Al-Mezel et al. *Journal of Inequalities and Applications* (2015) 2015:33
DOI 10.1186/s13660-015-0550-6

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A note on fixed point results in complex-valued metric spaces

Saleh A Al-Mezel[1], Hamed H Alsulami[1], Erdal Karapınar[1,2*] and Farshid Khojasteh[3]

*Correspondence: erdalkarapinar@yahoo.com; ekarapinar@atilim.edu.tr
[1]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, 21491, Saudi Arabia
[2]Department of Mathematics, Atilim University, Incek, Ankara, 06836, Turkey
Full list of author information is available at the end of the article

Abstract
In this paper, we prove that the fixed point results in the context of complex-valued metric spaces can be obtained as a consequence of corresponding existing results in the literature in the setting of associative metric spaces. In particular, we deduce that any complex metric space is a special case of cone metric spaces with a normal cone.
MSC: 46T99; 47H10; 54H25; 54C30

Keywords: contraction mapping; cone metric spaces; quaternion-valued metric space; complex-valued metric space; fixed point

1 Introduction and preliminaries

The notion of complex-valued metric spaces was introduced by Azam *et al.* [1], as a generalization of metric spaces to investigate the existence and uniqueness of fixed point results for mappings satisfying a rational inequalities. Following this paper, a number of authors have reported several fixed point results for various mapping satisfying a rational inequalities in the context of complex-valued metric spaces; see *e.g.* [1–3] and the related references therein.

The aim of this short note is to emphasize that the complex-valued metric space is an example of the cone metric space that was introduced in [4–6] under the name K-metric and K-normed spaces and re-introduced by Huang and Zhang [7]. It is well known that if the cone is normal then the corresponding cone metric associates a metric. There are some other approaches to induce a metric from cone metric; see *e.g.* [8–16]. As a consequence of these observations, we notice that fixed point results in the context of complete complex-valued metric spaces can be deduced the corresponding fixed point results on (associative) complete metric space. Based on the discussion above, for our purpose, we first prove the existence of common fixed point theorems for multi-valued mapping in the context of complete metric space. Then we derive the main results of the recent paper of Ahmad *et al.* [2] as corollaries of our results.

For the sake of completeness we recollect some basic definitions and fundamentals results on the topic in the literature. We mainly follow the notions and notations of Azam *et al.* in [1].

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad \operatorname{Re}(z_1) \le \operatorname{Re}(z_2), \operatorname{Im}(z_1) \le \operatorname{Im}(z_2).$$

It follows that

$$z_1 \precsim z_2$$

if one of the following conditions is satisfied:

(h_1) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(h_2) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(h_3) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(h_4) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$; $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we shall write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (h_1), (h_2), and (h_3) is satisfied. Further we write $z_1 \prec z_2$ if only (h_3) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \quad \Longrightarrow \quad |z_1| < |z_2|,$$

where $|\cdot|$ represents the modulus or magnitude of z, and

$$z_1 \precsim z_2, \qquad z_2 \prec z_3 \quad \Longrightarrow \quad z_1 \prec z_3.$$

Definition 1 [1] Let X be a nonempty set. A function $d : X \times X \to \mathbb{C}$ is called a complex-valued metric on X, if it satisfies the following conditions:

(b_1) $0 \precsim d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$, if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$, for all $x, y \in X$,
(b_3) $d(x,y) \precsim d(x,z) + d(y,z)$, for all $x, y, z \in X$.

Furthermore, the pair (X,d) is called a complex-valued metric space.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim_n x_n = x$, or $x_n \to x$, as $n \to \infty$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_{n+m}) \prec c$, then $\{x_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex-valued metric space.

Lemma 2 [1, Lemma 2, Azam *et al.*] *Let (X,d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 3 [1, Lemma 3, Azam *et al.*] *Let (X,d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$.*

Let E be a real Banach space. A subset P of E is called a cone, if the followings hold:

(a_1) P is closed, nonempty, and $P \neq \{0\}$,
(a_2) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,
(a_3) $x \in P$ and $-x \in P$ imply that $x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$, if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ stands for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal, if there exist a number $K \geq 1$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$, for all $x, y \in E$. The least positive number satisfying this, called the normal constant [7, 17].

