)f(0)+x^\alpha f(1)\geq t_1\}\bigr)\\
&\quad \wedge \mu\bigl([0,1]\cap\{m(1-x^\alpha)g(0)+x^\alpha g(1)\geq t_2\}\bigr)\\
&= t_1t_2 \wedge \mu\bigl([0,1]\cap\{mf(0)+(f(1)-mf(0))x^\alpha\geq t_1\}\bigr)\\
&\quad \wedge \mu\bigl([0,1]\cap\{mg(0)+(g(1)-mg(0))x^\alpha\geq t_2\}\bigr).
\end{aligned}
\tag{2.24}
$$

Case (a). If $\frac{f(1)}{f(0)} < \frac{g(1)}{g(0)}$, then by (2.24) we obtain
Case 1. If $m \in (0, \frac{f(1)}{f(0)})$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right)\wedge\left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right). \tag{2.25}$$

Case 2. If $m = \frac{f(1)}{f(0)}$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge 1 \wedge f(1) \wedge \left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right). \tag{2.26}$$

Case 3. If $m \in (\frac{f(1)}{f(0)}, \frac{g(1)}{g(0)})$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\wedge\left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right). \tag{2.27}$$

Case 4. If $m = \frac{g(1)}{g(0)}$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\wedge g(1)\wedge 1. \tag{2.28}$$

Case 5. If $m \in (\frac{g(1)}{g(0)}, 1)$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\wedge\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.29}$$

Case (b). If $\frac{f(1)}{f(0)} = \frac{g(1)}{g(0)}$, then by (2.24) we obtain
Case 1. If $m \in (0, \frac{f(1)}{f(0)})$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1-\left(\frac{t_1-mf(0)}{f(1)-mf(0)}\right)^{\frac{1}{\alpha}}\right)\wedge\left(1-\left(\frac{t_2-mg(0)}{g(1)-mg(0)}\right)^{\frac{1}{\alpha}}\right). \tag{2.30}$$

Case 2. If $m = \frac{f(1)}{f(0)}$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge f(1) \wedge g(1) \wedge 1. \tag{2.31}$$

Case 3. If $m \in (\frac{f(1)}{f(0)}, 1)$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.32}$$

Case (c). If $\frac{f(1)}{f(0)} > \frac{g(1)}{g(0)}$, then by (2.24) we obtain
Case 1. If $m \in (0, \frac{g(1)}{g(0)})$, then

$$\begin{aligned}(S)\int_0^1 f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left(1 - \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right) \\ &\wedge \left(1 - \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}\right).\end{aligned} \tag{2.33}$$

Case 2. If $m = \frac{g(1)}{g(0)}$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1 - \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge g(1) \wedge 1. \tag{2.34}$$

Case 3. If $m \in (\frac{g(1)}{g(0)}, \frac{f(1)}{f(0)})$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(1 - \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}}\right) \wedge \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.35}$$

Case 4. If $m = \frac{f(1)}{f(0)}$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge 1 \wedge f(1) \wedge \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.36}$$

Case 5. If $m \in (\frac{f(1)}{f(0)}, 1)$, then

$$(S)\int_0^1 f(x)g(x)\,dx \geq t_1t_2 \wedge \left(\frac{t_1 - mf(0)}{f(1) - mf(0)}\right)^{\frac{1}{\alpha}} \wedge \left(\frac{t_2 - mg(0)}{g(1) - mg(0)}\right)^{\frac{1}{\alpha}}. \tag{2.37}$$

