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New inequalities for the Hadamard product of an M-matrix and its inverse

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Abstract

For the Hadamard product $A \circ A^{-1}$ of an M-matrix A and its inverse A^{-1}, some new inequalities for the minimum eigenvalue of $A \circ A^{-1}$ are derived. Numerical example is given to show that the inequalities are better than some known results.

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1 Introduction

The set of all $n \times n$ real matrices is denoted by $\mathbb{R}^{n\times n}$, and $\mathbb{C}^{n\times n}$ denotes the set of all $n \times n$ complex matrices.

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an M-matrix [1] if there exists a nonnegative matrix B and a nonnegative real number λ such that

$$A = \lambda I - B, \quad \lambda \geq \rho(B),$$

where I is an identity matrix, $\rho(B)$ is a spectral radius of the matrix B. If $\lambda = \rho(B)$, then A is a singular M-matrix; if $\lambda > \rho(B)$, then A is called a nonsingular M-matrix. Denote by M_n the set of all $n \times n$ nonsingular M-matrices. Let us denote

$$\tau(A) = \min\{\operatorname{Re}(\lambda) : \lambda \in \sigma(A)\},$$

and $\sigma(A)$ denotes the spectrum of A. It is known that [2] $\tau(A) = \frac{1}{\rho(A^{-1})}$ is a positive real eigenvalue of $A \in M_n$.

The Hadamard product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is the matrix $A \circ B = (a_{ij}b_{ij})$. If A and B are M-matrices, then it is proved in [3] that $A \circ B^{-1}$ is also an M-matrix.

A matrix A is irreducible if there does not exist any permutation matrix P such that

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square matrices.

For convenience, for any positive integer n, N denotes the set $\{1, 2, \dots, n\}$. Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant by row, for any $i \in N$, denote

$$R_i = \sum_{k\neq i} |a_{ik}|, \qquad C_i = \sum_{k\neq i} |a_{ki}|, \qquad d_i = \frac{R_i}{|a_{ii}|}, \qquad c_i = \frac{C_i}{|a_{ii}|}, \quad i \in N;$$

$$s_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| d_k}{|a_{jj}|}, \quad j \neq i, j \in N; \qquad s_i = \max_{j\neq i}\{s_{ij}\}, \quad i \in N;$$

$$m_{ji} = \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{|a_{jj}|}, \quad j \neq i, j \in N; \qquad m_i = \max_{j\neq i}\{m_{ij}\}, \quad i \in N.$$

Recently, some lower bounds for the minimum eigenvalue of the Hadamard product of an M-matrix and its inverse have been proposed. Let $A \in M_n$, it was proved in [4] that

$$0 < \tau\left(A \circ A^{-1}\right) \leq 1.$$

Subsequently, Fiedler and Markham [3] gave a lower bound on $\tau(A \circ A^{-1})$,

$$\tau\left(A \circ A^{-1}\right) \geq \frac{1}{n},$$

and conjectured that

$$\tau\left(A \circ A^{-1}\right) \geq \frac{2}{n}.$$

Chen [5], Song [6] and Yong [7] have independently proved this conjecture.

In [8], Li *et al.* gave the following result:

$$\tau\left(A \circ A^{-1}\right) \geq \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}} \right\}.$$

Furthermore, if $a_{11} = a_{22} = \cdots = a_{nn}$, they have obtained

$$\min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}} \right\} \geq \frac{2}{n}.$$

In this paper, we present some new lower bounds for $\tau(A \circ A^{-1})$. These bounds improve the results in [8–11].

2 Preliminaries and notations

In this section, we give some lemmas that involve inequalities for the entries of A^{-1}. They will be useful in the following proofs.

Lemma 2.1 [7] *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a strictly row diagonally dominant matrix, that is,*

$$|a_{ii}| > \sum_{j\neq i} |a_{ij}|, \quad i \in N,$$

then $A^{-1} = (b_{ij})$ exists, and

$$|b_{ji}| \le \frac{\sum_{k\neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}|, \quad j \neq i.$$

Lemma 2.2 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly diagonally dominant M-matrix by row. Then, for $A^{-1} = (b_{ij})$, we have*

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{a_{jj}} b_{ii} \le m_j b_{ii}, \quad j \neq i, i \in N.$$