In this paper, E denotes a real Banach space, P denotes a cone in E with $\operatorname{int} P \neq \emptyset$, and $\leq$ denotes a partial ordering with respect to P. For more details on the cone metric, we refer *e.g.* to [7, 17, 18].

Definition 4 [7] Let X be a nonempty set. A function $d : X \times X \to E$ is called a cone metric on X, if it satisfies the following conditions:

(b_1) $d(x, y) \geq 0$ for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$,
(b_2) $d(x, y) = d(y, x)$, for all $x, y \in X$,
(b_3) $d(x, y) \leq d(x, z) + d(y, z)$, for all $x, y, z \in X$.

Then (X, d) is called a cone metric space.

The following definitions and lemmas have been taken from [7, 18].

Definition 5 Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $d(x_n, x_0) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent and $\{x_n\}_{n\in\mathbb{N}}$ converges to x and x is the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 6 Let (X, d) be a cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If for all $c \in E$ with $0 \ll c$, there is $n_0 \in \mathbb{N}$ such that for all $m, n > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}_{n\in\mathbb{N}}$ is called a Cauchy sequence in X.

Definition 7 Let (X, d) be a cone metric space. If every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Definition 8 Let (X, d) be a cone metric space. A self-map T on X is said to be continuous, if $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} T(x_n) = T(x)$ for all sequences $\{x_n\}_{n\in\mathbb{N}}$ in X.

Lemma 9 *Let (X, d) be a normal cone metric space and P be a cone. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\{x_n\}_{n\in\mathbb{N}}$ converges to x, if and only if*

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{1.1}$$

Lemma 10 *Let (X, d) be a normal cone metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is convergent, then it is a Cauchy sequence.*

Lemma 11 *Let (X, d) be a cone metric space and P be a cone in E. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if and only if* $\lim_{m,n\to\infty} d(x_m, x_n) = 0$.

2 Main result

In this section, we represent our main results. First of all, we represent some simple observations. Let $(X, d_{\mathbb{C}})$ be a complex-valued metric space. Now, we define the following set:

$$\mathcal{P}_{\mathbb{C}} = \{x + iy : x \geq 0, y \geq 0\}.$$

It is apparent that $\mathcal{P}_{\mathbb{C}} \subset \mathbb{C}$. Note that $(\mathbb{C}, |\cdot|)$ is a real Banach space.

Lemma 12 *$\mathcal{P}_{\mathbb{C}}$ is a normal cone in a real Banach space $(\mathbb{C}, |\cdot|)$.*

Proof Precisely, $\mathcal{P}_{\mathbb{C}}$ is nonempty, closed and $\mathcal{P}_{\mathbb{C}} \neq (\mathbf{0}_{\mathbb{C}})$. Also for all $\alpha, \beta \in \mathbb{R}^+$ and $p, q \in \mathcal{P}_{\mathbb{C}}$ we have $\alpha p + \beta q \in \mathcal{P}_{\mathbb{C}}$ and $\mathcal{P}_{\mathbb{C}} \cap (-\mathcal{P}_{\mathbb{C}}) = (\mathbf{0}_{\mathbb{C}})$. Also the normality of $\mathcal{P}_{\mathbb{C}}$ is apparent. □

Lemma 13 *Any complex-valued metric space $(X, d_{\mathbb{C}})$ is a cone metric space.*

Proof For all $p, q \in \mathbb{C}$ define

$$p \subseteqq q \quad \text{if and only if} \quad q - p \in \mathcal{P}_{\mathbb{C}}.$$

$\subseteqq$ defines a partial ordered on $\mathbb{C}$ and one can easily verify that $(X, d_{\mathbb{C}})$ is a cone metric space with respect to $\subseteqq$. □