This completes the proof. □

Example Consider $X = [0,1]$ and $p = 2$, $q = 4$. If we take the functions $f(x) = \sqrt[2]{x}$, $g(x) = \sqrt[4]{x}$, then $f(x)$, $g(x)$ are two $(\frac{2}{3}, \frac{1}{3})$ -concave functions. In fact, $\sqrt[t]{x} = f(x \cdot 1 + \frac{1}{3}(1-x) \cdot 0) \geq x^{\frac{2}{3}}f(1) + \frac{1}{3}(1 - x^{\frac{2}{3}})f(0) = x^{\frac{2}{3}}$ for $t \geq \frac{3}{2}$. Let m be the Lebesgue measure on X. A straightforward calculus shows that

$$(S)\int_0^1 f^2(x)\,dm = (S)\int_0^1 g^4(x)\,dm = 0.5, \qquad (S)\int_0^1 f(x)g(x)\,dm = 0.5497.$$

By Theorem 2.6, we have

$$\begin{aligned} 0.5497 = (S)\int_0^1 f(x)g(x)\,dx &\geq \left((S)\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}}\left((S)\int_0^1 g^4(x)\,dx\right)^{\frac{1}{4}} \\ &\quad \wedge \left(1-\left(\left((S)\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}}\right)^{\frac{3}{2}}\right) \wedge \left(1-\left(\left((S)\int_0^1 g^4(x)\,dx\right)^{\frac{1}{4}}\right)^{\frac{3}{2}}\right) \\ &= 0.0156 \wedge 0.8750 \wedge 0.9844 = 0.0156. \end{aligned} \tag{2.38}$$

Now, we will prove the general cases of Theorem 2.6.

Theorem 2.7 *Let $X = [a,b]$, $\alpha, m \in (0,1)$ and f, g be (α, m)-concave functions for all $x \in X$. If μ is a Lebesgue measure on X, then*

Case (i). *If $f(a) \leq f(b)$ and $g(a) \leq g(b)$, then*

$$\begin{aligned} (S)\int_a^b f(x)g(x)\,dx &\geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \\ &\quad \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \end{aligned} \tag{2.39}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (ii). *If $f(a) > f(b)$ and $g(a) > g(b)$, then*

Case (a). *If $\frac{f(b)}{f(a)} < \frac{g(b)}{g(a)}$, then*

Case 1. *If $m \in (0, \frac{f(b)}{f(a)})$, then*

$$\begin{aligned} (S)\int_a^b f(x)g(x)\,dx &\geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \\ &\quad \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \end{aligned} \tag{2.40}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. *If $m = \frac{f(b)}{f(a)}$, then*

$$\begin{aligned} (S)\int_a^b f(x)g(x)\,dx &\geq t_1t_2 \wedge (b-a) \wedge f(b) \\ &\quad \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \end{aligned} \tag{2.41}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. *If $m \in (\frac{f(b)}{f(a)}, \frac{g(b)}{g(a)})$, then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \tag{2.42}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 4. *If* $m = \frac{g(b)}{g(a)}$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge g(b) \wedge (b-a) \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.43}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 5. *If* $m \in (\frac{g(b)}{g(a)}, 1)$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.44}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (b). *If* $\frac{f(b)}{f(a)} = \frac{g(b)}{g(a)}$, *then*

Case 1. *If* $m \in (0, \frac{f(b)}{f(a)})$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \tag{2.45}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. *If* $m = \frac{f(b)}{f(a)}$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge f(b) \wedge g(b) \wedge (b-a), \tag{2.46}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. *If* $m \in (\frac{f(b)}{f(a)}, 1)$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.47}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (c). *If* $\frac{f(b)}{f(a)} > \frac{g(b)}{g(a)}$, *then*
Case 1. *If* $m \in (0, \frac{g(b)}{g(a)})$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right), \tag{2.48}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.
Case 2. *If* $m = \frac{g(b)}{g(a)}$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge g(b) \wedge (b-a), \tag{2.49}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.
Case 3. *If* $m \in (\frac{g(b)}{g(a)}, \frac{f(b)}{f(a)})$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.50}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.
Case 4. *If* $m = \frac{f(b)}{f(a)}$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge (b-a) \wedge f(b) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.51}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.
Case 5. *If* $m \in (\frac{f(b)}{f(a)}, 1)$, *then*