Proof For $i \in N$, let

$$d_k(\varepsilon) = \frac{\sum_{l\neq k} |a_{kl}| + \varepsilon}{a_{kk}},$$

and

$$s_{ji}(\varepsilon) = \frac{|a_{ji}| + (\sum_{k\neq j,i} |a_{jk}| + \varepsilon) d_k(\varepsilon)}{|a_{jj}|}, \quad j \neq i.$$

Since A is strictly diagonally dominant, then $0 < d_k < 1$ and $0 < s_{ji} < 1$. Therefore, there exists $\varepsilon > 0$ such that $0 < d_k(\varepsilon) < 1$ and $0 < s_{ji}(\varepsilon) < 1$. For any $i \in N$, let

$$S_i(\varepsilon) = \operatorname{diag}\bigl(s_{1i}(\varepsilon), \ldots, s_{i-1,i}(\varepsilon), 1, s_{i+1,i}(\varepsilon), \ldots, s_{ni}(\varepsilon)\bigr).$$

Obviously, the matrix $AS_i(\varepsilon)$ is also a strictly diagonally dominant M-matrix by row. Therefore, by Lemma 2.1, we derive the following inequality:

$$\frac{b_{ji}}{s_{ji}(\varepsilon)} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}(\varepsilon)}{s_{ji}(\varepsilon) a_{jj}} b_{ii}, \quad j \neq i, j \in N,$$

i.e.,

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}(\varepsilon)}{a_{jj}} b_{ii}, \quad j \neq i, j \in N.$$

Let $\varepsilon \longrightarrow 0$ to obtain

$$|b_{ji}| \le \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{a_{jj}} b_{ii} \le m_j b_{ii}, \quad j \neq i, i \in N.$$

This proof is completed. □

Lemma 2.3 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a strictly row diagonally dominant M-matrix. Then, for $A^{-1} = (b_{ij})$, we have*

$$\frac{1}{a_{ii} - \sum_{j\neq i} |a_{ij}| m_{ji}} \ge b_{ii} \ge \frac{1}{a_{ii}}, \quad i \in N.$$

Proof Let $B = A^{-1}$. Since A is an M-matrix, then $B \geq 0$. By $AB = I$, we have

$$1 = \sum_{j=1}^{n} a_{ij} b_{ji} = a_{ii} b_{ii} - \sum_{j \neq i} |a_{ij}| b_{ji}, \quad i \in N.$$

Hence

$$a_{ii} b_{ii} \geq 1, \quad i \in N,$$

that is,

$$b_{ii} \geq \frac{1}{a_{ii}}, \quad i \in N.$$

By Lemma 2.2, we have

$$\begin{aligned} 1 &= a_{ii} b_{ii} - \sum_{j \neq i} |a_{ij}| b_{ji} \\ &\geq a_{ii} b_{ii} - \sum_{j \neq i} |a_{ij}| \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| s_{ki}}{a_{jj}} b_{ii} \\ &= \left(a_{ii} - \sum_{j \neq i} |a_{ij}| m_{ji} \right) b_{ii}, \end{aligned}$$

i.e.,

$$\frac{1}{a_{ii} - \sum_{j \neq i} |a_{ij}| m_{ji}} \geq b_{ii}, \quad i \in N.$$

Thus the proof is completed. □

Lemma 2.4 [12] *If A^{-1} is a doubly stochastic matrix, then $Ae = e$, $A^T e = e$, where $e = (1, 1, \ldots, 1)^T$.*

Lemma 2.5 [13] *Let $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ and $x_1, x_2, \ldots, x_n$ be positive real numbers. Then all the eigenvalues of A lie in the region*

$$\bigcup_{\substack{i,j=1 \\ i \neq j}}^{n} \left\{ z \in C : |z - a_{ii}||z - a_{jj}| \leq \left(x_i \sum_{k \neq i} \frac{1}{x_k} |a_{ki}| \right) \left(x_j \sum_{k \neq j} \frac{1}{x_k} |a_{kj}| \right) \right\}.$$

Lemma 2.6 [3] *If P is an irreducible M-matrix, and $Pz \geq kz$ for a nonnegative nonzero vector z, then $\tau(P) \geq k$.*

3 Main results

In this section, we give two new lower bounds for $\tau(A \circ A^{-1})$ which improve some previous results.