Lemma 14 *The partial ordered $\subseteqq$ defined in Lemma 13 is equivalent to $\precsim$.*

Proof Assume $p = p_1 + ip_2$ and $q = q_1 + iq_2$. $p \subseteqq q$, if and only if $q - p \in \mathcal{P}_{\mathbb{C}}$, if and only if $q_1 - p_1 \geq 0$, $q_2 - p_2 \geq 0$. In other words, $\mathrm{Re}(p) \leq \mathrm{Re}(q)$, $\mathrm{Im}(p) \leq \mathrm{Im}(q)$, if and only if $p \precsim q$. □

Lemma 15 *A sequence $\{x_n\}$ in $(X, d_{\mathbb{C}})$ is convergent according to the concept of complex-valued metric space if and only if $\{x_n\}$ is convergent according to the concept of a cone metric space.*

Proof Let $\{x_n\}$ be sequence in X. Sequence $\{x_n\}$ converges to $x \in X$ according to the concept of complex-valued metric space if and only if $|d(x_n, x)| \to 0$ as $n \to \infty$ if and only if $\{x_n\}$ converges to x according to the concept of a cone metric space by considering $\mathbb{C}$ as the Banach space endowed with the cone $\mathcal{P}_{\mathbb{C}}$ (see Lemma 9). □

Finally, we recall some fundamental definition for multi-valued mappings and related metric spaces. Let (X, d) be a metric space. Let $\mathcal{P}(X) = \{Y \mid Y \subset X\}$ and $P(X) := \{Y \in \mathcal{P}(X) \mid Y \neq \emptyset\}$. Let us define *the gap functional* $D : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$, as

$$D(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}.$$

In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

We denote by $\mathcal{C}(X)$ the family of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$

defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in B} D(x,A), \sup_{x\in A} D(x,B)\right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X where $D(x,A) = \inf\{d(x,y) : y \in A\}$ for each $A \in \mathcal{CB}(X)$. A point v in X is a fixed point of a map T if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{P}(X)$ is a multi-valued map). We say that T has an endpoint if there exists $v \in X$ such that $Tv = \{v\}$. The set of fixed points of T is denoted by $\mathcal{F}(T)$ and the set of common fixed points of two multi-valued mappings T, S is denoted by $\mathcal{F}(T,S)$.

Definition 16 For two multi-valued mappings $T,S : X \to \mathcal{CB}(X)$, we say that T, S satisfy the common approximate endpoint property if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} \mathcal{H}\big(\{x_n\}, Tx_n\big) = \lim_{n\to\infty} \mathcal{H}\big(\{x_n\}, Sx_n\big) = 0.$$

Definition 17 For two mappings $T,S : X \to X$, we say that T, S have a common approximate fixed point if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} d(x_n, Tx_n) = \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

Throughout the paper, we assume that $\{a,b,c,d,e\} \subset [0,1)$.

The following is the fundamental theorem of this paper.

Theorem 18 *Let (X,d) be a complete metric space and let $T,S : X \to \mathcal{CB}(X)$ be two multi-valued functions such that for each $x,y \in X$,*

$$\begin{aligned}\mathcal{H}(Tx,Sy) \leq{}& ad(x,y) + bD(x,Sx)D(y,Ty)\\ &+ c\sqrt{D(y,Sx)D(x,Ty)} + dD(x,Sx)D(x,Ty)\\ &+ eD(y,Sx)D(y,Ty),\end{aligned}$$

where $a + b + c + 2d + 2e < 1$. Then T, S have a unique endpoint, if and only if they satisfy the common approximate endpoint property.