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right), \tag{2.52}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Proof Let $p,q \in (0,\infty)$, $((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}} = t_1$ and $((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}} = t_2$. Since $f,g : [a,b] \to [0,\infty)$ are two (α,m)-concave functions for $x \in [a,b]$, we have

$$\begin{aligned} f(x) &= f\left(m\left(1-\frac{x-ma}{b-ma}\right)\cdot a+\frac{x-ma}{b-ma}\cdot b\right) \\ &\geq m\left(1-\left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)f(a)+\left(\frac{x-ma}{b-ma}\right)^{\alpha}f(b)=h_1(x), \end{aligned} \tag{2.53}$$

$$\begin{aligned} g(x) &= g\left(m\left(1-\frac{x-ma}{b-ma}\right)\cdot a+\frac{x-ma}{b-ma}\cdot b\right) \\ &\geq m\left(1-\left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)g(a)+\left(\frac{x-ma}{b-ma}\right)^{\alpha}g(b)=h_2(x). \end{aligned} \tag{2.54}$$

Case (i). If $f(a) \leq f(b)$ and $g(a) \leq g(b)$, then by (3) of Proposition 2.3 and the comonotonicity of $h_1(x)$ and $h_2(x)$, we have

$$\begin{aligned} &(S)\int_a^b f(x)g(x)\,dx \\ &\quad\geq (S)\int_a^b h_1(x)h_2(x)\,dx \\ &\quad= \bigvee_{\beta\geq 0}\bigl(\beta\wedge\mu\bigl([a,b]\cap\{h_1(x)h_2(x)\geq\beta\}\bigr)\bigr) \\ &\quad\geq t_1t_2\wedge\mu\bigl([a,b]\cap\{h_1(x)h_2(x)\geq t_1t_2\}\bigr) \\ &\quad\geq t_1t_2\wedge\mu\bigl([a,b]\cap\{h_1(x)\geq t_1\}\cap\{h_2(x)\geq t_2\}\bigr) \\ &\quad= t_1t_2\wedge\mu\bigl([a,b]\cap\{h_1(x)\geq t_1\}\bigr)\wedge\mu\bigl([a,b]\cap\{h_2(x)\geq t_2\}\bigr) \\ &\quad= t_1t_2\wedge\mu\bigl([a,b]\cap\{h_1(x)\geq t_1\}\bigr)\wedge\mu\bigl([a,b]\cap\{h_2(x)\geq t_2\}\bigr) \\ &\quad= t_1t_2\wedge\mu\left([a,b]\cap\left\{m\left(1-\left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)f(a)+\left(\frac{x-ma}{b-ma}\right)^{\alpha}f(b)\geq t_1\right\}\right) \\ &\qquad\wedge\mu\left([a,b]\cap\left\{m\left(1-\left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)g(a)+\left(\frac{x-ma}{b-ma}\right)^{\alpha}g(b)\geq t_2\right\}\right) \\ &\quad= t_1t_2\wedge\mu\left([a,b]\cap\left\{x\geq\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}(b-ma)+ma\right\}\right) \\ &\qquad\wedge\mu\left([a,b]\cap\left\{x\geq\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}(b-ma)+ma\right\}\right) \\ &\quad= t_1t_2\wedge\left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \\ &\qquad\wedge\left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right). \end{aligned} \tag{2.55}$$

Case (ii). If $f(a) > f(b)$ and $g(a) > g(b)$, then by (3) of Proposition 2.3 and the comonotonicity of $h_1(x)$ and $h_2(x)$, we have