Theorem 3.1 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix, and suppose that $A^{-1} = (b_{ij})$ is doubly stochastic. Then*

$$b_{ii} \geq \frac{1}{1 + \sum_{j\neq i} m_{ji}}, \quad i \in N.$$

Proof Since A^{-1} is doubly stochastic and A is an M-matrix, by Lemma 2.4, we have

$$a_{ii} = \sum_{k\neq i} |a_{ik}| + 1 = \sum_{k\neq i} |a_{ki}| + 1, \quad i \in N,$$

and

$$b_{ii} + \sum_{j\neq i} b_{ji} = 1, \quad i \in N.$$

The matrix A is strictly diagonally dominant by row. Then, by Lemma 2.2, for $i \in N$, we have

$$\begin{aligned}
1 = b_{ii} + \sum_{j\neq i} b_{ji} &\leq b_{ii} + \sum_{j\neq i} \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{a_{jj}} b_{ii} \\
&= \left(1 + \sum_{j\neq i} \frac{|a_{ji}| + \sum_{k\neq j,i} |a_{jk}| s_{ki}}{a_{jj}}\right) b_{ii} \\
&= \left(1 + \sum_{j\neq i} m_{ji}\right) b_{ii},
\end{aligned}$$

i.e.,

$$b_{ii} \geq \frac{1}{1 + \sum_{j\neq i} m_{ji}}, \quad i \in N.$$

This proof is completed. □

Theorem 3.2 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix, and let $A^{-1} = (b_{ij})$ be doubly stochastic. Then*

$$\begin{aligned}
\tau\left(A \circ A^{-1}\right) \geq \min_{i\neq j} \frac{1}{2} \Bigg\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
& + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\left(m_j \sum_{k\neq j} |a_{kj}| b_{jj}\right)\Bigg]^{\frac{1}{2}} \Bigg\}. \qquad (3.1)
\end{aligned}$$

Proof It is evident that (3.1) is an equality for $n = 1$.

We next assume that $n \geq 2$.

Firstly, we assume that A^{-1} is irreducible. By Lemma 2.4, we have

$$a_{ii} = \sum_{j\neq i} |a_{ij}| + 1 = \sum_{j\neq i} |a_{ji}| + 1, \quad i \in N,$$

and

$$a_{ii} > 1, \quad i \in N.$$

Let

$$m_j = \max_{i \neq j}\{m_{ji}\} = \max_{i \neq j}\left\{\frac{|a_{ji}| + \sum_{k \neq j,i}|a_{jk}|s_{ki}}{a_{jj}}\right\}, \quad j \in N.$$

Since A is an irreducible matrix, then $0 < m_j \leq 1$. Let $\tau(A \circ A^{-1}) = \lambda$, so that $0 < \lambda < a_{ii}b_{ii}$, $i \in N$. Thus, by Lemma 2.5, there is a pair (i,j) of positive integers with $i \neq j$ such that

$$\begin{aligned}
|\lambda - a_{ii}b_{ii}||\lambda - a_{jj}b_{jj}| &\leq \left(m_i \sum_{k \neq i} \frac{1}{m_k}|a_{ki}b_{ki}|\right)\left(m_j \sum_{k \neq j} \frac{1}{m_k}|a_{kj}b_{kj}|\right) \\
&\leq \left(m_i \sum_{k \neq i} \frac{1}{m_k}|a_{ki}|m_k b_{ii}\right)\left(m_j \sum_{k \neq i} \frac{1}{m_k}|a_{kj}|m_k b_{jj}\right) \\
&= \left(m_i \sum_{k \neq i}|a_{ki}|b_{ii}\right)\left(m_j \sum_{k \neq j}|a_{kj}|b_{jj}\right). \qquad (3.2)
\end{aligned}$$

From inequality (3.2), we have

$$(\lambda - a_{ii}b_{ii})(\lambda - a_{jj}b_{jj}) \leq \left(m_i \sum_{k \neq i}|a_{ki}|b_{ii}\right)\left(m_j \sum_{k \neq j}|a_{kj}|b_{jj}\right). \qquad (3.3)$$

