Wu et al. *Journal of Inequalities and Applications* (2015) 2015:20
DOI 10.1186/s13660-015-0553-3

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

An inequality on derived length of a solvable group

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Abstract
Let G be a finite solvable group. Write $\delta^*(G)$ for the number of conjugacy classes of non-abelian subgroups of G, and by $d(G)$ denote the length of derived subgroups. In this paper an upper bound of $d(G)$ is given in terms of $\delta^*(G)$.
MSC: 20D10; 20D20

Keywords: solvable group; non-abelian subgroup; derived length

1 Introduction

In this paper G is a solvable group of finite order and let $d(G)$ denote the derived length of G. By $\delta(G)$ denote the number of conjugacy classes of non-cyclic subgroups of G. It can be proved (see [1, Theorem 4.2]) that

$$d(G) \le 2\bigl(\delta(G)-1\bigr)^{1/2}+1.$$

This gives an upper bound of the derived length of G. Note that this bound is not nice, for which we have an improvement in this note.

In this paper by $\delta^*(G)$ denote the number of conjugacy classes of non-abelian subgroups of G, which will replace $\delta(G)$.

Recall some information about a formation which is required in this note. A class $\mathcal{F}$ of finite groups is called a formation if $G \in \mathcal{F}$ and $N \trianglelefteq G$ then $G/N \in \mathcal{F}$, and if G/N_i $(i = 1,2) \in \mathcal{F}$ then $G/N_1 \cap N_2 \in \mathcal{F}$. If, in addition, $G/\Phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$, we say that $\mathcal{F}$ to be saturated. The class of all abelian groups is a formation but not saturated; the class of all nilpotent groups is a formation and saturated [2, 9.5 Formations].

Let $\mathcal{F}$ be a subgroup-closed formation. and let $G^{\mathcal{F}}$ be the $\mathcal{F}$-residual of G, that is,

$$G^{\mathcal{F}} = \bigcap_{N:N\trianglelefteq G, G/N\in\mathcal{F}} (N).$$

Define

$$G = G^{\mathcal{F}^0},$$
$$G^{\mathcal{F}^i} = \bigl(G^{\mathcal{F}^{i-1}}\bigr)^{\mathcal{F}}, \quad i = 1,2,\ldots.$$

Consider the series of characteristic (normal) subgroups of G:

$$G = G^{\mathcal{F}^0} \ge G^{\mathcal{F}^1} \ge G^{\mathcal{F}^2} \ge \cdots \ge G^{\mathcal{F}^r} = 1.$$

Here, if r satisfies $G^{\mathcal{F}^r} = 1$ but $G^{\mathcal{F}^{(r-1)}} \neq 1$, we say that r is the $\mathcal{F}$-length of G. As G is solvable, r must exist and $r \geq 1$.

Note that $G/G^{\mathcal{F}}$ and $G^{\mathcal{F}^{(r-1)}}$ both are in $\mathcal{F}$, and

$$G^{\mathcal{F}^i}/G^{\mathcal{F}^{(i+1)}} > 1, \quad i = 0, 1, \dots, r-1.$$

Suppose that $r \geq 4$. Now, we are able to choose elements $x_0, x_1, \dots, x_{r-4}$ in $G = G^{\mathcal{F}^0}, G^{\mathcal{F}^1}, \dots, G^{\mathcal{F}^{(r-4)}}$ satisfying the following condition:

$$x_i \in G^{\mathcal{F}^i} \quad \text{but } x_i \notin G^{\mathcal{F}^{(i+1)}}, i = 0, 1, \dots, r-4.$$

Define subgroups as follows:

$$M_{i,j} = \langle x_i \rangle G^{\mathcal{F}^j}, \quad i = 0, 1, \dots, r-4, i+2 \leq j \leq r-2.$$

Note that:

(1) In the definition of $M_{i,j}$, $j \geq i+2$ is required.