$$\begin{aligned} &(S)\int_a^b f(x)g(x)\,dx \\ &\quad\geq (S)\int_a^b h_1(x)h_2(x)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\beta \geq 0} \bigl(\beta \wedge \mu\bigl([a,b] \cap \{h_1(x)h_2(x) \geq \beta\}\bigr)\bigr) \\
&\geq t_1t_2 \wedge \mu\bigl([a,b] \cap \{h_1(x)h_2(x) \geq t_1t_2\}\bigr) \\
&\geq t_1t_2 \wedge \mu\bigl([a,b] \cap \{h_1(x) \geq t_1\} \cap \{h_2(x) \geq t_2\}\bigr) \\
&= t_1t_2 \wedge \mu\bigl([a,b] \cap \{h_1(x) \geq t_1\}\bigr) \wedge \mu\bigl([a,b] \cap \{h_2(x) \geq t_2\}\bigr) \\
&= t_1t_2 \wedge \mu\bigl([a,b] \cap \{h_1(x) \geq t_1\}\bigr) \wedge \mu\bigl([a,b] \cap \{h_2(x) \geq t_2\}\bigr) \\
&= t_1t_2 \wedge \mu\left([a,b] \cap \left\{m\left(1 - \left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)f(a) + \left(\frac{x-ma}{b-ma}\right)^{\alpha} f(b) \geq t_1\right\}\right) \\
&\quad \wedge \mu\left([a,b] \cap \left\{m\left(1 - \left(\frac{x-ma}{b-ma}\right)^{\alpha}\right)g(a) + \left(\frac{x-ma}{b-ma}\right)^{\alpha} g(b) \geq t_2\right\}\right) \\
&= t_1t_2 \wedge \mu\left([a,b] \cap \left\{mf(a) + \bigl(f(b) - mf(a)\bigr)\left(\frac{x-ma}{b-ma}\right)^{\alpha} \geq t_1\right\}\right) \\
&\quad \wedge \mu\left([a,b] \cap \left\{mg(a) + \bigl(g(b) - mg(a)\bigr)\left(\frac{x-ma}{b-ma}\right)^{\alpha} \geq t_2\right\}\right).
\end{aligned} \tag{2.56}
$$

Case (a). If $\frac{f(b)}{f(a)} < \frac{g(b)}{g(a)}$, then by (2.56) we obtain
Case 1. If $m \in (0, \frac{f(b)}{f(a)})$, then

$$
\begin{aligned}
(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left((b-ma)\left(1 - \left(\frac{t_1 - mf(a)}{f(b) - mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \\
&\wedge \left((b-ma)\left(1 - \left(\frac{t_2 - mg(a)}{g(b) - mg(a)}\right)^{\frac{1}{\alpha}}\right)\right).
\end{aligned} \tag{2.57}
$$

Case 2. If $m = \frac{f(b)}{f(a)}$, then

$$
\begin{aligned}
(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge (b-a) \wedge f(b) \\
&\wedge \left((b-ma)\left(1 - \left(\frac{t_2 - mg(a)}{g(b) - mg(a)}\right)^{\frac{1}{\alpha}}\right)\right).
\end{aligned} \tag{2.58}
$$

Case 3. If $m \in (\frac{f(b)}{f(a)}, \frac{g(b)}{g(a)})$, then

$$
\begin{aligned}
(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b) - mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \\
&\wedge \left((b-ma)\left(1 - \left(\frac{t_2 - mg(a)}{g(b) - mg(a)}\right)^{\frac{1}{\alpha}}\right)\right).
\end{aligned} \tag{2.59}
$$

Case 4. If $m = \frac{g(b)}{g(a)}$, then

$$
\begin{aligned}
(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge g(b) \wedge (b-a) \\
&\wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b) - mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right).
\end{aligned} \tag{2.60}
$$

Case 5. If $m \in (\frac{g(b)}{g(a)}, 1)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1 - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right). \tag{2.61}$$

Case (b). If $\frac{f(b)}{f(a)} = \frac{g(b)}{g(a)}$, then by (2.56) we obtain
Case 1. If $m \in (0, \frac{f(b)}{f(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right). \tag{2.62}$$

Case 2. If $m = \frac{f(b)}{f(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge f(b) \wedge g(b) \wedge (b-a). \tag{2.63}$$