Proof If T, S have a unique endpoint then they precisely satisfy the common approximate endpoint property. Conversely, suppose that T, S satisfy the common approximate endpoint property, then there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} \mathcal{H}(\{x_n\}, Tx_n) = \lim_{n\to\infty} \mathcal{H}(\{x_n\}, Sx_n) = 0$. We claim that $\{x_n\}$ is a Cauchy sequence. For convenience suppose that $\alpha_n = \mathcal{H}(\{x_n\}, Tx_n)$ and $\beta_n = \mathcal{H}(\{x_n\}, Sx_n)$; we have

$$\begin{aligned}d(x_n,x_m) &\leq \alpha_n + \beta_m + \mathcal{H}(Tx_n, Sx_m)\\ &\leq \alpha_n + \beta_m + ad(x_n,x_m) + bD(x_n,Sx_n)D(x_m,Tx_m)\\ &\quad + c\sqrt{D(x_m,Sx_n)D(x_n,Tx_m)} + dD(x_n,Sx_n)D(x_n,Tx_m)\\ &\quad + eD(x_m,Sx_n)D(x_m,Tx_m)\end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n + \beta_m + ad(x_n,x_m) + b\beta_n\alpha_m + c\sqrt{\big(d(x_n,x_m)+\beta_n\big)\big(d(x_n,x_m)+\alpha_m\big)} \\
&\quad + d\big(\beta_n\big(d(x_n,x_m)+\alpha_m\big)\big) + e\big(\alpha_m\big(d(x_n,x_m)+\beta_n\big)\big) \\
&\le \alpha_n + \beta_m + ad(x_n,x_m) + c\frac{(d(x_n,x_m)+\beta_n)+(d(x_n,x_m)+\alpha_m)}{2} \\
&\quad + d\beta_n d(x_n,x_m) + d\beta_n\alpha_m + e\alpha_m d(x_n,x_m) + e\alpha_m\beta_n.
\end{aligned}
$$

It means that

$$d(x_n,x_m)(1-a-c-d\beta_n-e\alpha_m) \le \alpha_n+\beta_m+(d+b)\beta_n\alpha_m+e\alpha_m\beta_n+\frac{c}{2}(\beta_n+\alpha_m).$$

In other words,

$$d(x_n,x_m) \le \frac{\alpha_n+\beta_m+(d+b)\beta_n\alpha_m+e\alpha_m\beta_n+\frac{c}{2}(\beta_n+\alpha_m)}{1-a-c-d\beta_n-e\alpha_m}.$$

Since $a+c<1$ we have $d(x_n,x_m)\to 0$ as $n,m\to\infty$. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete metric space, it converges to some $z\in X$. Also we have

$$
\begin{aligned}
\mathcal{H}(z,Tz) &\le d(z,x_n)+\beta_n+\mathcal{H}(Tz,Sx_n) \\
&\le d(z,x_n)+\beta_n+ad(x_n,z)+bD(z,Sz)D(x_n,Tx_n) \\
&\quad + c\sqrt{D(x_n,Sz)D(z,Tx_n)} + dD(z,Sz)D(z,Tx_n) + eD(x_n,Sz)D(x_n,Tx_n) \\
&\le d(z,x_n)+\beta_n+ad(x_n,z)+b\alpha_n D(z,Sz) \\
&\quad + c\sqrt{D(x_n,Sz)\big(d(x_n,z)+\alpha_n\big)} + dD(z,Sz)\big(d(z,x_n)+\alpha_n\big) \\
&\quad + eD(x_n,Sz)\alpha_n.
\end{aligned}
\tag{2.1}
$$

By taking the limit on both sides of (2.1) we have $\mathcal{H}(\{z\},Tz)=0$ and so $Tz=\{z\}$. By a similar argument we deduce that $Sz=\{z\}$. □

Theorem 19 *Let (X,d) be a complete metric space and let $T,S:X\to\mathcal{CB}(X)$ be two multi-valued functions such that for each $x,y\in X$,*

$$
\begin{aligned}
\mathcal{H}(Tx,Sy) &\le ad(x,y) + b\frac{D(x,Sx)D(y,Ty)}{1+d(x,y)} \\
&\quad + c\frac{\sqrt{D(y,Sx)D(x,Ty)}}{1+d(x,y)} + d\frac{D(x,Sx)D(x,Ty)}{1+d(x,y)} + e\frac{D(y,Sx)D(y,Ty)}{1+d(x,y)},
\end{aligned}
$$

where $a+b+c+2d+2e<1$. Then T, S have a unique endpoint, if and only if they satisfy the common approximate endpoint property.