(2) For every $i \in \{0, 1, \dots, r-4\}$, the set $\{y_i : y_i \in G^{(i)} - G^{\mathcal{F}^{(i+1)}}\}$ is non-empty, the $(x_0, x_1, \dots, x_{r-4})$ may be replaced by $(y_0, y_1, \dots, y_{r-4})$ in the note.

(3) If for some i, $G^{(i)}/G^{(i+1)}$ is a non-abelian 2-group, we say that the i is a $\lambda(G)$. The investigation of the non-$\mathcal{F}$-subgroups of G is an interesting problem.

2 Preliminaries

In this section we list some known results which are needed in the sequel.

Lemma 1 *For all possible i and j we have*:

(1) *All $M_{i,j}$ are subgroups of G.*

(2) *No $M_{i,j}$ is in $\mathcal{F}$.*

Proof (1) As all $G^{\mathcal{F}^j}$ are characteristic (normal) subgroups of G, all $M_{i,j}$ must be subgroups of G.

(2) As $G^{\mathcal{F}^r} = 1$ but $G^{\mathcal{F}^{(r-1)}} \neq 1$, we can see that $G^{\mathcal{F}^{(r-2)}}$ is not in $\mathcal{F}$. Next, by the definition of $M_{i,j}$, we have $r-2 \geq j$, so $G^{\mathcal{F}^{(r-2)}} \leq G^{\mathcal{F}^j} \leq M_{i,j}$. As $\mathcal{F}$ is subgroup-closed, we conclude that $M_{i,j} \notin \mathcal{F}$. □

In the following part $\mathcal{F}$ is assumed to be the class of all abelian groups. Then $G^{\mathcal{F}} = G'$ and $G^{\mathcal{F}^{(i)}} = G^{(i)}$ for all i. Lemma 2 is valuable for the following proofs.

Lemma 2 *Let G be a nilpotent group. Then that G/G' is cyclic implies G is cyclic.*

Proof In a nilpotent group, the derived subgroup is contained in the Frattini subgroup [2, 5.2.16], so $G' \leq \Phi(G)$ and hence $G/\Phi(G)$ is cyclic. Thus G is cyclic. □

Lemma 3 *Suppose that G is a nilpotent group. Then no two of $M_{i,j}$ for all possible i and j are conjugate in G.*

Proof Assume the lemma is false. So that there exist $M_{i,j}$ with $j \geq r+2$and $M_{i',j'}$ with $j' \geq r+2$ which are conjugate, that is, there exists a $y \in G$ such that $M_{i,j}^y = M_{i',j'}$. By definition of $M_{i,j}$ we have $G^{(j)} \leq G^{(i)}$. Thus, we have $M_{i,j}(= \langle x_i \rangle G^{(j)}) \leq G^{(i)}$. It follows that

$$M_{i',j'} = M_{i,j}^{y} \leq (G^{(i)})^{y} = G^{(i)}.$$

Hence $x_{i'} \in G^{(i)}$. By the choice of x_i, we have $i' \geq i$. Similarly, $i \geq i'$. It follows that $i = i'$ and $x_i = x_{i'}$.

In order to finish the proof, we also claim that $j = j'$. Suppose that $j \neq j'$. Without loss of generality, let $j < j'$. Then $j + 1 \leq j'$ and

$$\begin{aligned} M_{i,j} &= \langle x_i \rangle G^{(j)} \geq \langle x_i \rangle G^{(j+1)} \\ &\geq \langle x_i \rangle G^{(j')} = \langle x_i' \rangle G^{(j')} = M_{I',J'}. \end{aligned}$$

We thus get

$$\langle x_i \rangle G^{(j)} = \langle x_i \rangle G^{(j+1)}.$$

Consequently, $G^{(j)}/G^{(j+1)}$ is cyclic.