Case 3. If $m \in (\frac{f(b)}{f(a)}, 1)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}} + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1 - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}} + ma - a\right). \tag{2.64}$$

Case (c). If $\frac{f(b)}{f(a)} > \frac{g(b)}{g(a)}$, then by (2.56) we obtain
Case 1. If $m \in (0, \frac{g(b)}{g(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}\right)\right). \tag{2.65}$$

Case 2. If $m = \frac{g(b)}{g(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge g(b) \wedge (b-a). \tag{2.66}$$

Case 3. If $m \in (\frac{g(b)}{g(a)}, \frac{f(b)}{f(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}+ma-a\right). \tag{2.67}$$

Case 4. If $m = \frac{f(b)}{f(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge f(b) \wedge (b-a) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}+ma-a\right). \tag{2.68}$$

Case 5. If $m \in (\frac{f(b)}{f(a)}, 1)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)^{\frac{1}{\alpha}}+ma-a\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)^{\frac{1}{\alpha}}+ma-a\right). \tag{2.69}$$

This completes the proof. □

Example Consider $X = [2,5]$ and $p = 2$, $q = 4$. Let m be the Lebesgue measure on X. If we take the function $f(x) = \sqrt[2]{6-x}$, $g(x) = \sqrt[4]{6-x}$, then $f(x)$, $g(x)$ are two $(\frac{1}{2}, \frac{\sqrt{5}}{2})$ -concave functions. In fact,

$$\begin{aligned}\sqrt{6-x} &= f\left(\frac{\sqrt{5}}{4}\cdot\left(1-\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)2+\left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)\cdot 5\right)\\ &\geq \left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)^{\frac{1}{2}}f(5)+\frac{\sqrt{5}}{4}\left(1-\left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)^{\frac{1}{2}}\right)f(2)\\ &= \frac{(2-\sqrt{5})x+4\sqrt{5}}{10-\sqrt{5}}\end{aligned} \tag{2.70}$$

and

$$\begin{aligned}\sqrt[4]{6-x} &= g\left(\frac{\sqrt{5}}{4}\cdot\left(1-\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)\times 2+\left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)\times 5\right)\\ &\geq \left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)^{\frac{1}{2}}g(5)+\frac{\sqrt{5}}{4}\left(1-\left(\frac{x-\frac{\sqrt{5}}{4}\times 2}{5-\frac{\sqrt{5}}{4}\times 2}\right)^{\frac{1}{2}}\right)g(2)\\ &= \frac{(4-\sqrt{10})x+5\sqrt{10}-5}{20-2\sqrt{5}}.\end{aligned} \tag{2.71}$$

A straightforward calculus shows that

$$(S)\int_2^5 f^2(x)\,dm = (S)\int_2^5 g^4(x)\,dm = 2, \qquad (S)\int_2^5 f(x)g(x)\,dm = 1.8040.$$

By Theorem 2.7, we have

$$\begin{aligned}
1.8040 = (S)\int_2^5 f(x)g(x)\,dx &\geq \left((S)\int_2^5 f^2(x)\,dx\right)^{\frac{1}{2}}\left((S)\int_2^5 g^4(x)\,dx\right)^{\frac{1}{4}} \\
&\wedge\left(\left(5-\frac{\sqrt{5}}{4}\times 2\right)\left(\frac{((S)\int_2^5 f^2(x)\,dx)^{\frac{1}{2}}-\frac{\sqrt{5}}{4}f(2)}{f(5)-\frac{\sqrt{5}}{4}f(2)}\right)^2+\frac{\sqrt{5}}{4}\times 2-2\right) \\
&\wedge\left(\left(5-\frac{\sqrt{5}}{4}\times 2\right)\left(\frac{((S)\int_2^5 g^4(x)\,dx)^{\frac{1}{4}}-\frac{\sqrt{5}}{4}g(2)}{g(5)-\frac{\sqrt{5}}{4}g(2)}\right)^2\right) \\
&= 1.6818 \wedge 23.5607 \wedge 34.4227 = 1.6818. \qquad (2.72)
\end{aligned}$$

As some special cases of (α, m)-concave functions in Theorem 2.7, we have the following results.