Proof By Theorem 18 since

$$
\begin{aligned}
\mathcal{H}(Tx,Sy) &\le ad(x,y) + b\frac{D(x,Sx)D(y,Ty)}{1+d(x,y)} \\
&\quad + c\frac{\sqrt{D(y,Sx)D(x,Ty)}}{1+d(x,y)} + d\frac{D(x,Sx)D(x,Ty)}{1+d(x,y)}
\end{aligned}
$$

$$
\begin{aligned}
&+ e\frac{D(y,Sx)D(y,Ty)}{1+d(x,y)} \\
\leq{}& ad(x,y) + bD(x,Sx)D(y,Ty) \\
&+ c\sqrt{D(y,Sx)D(x,Ty)} + dD(x,Sx)D(x,Ty) \\
&+ eD(y,Sx)D(y,Ty),
\end{aligned}
$$

we conclude the desired result. □

Theorem 20 *Let (X,d) be a complete metric space and let $f,g : X \to X$ be two multi-valued functions such that for each $x,y \in X$,*

$$
\begin{aligned}
d(fy,gx) \leq{}& ad(x,y) + bd(x,gx)d(y,fy) \\
&+ cd(y,gx)d(x,fy) + dd(x,gx)d(x,fy) \\
&+ ed(y,gx)d(y,fy),
\end{aligned}
$$

where $a + b + c + 2d + 2e < 1$. Then f, g have a common fixed point.

Proof Let $x_0 \in X$ be arbitrary and let $x_{2n-1} = fx_{2n-2}$ and $x_{2n} = gx_{2n-1}$. We have

$$
\begin{aligned}
d(x_{2n+1},x_{2n}) ={}& d(fx_{2n},gx_{2n-1}) \\
\leq{}& ad(x_{2n},x_{2n-1}) + bd(x_{2n-1},gx_{2n-1})d(x_{2n},fx_{2n}) \\
&+ cd(x_{2n},gx_{2n-1})d(x_{2n-1},fx_{2n}) + dd(x_{2n-1},gx_{2n-1})d(x_{2n-1},fx_{2n}) \\
&+ ed(x_{2n},gx_{2n-1})d(x_{2n},fx_{2n}) \\
={}& ad(x_{2n},x_{2n-1}) + bd(x_{2n-1},x_{2n})d(x_{2n},x_{2n+1}) \\
&+ cd(x_{2n},x_{2n})d(x_{2n-1},x_{2n+1}) + dd(x_{2n-1},x_{2n})d(x_{2n-1},x_{2n+1}) \\
&+ ed(x_{2n},x_{2n})d(x_{2n},x_{2n+1}) \\
={}& ad(x_{2n},x_{2n-1}) + bd(x_{2n-1},x_{2n})d(x_{2n},x_{2n+1}) \\
&+ dd(x_{2n-1},x_{2n})d(x_{2n-1},x_{2n+1}) \\
\leq{}& ad(x_{2n},x_{2n-1}) + bd(x_{2n},x_{2n+1}) \\
&+ dd(x_{2n-1},x_{2n})\big(d(x_{2n-1},x_{2n}) + d(x_{2n},x_{2n+1})\big) \\
\leq{}& ad(x_{2n},x_{2n-1}) + bd(x_{2n},x_{2n+1}) + d\big(d(x_{2n-1},x_{2n}) + d(x_{2n},x_{2n+1})\big).
\end{aligned}
$$