Now, applying the hypothesis that G is a nilpotent group, by Lemma 2 we find that $G^{(j)}$ is cyclic, consequently $j \leq r - 2$, which is a contradiction (see the definition of $M_{i,j}$). □

Lemma 4 ([1, Lemma 4.1]) *Let G be a solvable group. Then the following statements are true*:

(1) *$M_{i,2j}$ and $M_{i',2j'}$ are conjugate if and only if $i = i'$ and $j = j'$.*
(2) *No $M_{i,j}$ is conjugate to some $G^{(k)}$.*

Please note Lemmas 6 and 7 below.

Lemma 5 *Let G be a nilpotent group and $G^{(2)} \neq 1$. Then there exists a non-abelian subgroup M which is a maximal subgroup in G, and the following statements are true*:

(1) *No subgroups $M_{i,j}$ for all possible i and j are conjugate to M.*
(2) *No subgroups $G^{(k)}$ for all possible k are conjugate to M.*

Proof By the condition that $G^{(2)} \neq 1$, so G' is non-abelian, and hence every maximal subgroup of G which contains G' is non-abelian, for which we write M.

As G is nilpotent, it follows that $G' \leq \Phi(G)$ and hence M is normal. Suppose some $M_{i,j}$ is conjugate to M. Then $M_{i,j} = \langle x_i \rangle G^{(j)} = M \trianglelefteq G$ and $G' \leq M = M_{i,j}$. If $x_i \in G'$, as G' contains $G^{(j)}$, we see that $M_{i,j} = G' < M < G$, which is a contradiction. Thus $x_i \notin G'$, and it follows that $x_i = x_0$, and $M_{i,j} = M_{0,j}$, $j \geq 2$. Now, both x_0 and G'' are in $M_{0,j}$, hence $M_{0,2} = \langle x_0 \rangle G'' = M_{0,j}$. It follows that $M_{0,2}/G''$ is cyclic, by applying Lemma 2, G' is cyclic, a contradiction. □

Lemma 6 *Let G be a solvable group with $d(G) = r \geq 4$. Suppose that $G^{(i)}/G^{(i+1)}$ is a noncyclic 2-group for some fixed $i \in \{0, 1, \dots, r-4\}$. Fix this i and take a_i for an element of $G^{(i)}$ but not in $G^{(i+1)}$ and let $K_i = \langle a_i \rangle G^{(i+1)}$ (note that $G^{(i)} > K_i > G^{(i+1)}$). If K_i is conjugate to some $M_{s,t}$ or some $G^{(k)}$, then $G^{(i)}/G^{(i+2)} \cong Q_8$, the quaternion group of order 8.*

Proof Fix i and write

$$G^{(i)}/G^{(i+1)} = \langle a_1 G^{(i+1)} \rangle \times \cdots \times \langle a_l G^{(i+1)} \rangle,$$

where all a_h are 2-element, and $l \geq 2$. $G^{(i)}$ is non-abelian, then $\langle a_h, G^{(i+1)}\rangle = \langle a_h\rangle G^{(i+1)} = K_h$ is non-abelian too.

As $G^{(i)} > K_h > G^{(i+1)}$, there is no $G^{(k)}$ which is conjugate to K_h. By condition, some $M_{s,t}$ is conjugate to K_h. Thus, for some $y \in G$ we have

$$K_h = M_{s,t}^y = \left(\langle x_s\rangle G^{(t)}\right)^y = \left\langle x_s^y\right\rangle G^{(t)}, \quad t \geq i+2.$$

As $K_h < G^{(i)}$, it follows that

$$\langle a_h\rangle G^{(i+1)} = K_h = \left\langle x_s^y\right\rangle G^{(t)} < G^{(i)}.$$

Now, as $G^{(i+2)} < G^{(i+1)} < K_h$ when $i < r-4$, we have $G^{(i+2)} < G^{(i+1)} < K_h = M_{s,t}$, it follows that $K_h/G^{(i+2)}$ is a cyclic group of $M_{s,t}/G^{(i+2)}$ which is generated by $a_hG^{(i+2)}$. Hence $G^{(i+1)}/G^{(i+2)}$ is cyclic.