Thus, (3.3) is equivalent to

$$\begin{aligned}
\lambda \geq \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
+ 4\left(m_i \sum_{k \neq i}|a_{ki}|b_{ii}\right)\left(m_j \sum_{k \neq j}|a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\},
\end{aligned}$$

that is,

$$\begin{aligned}
\tau\left(A \circ A^{-1}\right) &\geq \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
&\quad + 4\left(m_i \sum_{k \neq i}|a_{ki}|b_{ii}\right)\left(m_j \sum_{k \neq j}|a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\} \\
&\geq \min_{i \neq j} \frac{1}{2}\Bigg\{a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\
&\quad + 4\left(m_i \sum_{k \neq i}|a_{ki}|b_{ii}\right)\left(m_j \sum_{k \neq j}|a_{kj}|b_{jj}\right)\Bigg]^{\frac{1}{2}}\Bigg\}.
\end{aligned}$$

If A is reducible, without loss of generality, we may assume that A has the following block upper triangular form:

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ & A_{22} & \cdots & A_{2s} \\ & & \cdots & \cdots \\ & & & A_{ss} \end{bmatrix}$$

with irreducible diagonal blocks A_{ii}, $i = 1, 2, \ldots, s$. Obviously, $\tau(A \circ A^{-1}) = \min_i \tau(A_{ii} \circ A_{ii}^{-1})$. Thus, the problem of the reducible matrix A is reduced to those of irreducible diagonal blocks A_{ii}. The result of Theorem 3.2 also holds. □

Theorem 3.3 *Let $A = (a_{ij}) \in M_n$ and $A^{-1} = b_{ij}$ be a doubly stochastic matrix. Then*

$$\begin{aligned} &\min_{i \neq j} \frac{1}{2} \Biggl\{ a_{ii}b_{ii} + a_{jj}b_{jj} - \Biggl[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ &\quad + 4\Biggl(m_i \sum_{k \neq i} |a_{ki}| b_{ii} \Biggr)\Biggl(m_j \sum_{k \neq j} |a_{kj}| b_{jj} \Biggr)\Biggr]^{\frac{1}{2}} \Biggr\} \\ &\quad \geq \min_i \biggl\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \neq i} s_{ji}} \biggr\}. \end{aligned}$$

Proof Since A^{-1} is a doubly stochastic matrix, by Lemma 2.4, we have

$$a_{ii} = \sum_{k \neq i} |a_{ik}| + 1 = \sum_{k \neq i} |a_{ki}| + 1, \quad i \in N.$$

For any $j \neq i$, we have

$$\begin{aligned} d_j - s_{ji} &= \frac{R_j}{a_{jj}} - \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{a_{jj}} \\ &= \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}|}{a_{jj}} - \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{a_{jj}} \\ &= \frac{(1 - d_k) \sum_{k \neq j,i} |a_{jk}|}{a_{jj}} \geq 0, \end{aligned}$$

or equivalently

$$d_j \geq s_{ji}, \quad j \neq i, j \in N. \tag{3.4}$$

So, we can obtain

$$m_{ji} = \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| s_{ki}}{a_{jj}} \leq \frac{|a_{ji}| + \sum_{k \neq j,i} |a_{jk}| d_k}{a_{jj}} = s_{ji}, \quad j \neq i, j \in N, \tag{3.5}$$

and

$$m_i \leq s_i, \quad i \in N.$$

Without loss of generality, for $i \neq j$, assume that

$$a_{ii}b_{ii} - m_i \sum_{k\neq i} |a_{ki}| b_{ii} \leq a_{jj}b_{jj} - m_j \sum_{k\neq j} |a_{kj}| b_{jj}. \tag{3.6}$$

Thus, (3.6) is equivalent to

$$m_j \sum_{k\neq j} |a_{kj}| b_{jj} \leq a_{jj}b_{jj} - a_{ii}b_{ii} + m_i \sum_{k\neq i} |a_{ki}| b_{ii}. \tag{3.7}$$