Remark 2.8 Let $X = [a, b]$, $\alpha = m = 0$ and f, g be two decreasing functions for all $x \in X$. If μ is a Lebesgue measure on X, then

$$(S)\int_a^b f(x)g(x)\,dx \geq \left((S)\int_a^b f^p(x)\,dx\right)^{\frac{1}{p}}\left((S)\int_a^b g^q(x)\,dx\right)^{\frac{1}{q}} \wedge f(b)\wedge g(b)\wedge(b-a). \qquad (2.73)$$

Remark 2.9 Let $X = [a, b]$, $\alpha = 1$, $m = 0$ and f, g be two starshaped functions for all $x \in X$. If μ is a Lebesgue measure on X, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge b\left(1-\frac{t_1}{f(b)}\right)\wedge b\left(1-\frac{t_2}{g(b)}\right), \qquad (2.74)$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Remark 2.10 Let $X = [a, b]$, $\alpha = 1$, $m \in (0, 1)$ and f, g be two m-concave functions for all $x \in X$. If μ is a Lebesgue measure on X, then

Case (i). If $f(a) \leq f(b)$ and $g(a) \leq g(b)$, then

$$\begin{aligned}
(S)\int_a^b f(x)g(x)\,dx &\geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)\right)\right) \\
&\wedge\left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)\right)\right), \qquad (2.75)
\end{aligned}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (ii). If $f(a) > f(b)$ and $g(a) > g(b)$, then

Case (a). If $\frac{f(b)}{f(a)} < \frac{g(b)}{g(a)}$, then

Case 1. If $m \in (0, \frac{f(b)}{f(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)\right)\right), \tag{2.76}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. If $m = \frac{f(b)}{f(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge (b-a) \wedge f(b) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)\right)\right), \tag{2.77}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. If $m \in (\frac{f(b)}{f(a)}, \frac{g(b)}{g(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right) + ma - a\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)\right)\right), \tag{2.78}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 4. If $m = \frac{g(b)}{g(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge g(b) \wedge (b-a) \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right) + ma - a\right), \tag{2.79}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 5. If $m \in (\frac{g(b)}{g(a)}, 1)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right) + ma - a\right) \wedge \left((b-ma)\left(\frac{t_1 - mg(a)}{g(b)-mg(a)}\right) + ma - a\right), \tag{2.80}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (b). If $\frac{f(b)}{f(a)} = \frac{g(b)}{g(a)}$, then

Case 1. If $m \in (0, \frac{f(b)}{f(a)})$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1 t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1 - mf(a)}{f(b)-mf(a)}\right)\right)\right) \wedge \left((b-ma)\left(1-\left(\frac{t_2 - mg(a)}{g(b)-mg(a)}\right)\right)\right), \tag{2.81}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. If $m = \frac{f(b)}{f(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge f(b) \wedge g(b) \wedge (b-a), \tag{2.82}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. If $m \in (\frac{f(b)}{f(a)}, 1)$, then

$$\begin{aligned}(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)+ma-a\right)\\ &\wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)+ma-a\right),\end{aligned} \tag{2.83}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (c). If $\frac{f(b)}{f(a)} > \frac{g(b)}{g(a)}$, then