It means that

$$
d(x_{2n+1},x_{2n}) \leq \frac{a+d}{1-b-d}d(x_{2n-1},x_{2n}).
$$

Also

$$
\begin{aligned}
d(x_{2n-1},x_{2n}) ={}& d(fx_{2n-2},gx_{2n-1}) \\
\leq{}& ad(x_{2n-2},x_{2n-1}) + bd(x_{2n-1},gx_{2n-1})d(x_{2n-2},fx_{2n-2})
\end{aligned}
$$

$$\begin{aligned}
&\quad + cd(x_{2n-2}, gx_{2n-1})d(x_{2n-1}, fx_{2n-2}) + dd(x_{2n-1}, gx_{2n-1})d(x_{2n-1}, fx_{2n-2}) \\
&\quad + ed(x_{2n-2}, gx_{2n-1})d(x_{2n-2}, fx_{2n-2}) \\
&= ad(x_{2n-2}, x_{2n-1}) + bd(x_{2n-1}, x_{2n})d(x_{2n-2}, x_{2n-1}) \\
&\quad + cd(x_{2n-2}, x_{2n})d(x_{2n-1}, x_{2n-1}) + dd(x_{2n-1}, x_{2n})d(x_{2n-1}, x_{2n-1}) \\
&\quad + ed(x_{2n-2}, x_{2n})d(x_{2n-2}, x_{2n-1}) \\
&= ad(x_{2n-2}, x_{2n-1}) + bd(x_{2n-1}, x_{2n})d(x_{2n-2}, x_{2n-1}) \\
&\quad + ed(x_{2n-2}, x_{2n})d(x_{2n-2}, x_{2n-1}) \\
&\le ad(x_{2n-2}, x_{2n-1}) + bd(x_{2n-1}, x_{2n}) \\
&\quad + e\big(d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})\big)d(x_{2n-2}, x_{2n-1}) \\
&\le ad(x_{2n-2}, x_{2n-1}) + bd(x_{2n-1}, x_{2n}) + e\big(d(x_{2n-2}, x_{2n-1}) + d(x_{2n-1}, x_{2n})\big).
\end{aligned}$$

It means that

$$d(x_{2n-1}, x_{2n}) \le \frac{a+e}{1-b-e} d(x_{2n-2}, x_{2n-1}).$$

Now taking $\lambda = \max\{\frac{a+d}{1-b-d}, \frac{a+e}{1-a-e}\} < 1$ we conclude that for each $n \in \mathbb{N}$

$$d(x_{n+1}, x_n) \le \lambda d(x_n, x_{n-1}).$$

By a standard technique, one can show that $\{x_n\}$ is Cauchy. Since (X, d) is a complete metric space, there exists $u \in X$ such that $x_n \to u$.

We claim that $fu = u$. By the triangle inequality, we have

$$\begin{aligned}
d(u, fu) &\le d(u, gx_{2n-1}) + d(gx_{2n-1}, fu) \\
&\le d(u, gx_{2n-1}) + ad(x_{2n-1}, u) + bd(x_{2n-1}, gx_{2n-1})d(u, fu) \\
&\quad + cd(u, gx_{2n-1})d(x_{2n-1}, fu) + dd(x_{2n-1}, gx_{2n-1})d(x_{2n-1}, fu) \\
&\quad + ed(u, gx_{2n-1})d(u, fu).
\end{aligned}$$

Letting $n \to \infty$ in the inequality above, we get

$d(u, fu) \le 0$ and hence we find $fu = u$.

Analogously, we derive that $gu = u$. Hence, we conclude that u is the common fixed point of f and g. □

Corollary 21 *Let (X, d) be a complete metric space and let $f, g : X \to X$ be two multivalued functions such that for each $x, y \in X$,*

$$\begin{aligned}
d(fy, gx) &\le ad(x, y) + bd(x, gx)d(y, fy) \\
&\quad + dd(x, gx)d(x, fy) + ed(y, gx)d(y, fy),
\end{aligned}$$

where $a + b + 2d + 2e < 1$. Then f, g have a unique common fixed point.