From (3.1) and (3.7), we have

$$\begin{aligned}
&\frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\left(m_j \sum_{k\neq j} |a_{kj}| b_{jj}\right)\right]^{\frac{1}{2}}\right\} \\
&\quad\geq \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2\right.\right. \\
&\qquad \left.\left. + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\left(a_{jj}b_{jj} - a_{ii}b_{ii} + m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\right]^{\frac{1}{2}}\right\} \\
&\quad= \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2\right.\right. \\
&\qquad \left.\left. + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)^2 + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)(a_{jj}b_{jj} - a_{ii}b_{ii})\right]^{\frac{1}{2}}\right\} \\
&\quad= \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[\left(a_{jj}b_{jj} - a_{ii}b_{ii} + 2m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)^2\right]^{\frac{1}{2}}\right\} \\
&\quad= \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left(a_{jj}b_{jj} - a_{ii}b_{ii} + 2m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\right\} \\
&\quad= a_{ii}b_{ii} - m_i \sum_{k\neq i} |a_{ki}| b_{ii} \\
&\quad= b_{ii}\left(a_{ii} - m_i \sum_{k\neq i} |a_{ki}|\right) \\
&\quad\geq \frac{a_{ii} - m_i R_i}{1 + \sum_{j\neq i} m_{ji}} \\
&\quad\geq \frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\min_{i\neq j} \frac{1}{2}\left\{a_{ii}b_{ii} + a_{jj}b_{jj} - \left[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 + 4\left(m_i \sum_{k\neq i} |a_{ki}| b_{ii}\right)\left(m_j \sum_{k\neq j} |a_{kj}| b_{jj}\right)\right]^{\frac{1}{2}}\right\} \\
&\quad\geq \min_i \left\{\frac{a_{ii} - s_i R_i}{1 + \sum_{j\neq i} s_{ji}}\right\}.
\end{aligned}$$

This proof is completed. □

Remark 3.1 According to inequality (3.4), it is easy to know that

$$b_{ji} \le \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}| s_{ki}}{a_{jj}} b_{ii} \le \frac{|a_{ji}| + \sum_{k \ne j,i} |a_{jk}| d_k}{a_{jj}} b_{ii}, \quad j \in N.$$

That is to say, the result of Lemma 2.2 is sharper than that of Theorem 2.1 in [8]. Moreover, the result of Theorem 3.2 is sharper than that of Theorem 3.1 in [8], respectively.

Theorem 3.4 *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an irreducible strictly row diagonally dominant M-matrix. Then*

$$\tau\left(A \circ A^{-1}\right) \ge \min_i \left\{1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| m_{ji}\right\}.$$

Proof Since A is irreducible, then $A^{-1} > 0$, and $A \circ A^{-1}$ is again irreducible. Note that

$$\tau\left(A \circ A^{-1}\right) = \tau\left(\left(A \circ A^{-1}\right)^T\right) = \tau\left(A^T \circ \left(A^T\right)^{-1}\right).$$

Let

$$\left(A^T \circ \left(A^T\right)^{-1}\right) e = (t_1, t_2, \ldots, t_n)^T,$$

where $e = (1,1,\ldots,1)^T$. Without loss of generality, we may assume that $t_1 = \min_i\{t_i\}$, by Lemma 2.2, we have

$$\begin{aligned} t_1 &= \sum_{j=1}^{n} |a_{j1} b_{j1}| = a_{11} b_{11} - \sum_{j \ne 1} |a_{j1}| b_{j1} \\ &\ge a_{11} b_{11} - \sum_{j \ne 1} |a_{j1}| \frac{|a_{j1}| + \sum_{k \ne j,1} |a_{jk}| s_{k1}}{a_{jj}} b_{11} \\ &= a_{11} b_{11} - \sum_{j \ne 1} |a_{j1}| m_{j1} b_{11} \\ &= \left(a_{11} - \sum_{j \ne 1} |a_{j1}| m_{j1}\right) b_{11} \\ &\ge \frac{a_{11} - \sum_{j \ne 1} |a_{j1}| m_{j1}}{a_{11}} \\ &= 1 - \frac{1}{a_{11}} \sum_{j \ne 1} |a_{j1}| m_{j1}. \end{aligned}$$

Therefore, by Lemma 2.6, we have

$$\tau\left(A \circ A^{-1}\right) \ge \min_i \left\{1 - \frac{1}{a_{ii}} \sum_{j \ne i} |a_{ji}| m_{ji}\right\}.$$

This proof is completed. □

Remark 3.2 According to inequality (3.5), we can get

$$1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| m_{ji} \geq 1 - \frac{1}{a_{ii}} \sum_{j \neq i} |a_{ji}| s_{ji}.$$

That is to say, the bound of Theorem 3.4 is sharper than the bound of Theorem 3.5 in [8].

Remark 3.3 If A is an M-matrix, we know that there exists a diagonal matrix D with positive diagonal entries such that $D^{-1}AD$ is a strictly row diagonally dominant M-matrix. So the result of Theorem 3.4 also holds for a general M-matrix.