Case 1. If $m \in (0, \frac{g(b)}{g(a)})$, then

$$\begin{aligned}(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)\right)\right)\\ &\wedge \left((b-ma)\left(1-\left(\frac{t_2-mg(a)}{g(b)-mg(a)}\right)\right)\right),\end{aligned} \tag{2.84}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 2. If $m = \frac{g(b)}{g(a)}$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)\right)\right) \wedge g(b) \wedge (b-a), \tag{2.85}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 3. If $m \in (\frac{g(b)}{g(a)}, \frac{f(b)}{f(a)})$, then

$$\begin{aligned}(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge \left((b-ma)\left(1-\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)\right)\right)\\ &\wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)+ma-a\right),\end{aligned} \tag{2.86}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 4. If $m = \frac{f(b)}{f(a)}$, then

$$\begin{aligned}(S)\int_a^b f(x)g(x)\,dx \geq{}& t_1t_2 \wedge (b-a) \wedge f(b)\\ &\wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)+ma-a\right),\end{aligned} \tag{2.87}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case 5. If $m \in (\frac{f(b)}{f(a)}, 1)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-ma)\left(\frac{t_1-mf(a)}{f(b)-mf(a)}\right)+ma-a\right) \wedge \left((b-ma)\left(\frac{t_1-mg(a)}{g(b)-mg(a)}\right)+ma-a\right), \tag{2.88}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Remark 2.11 [22] Let $X = [a,b]$, $\alpha = 1$, $m = 1$ and f, g be two concave functions for all $x \in X$. If μ is a Lebesgue measure on X, then

Case (i). If $f(a) < f(b)$ and $g(a) < g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-a)\left(1-\left(\frac{t_1-f(a)}{f(b)-f(a)}\right)\right)\right) \wedge \left((b-a)\left(1-\left(\frac{t_2-g(a)}{g(b)-g(a)}\right)\right)\right), \tag{2.89}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (ii). If $f(a) = f(b)$ and $g(a) = g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq f(a)g(a) \wedge (b-a). \tag{2.90}$$

Case (iii). If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-a)\left(\frac{t_1-f(a)}{f(b)-f(a)}\right)\right) \wedge \left((b-a)\left(\frac{t_1-g(a)}{g(b)-g(a)}\right)\right), \tag{2.91}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Remark 2.12 Let $X = [a,b]$, $\alpha \in (0,1)$, $m = 0$ and f, g be two α-starshaped functions for all $x \in X$. If μ is a Lebesgue measure on X, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge b\left(1-\left(\frac{t_1}{f(b)}\right)^{\frac{1}{\alpha}}\right) \wedge b\left(1-\left(\frac{t_2}{g(b)}\right)^{\frac{1}{\alpha}}\right), \tag{2.92}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Remark 2.13 Let $X = [a,b]$, $\alpha \in (0,1)$, $m = 1$ and f, g be two α-concave functions for all $x \in X$. If μ is a Lebesgue measure on X, then

Case (i). If $f(a) < f(b)$ and $g(a) < g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-a)\left(1-\left(\frac{t_1-f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}\right)\right) \wedge \left((b-a)\left(1-\left(\frac{t_2-g(a)}{g(b)-g(a)}\right)^{\frac{1}{\alpha}}\right)\right), \tag{2.93}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

Case (ii). If $f(a) = f(b)$ and $g(a) = g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq f(a)g(a) \wedge (b-a). \tag{2.94}$$

Case (iii). If $f(a) > f(b)$ and $g(a) > g(b)$, then

$$(S)\int_a^b f(x)g(x)\,dx \geq t_1t_2 \wedge \left((b-a)\left(\frac{t_1 - f(a)}{f(b)-f(a)}\right)^{\frac{1}{\alpha}}\right) \wedge \left((b-a)\left(\frac{t_1 - g(a)}{g(b)-g(a)}\right)^{\frac{1}{\alpha}}\right), \tag{2.95}$$

where $t_1 = ((S)\int_a^b f^p(x)\,dx)^{\frac{1}{p}}$, $t_2 = ((S)\int_a^b g^q(x)\,dx)^{\frac{1}{q}}$.

3 Conclusion

In this paper, we have investigated the Barnes-Godunova-Levin type inequality of the Sugeno integral with respect to an (α, m)-concave function. For further investigations, we will continue to explore other integral inequalities for the Sugeno integral related to (α, m)-concavity.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper and they read and approved the final manuscript.

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