Proof By following the lines in the proof of Theorem 20 we get that f and g have a common fixed point, say $u \in X$. We shall show that u is the unique common fixed point of f and g. Suppose, on the contrary, that u and v are distinct common fixed points of f and g. Hence, we have

$$\begin{aligned} d(u,v) &= d(fu,gv) \leq ad(v,u) + bd(v,gv)d(u,fu) \\ &\quad + dd(v,gv)d(v,fu) + ed(u,gv)d(u,fu) \\ &\leq ad(v,u) + bd(v,v)d(u,u) \\ &\quad + dd(v,v)d(v,u) + ed(u,v)d(u,u), \end{aligned}$$

which implies that

$$d(u,v) \leq ad(v,u).$$

Since $a < 1$, $d(u,v) = 0$, which is a contradiction. Hence, u is the unique common fixed point of f and g. □

Theorem 22 *Let (X,d) be a complex-valued metric space and let $f,g : X \to X$ be two functions and a, b, c, d, e be such that $a + b + c + 2d + 2e < 1$. Let*

$$\begin{aligned} d(fy,gx) \precsim{} & ad(x,y) + bd(x,gx)d(y,fy) \\ & + cd(y,gx)d(x,fy) + dd(x,gx)d(x,fy) \\ & + ed(y,gx)d(y,fy) \end{aligned}$$

for all $x,y \in X$. Then f, g have a unique common fixed point.

Proof Taking $\rho(x,y) = |d(x,y)|$, (X,ρ) is a complete metric space and applying Theorem 20 we conclude that f and g have a common fixed point. As in the proof of Corollary 21, uniqueness of the common fixed point of f and g can be derived easily by *reductio ad absurdum.* □

The following results, the main results of Ahmad *et al.* [2], can be considered as a consequence of Theorem 20.

Theorem 23 *Let (X,d) be a complex-valued metric space and let $f,g : X \to X$ be two functions and a, b, c, d, e be such that $a + b + c + 2d + 2e < 1$. Let*

$$\begin{aligned} d(fy,gx) \precsim{} & ad(x,y) + b\frac{d(x,gx)d(y,fy)}{1 + d(x,y)} \\ & + c\frac{d(y,gx)d(x,fy)}{1 + d(x,y)} + d\frac{d(x,gx)d(x,fy)}{1 + d(x,y)} \\ & + e\frac{d(y,gx)d(y,fy)}{1 + d(x,y)} \end{aligned}$$

for all $x,y \in X$. Then f, g have a unique common fixed point.

Proof We have $a + b + c + d + e < a + b + c + 2d + 2e < 1$ and

$$\begin{aligned} d(fy,gx) &\precsim ad(x,y) + b\frac{d(x,gx)d(y,fy)}{1+d(x,y)} \\ &\quad + c\frac{d(y,gx)d(x,fy)}{1+d(x,y)} + d\frac{d(x,gx)d(x,fy)}{1+d(x,y)} \\ &\quad + e\frac{d(y,gx)d(y,fy)}{1+d(x,y)} \\ &\precsim ad(x,y) + bd(x,gx)d(y,fy) \\ &\quad + cd(y,gx)d(x,fy) + dd(x,gx)d(x,fy) \\ &\quad + ed(y,gx)d(y,fy). \end{aligned}$$

By Theorem 20 we conclude that f and g have a common fixed point. Uniqueness can be derived easily verbatim as in the proof of Corollary 21. □

Remark 24 By Theorem 20, one can derive the other results in [2] but we prefer not to list these here.

In what follows we state a theorem that is just a variation of Theorem 20.

Corollary 25 *Let (X,d) be a complete metric space and let $f,g : X \to X$ be two multi-valued functions such that for each $x,y \in X$,*

$$\begin{aligned} d(fy,gx) &\le ad(x,y) + bd(x,gx)d(y,fy) \\ &\quad + c\sqrt{d(y,gx)d(x,fy)} + dd(x,gx)d(x,fy) \\ &\quad + ed(y,gx)d(y,fy), \end{aligned}$$

where $a + b + c + 2d + 2e < 1$. Then f, g have a unique common fixed point.