4 Example

Consider the following M-matrix:

$$A = \begin{bmatrix} 4 & -1 & -1 & -1 \\ -2 & 5 & -1 & -1 \\ 0 & -2 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}.$$

Since $Ae = e$ and $A^T e = e$, A^{-1} is doubly stochastic. By calculations we have

$$A^{-1} = \begin{bmatrix} 0.4000 & 0.2000 & 0.2000 & 0.2000 \\ 0.2333 & 0.3667 & 0.2000 & 0.2000 \\ 0.1667 & 0.2333 & 0.4000 & 0.2000 \\ 0.2000 & 0.2000 & 0.2000 & 0.4000 \end{bmatrix}.$$

(1) Estimate the upper bounds for entries of $A^{-1} = (b_{ij})$. If we apply Theorem 2.1(a) of [8], we have

$$A^{-1} \leq \begin{bmatrix} 1 & 0.6250 & 0.6375 & 0.6375 \\ 0.7000 & 1 & 0.6500 & 0.6500 \\ 0.5875 & 0.6875 & 1 & 0.6500 \\ 0.6375 & 0.6250 & 0.6375 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

If we apply Lemma 2.2, we have

$$A^{-1} \leq \begin{bmatrix} 1 & 0.5781 & 0.5718 & 0.5750 \\ 0.6450 & 1 & 0.5825 & 0.5850 \\ 0.5093 & 0.6562 & 1 & 0.5750 \\ 0.5718 & 0.5781 & 0.5718 & 1 \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \\ b_{11} & b_{22} & b_{33} & b_{44} \end{bmatrix}.$$

Combining the result of Lemma 2.2 with the result of Theorem 2.1(a) of [8], we see that the result of Lemma 2.2 is the best.

By Theorem 2.3 and Lemma 3.2 of [8], we can get the following bounds for the diagonal entries of A^{-1}:

$$0.3419 \leq b_{11} \leq 0.5882; \qquad 0.3404 \leq b_{22} \leq 0.5128,$$

$$0.3419 \leq b_{33} \leq 0.6061; \qquad 0.3404 \leq b_{44} \leq 0.5882.$$

By Lemma 2.3 and Theorem 3.1, we obtain

$$0.3668 \le b_{11} \le 0.4397; \qquad 0.3556 \le b_{22} \le 0.3832,$$
$$0.3668 \le b_{33} \le 0.4419; \qquad 0.3656 \le b_{44} \le 0.4415.$$

(2) Lower bounds for $\tau(A \circ A^{-1})$.

By the conjecture of Fiedler and Markham, we have

$$\tau\left(A \circ A^{-1}\right) \ge \frac{2}{n} = \frac{1}{2} = 0.5.$$

By Theorem 3.1 of [8], we have

$$\tau\left(A \circ A^{-1}\right) \ge \min_i \left\{ \frac{a_{ii} - s_i R_i}{1 + \sum_{j \ne i} s_{ji}} \right\} = 0.6624.$$

By Corollary 2.5 of [9], we have

$$\tau\left(A \circ A^{-1}\right) \ge 1 - \rho^2(J_A) = 0.4145.$$

By Theorem 3.1 of [10], we have

$$\tau\left(A \circ A^{-1}\right) \ge \min_i \left\{ \frac{a_{ii} - u_i R_i}{1 + \sum_{j \ne i} u_{ji}} \right\} = 0.8250.$$

By Corollary 2 of [11], we have

$$\tau\left(A \circ A^{-1}\right) \ge \min_i \left\{ \frac{a_{ii} - w_i \sum_{j \ne i} | a_{ji} |}{1 + \sum_{j \ne i} w_{ji}} \right\} = 0.8321.$$

If we apply Theorem 3.2, we have

$$\begin{aligned} \tau\left(A \circ A^{-1}\right) \ge \min_{i \ne j} \frac{1}{2} \Bigg\{ & a_{ii}b_{ii} + a_{jj}b_{jj} - \Bigg[(a_{ii}b_{ii} - a_{jj}b_{jj})^2 \\ & + 4\Bigg(m_i \sum_{k \ne i} |a_{ki}| b_{ii} \Bigg) \Bigg(m_j \sum_{k \ne j} |a_{kj}| b_{jj} \Bigg) \Bigg]^{\frac{1}{2}} \Bigg\} = 0.8456. \end{aligned}$$

The numerical example shows that the bound of Theorem 3.2 is better than these corresponding bounds in [8–11].

Competing interests
The author declares that he has no competing interests.

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