For any element a of $G^{(i)}$ with order 2 (mould $G^{(i+2)}$), if $a \notin G^{(i+1)}$, then $\langle a\rangle \cap G^{(i+1)} = 1$, contrary to $G^{(i+1)}/G^{(i+2)}$ being cyclic. Thus, $a \in G^{(i+1)}$ and it follows that $\langle a\rangle$ is a unique subgroup of order 2 (mould $G^{(i+2)}$), consequently $G^{(i)} \cong Q_8$ of order 8 [2, 5.3.6], as desired. □

Lemma 7 *Let G be a solvable group with $d(G) = r \geq 4$. Suppose that for some fixed $i \in \{0,1,\dots,r-4\}$, $G^{(i)}/G^{(i+2)} \cong Q_8$ of order 8. Then $G^{(i)}$ is non-nilpotent and contains an abnormal maximal subgroup K which is non-abelian such that $K \in \delta^*(G)$.*

Proof The condition that $d(G) = r \geq 4$ shows that $G^{(r-4)}$ is non-abelian. By the condition that $G^{(i)}/G^{(i+2)} \cong Q_8$, $G^{(i+1)}/G^{(i+2)}$ is cyclic of order 2, by Lemma 2 we see $G^{(i+1)}$ is cyclic, hence $G^{(i+2)} = 1$. It follows that $r \leq i+2 \leq r-3+2 = r-1$, a contradiction. Now we find that $G^{(i)}$ is non-nilpotent. Then there exists an abnormal maximal subgroup K_i of $G^{(i)}$. If K_i is abelian, then $G^{(r-3)}K_i = G$, this implies that $G' \leq G^{(r-3)}$, contrary to $r \geq 4$. Now, we conclude that K_i is non-abelian, as desired.

Obviously, no $G^{(s)}$ is conjugate to K_i for all possible s. Suppose that some $M_{s,t}$ is conjugate to K_i. So, $M_{s,t}^y = K_i < G^{(i)}$ for some $y \in G$. By definition, $M_{s,t} = \langle x_s\rangle G^{(t)}$ with $t \geq s+2$. When $s \geq i+1$, then x_s and $G^{(t)}$ both are in $G^{(i+1)}$, so $K_i = M_{s,t}^y \leq G^{(i+1)} = (G^{(i)})'$, contrary to K_i being a maximal subgroup of $G^{(i)}$. Thus $s = i$ and $M_{s,t} = M_{i,t}$ with $t \geq i+2$. Now, $M_{s,t} = M_{i,t} = \langle x_i\rangle G^{(t)} \leq G^{(i)}$, so $G^{(i+2)} \geq G^{(t)}$. If $G^{(i+2)} > G^{(t)}$, we have $\langle x_i\rangle G^{(t)} > K$, consequently, $\langle x_i\rangle G^{(i+2)} = G^{(i)}$, and hence $\langle x_i\rangle G^{(i+1)} = G^{(i)}$, contrary to Q_8 ($\cong G^{(i)}/G^{(i+2)}$). Thus, $M_{s,t} = M_{i,i+2} = \langle x_i\rangle G^{(i+2)}$, which is normal in $G^{(i)}$, consequently, K_i would be normal in $G^{(i)}$, a contradiction. We conclude that no $M_{s,t}$ is conjugate to K_i. □

3 Main results

Now, we are able to give the main theorems of this note as follows:

Theorem 8 *Let G be a solvable group with $\delta^*(G) \geq 1$. Write $\lambda(G) = c+1$. Then*

$$d(G) \leq 2\left(\delta^*(G) - c - 2\right)^{1/2} + 1.$$

Proof In this section $\mathcal{F}$ denotes the class of all abelian groups, then $G^{\mathcal{F}} = G'$ and $G^{\mathcal{F}^i} = G^{(i)}$. If $d(G) = 1,2,3$, then the theorem holds obviously. Let $d(G) = r \geq 4$. By Lemmas 1 and 3, there exist the following non-abelian subgroups in G:

(a) $G(=G^{(0)}), G^{(1)}, G^{(2)}, \dots, G^{(r-2)}$;
(b) $M_{0,2}$; $M_{0,4}, M_{1,4}, M_{2,4}$; $M_{0,6}, M_{1,6}, M_{2,6}, M_{3,6}, M_{4,6}$; $\dots$;

$$M_{0,2k}, M_{1,2k}, \dots, M_{2k-2,2k}, \quad r-2 \geq 2k \geq r-3.$$

By Lemmas 6 and 7 for every $\lambda(G)$, G^{i_l} contains at least a non-abelian subgroup K_{i_l}, which belongs to $\delta^*(G)$, so we have
(c) $K_{i_0}, K_{i_1}, \dots, K_{i_c}$.
No two of these subgroups are conjugate in G, therefore

$$\begin{aligned}\delta^*(G) &\geq (r-1) + \bigl(1+3+5+(2k-1)\bigr) + (c+1)\\ &= r + k^2 + c\\ &\geq (r+c) + \bigl((r-3)/2\bigr)^2.\end{aligned}$$

That is, $4\delta^*(G) \geq (r-1)^2 + 4c + 9$,

$$d(G) \leq 2\bigl(\delta^*(G) - c - 2\bigr)^{1/2} + 1. \qquad \square$$

In this case when G is nilpotent, we have the following.

Theorem 9 *Let G be a nilpotent group with $d(G) = r$. Then*

$$d(G) \leq \bigl(2\delta^*(G) + 9/4\bigr)^{1/2} + 1/2.$$

Proof If $d(G) = r = 1, 2, 3$, then the theorem holds obviously. Let $d(G) \geq 4$. By Lemmas 1 and 4, there exist the following non-abelian subgroups in G:
(a) $G\ (=G^{(0)}), G^{(1)}, G^{(2)}, \dots, G^{(r-2)}$;
(b) $M_{i,j}$, $i \in \{0, 1, \dots, r-4\}$, $r+2 \leq j \leq r-2$.
By Lemma 5, every $G^{(i)}$ contains a non-abelian subgroup K_i which is in $\delta^*(G)$, so we have
(c) $K_0, K_1, \dots, K_{r-4}$.
No two of these subgroups are conjugate in G, therefore

$$\begin{aligned}\delta^*(G) &\geq (r-1) + (r-3)(r-2)/2 + (r-3)\\ &= (2r-4) + (r-3)(r-2)/2,\end{aligned}$$

$$d(G) \leq \bigl(2\delta^*(G) + 9/4\bigr)^{1/2} + 1/2. \qquad \square$$

Example 10 Let $G = S_4$. Then $d(G) = 3$ and $\delta^*(G) = 4(S_4, S_3, A_4, Q_8)$.
(1) By Theorem 8, we have ($\lambda(G) = c + 1 = 0$)

$$2\bigl(\delta^*(G) - 2 - c\bigr)^{1/2} + 1 = 2(4-1)^{1/2} + 1 < 4.47.$$

We conclude that $d(G) = 3 < 4.47$ and $4.47 - r = 1.47$.

(2) $\delta(G) = 5(S_4, S_3, A_4, Q_8, C_2 \times C_2)$, by [1, Theorem 4.1], we have

$$4.4 \leq 2\big(\delta(G) - 1\big)^{1/2} + 1 = 2(4)^{1/2} + 1 \leq 5.$$

We conclude that $d(G) = 3 < 5$ and $5 - r = 2$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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Acknowledgements
The authors are grateful to the editors and the referees, who provided their detailed reports. The research of the work was supported by NSFC (Grant Nos. 11171364, 11271301, U1204101, 11471266), Fundamental Research Funds for the Central Universities (No. XDJK2014C163), National Youth Science Foundation (No. 11201385) and the Major Project of Education Department of Henan Province (No. 13B110085).

Received: 20 October 2014 Accepted: 7 January 2015 Published online: 21 January 2015

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