Proof By following the lines in the proof of Theorem 20, one can easily observe that

$$\lim_{n\to\infty} d(x_n,fx_n) = \lim_{n\to\infty} d(x_n,gx_n) = 0.$$

Therefore, f, g have a common approximate fixed point. Thus taking $Tx = \{fx\}$ and $Sx = \{gx\}$ in Theorem 18 we conclude that T, S satisfy the common approximate endpoint property and so f, g have a unique common fixed point. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, 21491, Saudi Arabia. [2]Department of Mathematics, Atilim University, Incek, Ankara, 06836, Turkey. [3]Department of Mathematics, Arak Branch, Islamic Azad University, Aral, Iran.

Received: 1 October 2014 Accepted: 6 January 2015 Published online: 28 January 2015

References

1. Azam, A, Fisher, B, Khan, M: Common fixed point theorems in complex-valued metric spaces. Numer. Funct. Anal. Optim. **32**, 243-253 (2011). doi:10.1080/01630563.2011.533046
2. Ahmad, J, Azam, A, Saejung, S: Common fixed point results for contractive mappings in complex valued metric spaces. Fixed Point Theory Appl. **2014**, Article ID 67 (2014). doi:10.1186/1687-1812-2014-67
3. Ozturk, M: Common fixed point theorems satisfying contractive type conditions in complex valued metric spaces. Abstr. Appl. Anal. **2014**, Article ID 598465 (2014)
4. Kantorovich, LV: The majorant principle and Newton's method. Dokl. Akad. Nauk SSSR **76**, 17-20 (1951)
5. Vandergraft, JS: Newton's method for convex operators in partially ordered spaces. SIAM J. Numer. Anal. **4**, 406-432 (1967)
6. Zabreiko, PP: *K*-Metric and *K*-normed spaces: survey. Collect. Math. **48**(4-6), 825-859 (1997)
7. Huang, LG, Zhang, X: Cone metric spaces and fixed point theorems of contractive mappings. J. Math. Anal. Appl. **332**, 1468-1476 (2007)
8. Du, W-S: A note on cone metric fixed point theory and its equivalence. Nonlinear Anal. **72**(5), 2259-2261 (2010)
9. Abdeljawad, T, Karapınar, E: A gap in the paper 'A note on cone metric fixed point theory and its equivalence' [Nonlinear Anal. 72(5), (2010), 2259-2261]. Gazi Univ. J. Sci. **24**(2), 233-234 (2011)
10. Khamsi, MA: Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. (2010). doi:10.1155/2010/315398
11. Amini-Harandi, A, Fakhar, M: Fixed point theory in cone metric spaces obtained via the scalarization method. Comput. Math. Appl. **59**, 3529-3534 (2010)
12. Feng, Y, Mao, W: The equivalence of cone metric spaces and metric spaces. Fixed Point Theory **11**(2), 259-264 (2010)
13. Jankovic, S, Kadelburg, Z, Radenovic, S: On cone metric spaces: a survey. Nonlinear Anal., Theory Methods Appl. **74**(7), 2591-2601 (2011)
14. Farajzadeh, AP: On the scalarization method in cone metric spaces. Positivity **18**, 703-708 (2014). doi:10.1007/s11117-013-0271-3
15. Agarwal, RP, Alsulami, H, Karapınar, E, Khojasteh, F: Remarks on some recent fixed point results in quaternion-valued metric spaces. Abstr. Appl. Anal. **2014**, Article ID 171624 (2014)
16. Kadelburg, Z, Radenovic, S, Rakocevic, V: A note on the equivalence of some metric and cone metric fixed point results. Appl. Math. Lett. **24**(3), 370-374 (2011)
17. Rezapour, S, Haghi, RH: Some notes on the paper 'Cone metric spaces and fixed point theorems of contractive mappings'. J. Math. Anal. Appl. **345**, 719-724 (2008)
18. Khojasteh, F, Razani, A, Moradi, S: A fixed point of generalized T_F-contraction mappings in cone metric spaces. Fixed Point Theory Appl. **2011**, Article ID 14 (2011). doi:10.1186/1687-1812-2011-14