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# Criteria for starlike and convex functions of order $\alpha$

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**Abstract**

Let $\mathcal{A}_n$ ($n \in \mathbb{N}$) be the class of certain analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ and $\mathcal{P}_n(\lambda)$ be the subclass of $\mathcal{A}_n$ consisting of $f(z)$ which satisfy $|f''(z)| \leqq \lambda$ ($\lambda > 0$) in $\mathbb{U}$. Some properties for the class $\mathcal{P}_n(\lambda)$, which are the improvements of the previous results due to Ponnusamy and Singh (Complex Var. Theory Appl. 34:276-291, 1997), are discussed.

**MSC:** Primary 30C45

**Keywords:** starlike function; convex function; strongly starlike function; subordination

## 1 Introduction

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad \bigl(n \in \mathbb{N} = \{1, 2, 3, \ldots\}\bigr), \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}_1 = \mathcal{A}$.

A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ in $\mathbb{U}$ if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \quad (z \in \mathbb{U}) \tag{1.2}$$

for some real $\alpha$ ($\alpha < 1$). If $f(z) \in \mathcal{S}^*(\alpha)$ with $0 \leqq \alpha < 1$, then $f(z)$ is said to be univalent and starlike of order $\alpha$ in $\mathbb{U}$. We denote $\mathcal{S}^*(0) = \mathcal{S}^*$. A function $f(z) \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha)$ if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (z \in \mathbb{U}) \tag{1.3}$$

for some real $\alpha$ ($\alpha < 1$). If $f(z) \in \mathcal{C}(\alpha)$ with $0 \leqq \alpha < 1$, then $f(z)$ is said to be univalent and convex of order $\alpha$ in $\mathbb{U}$. We write $\mathcal{C}(0) = \mathcal{C}$.

Let $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ which satisfies $w(0) = 0$, $|w(z)| < 1$ ($z \in \mathbb{U}$) and $f(z) = g(w(z))$ for $z \in \mathbb{U}$. If $g(z)$ is univalent in $\mathbb{U}$, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ (*cf.* Duren [1]).

A function $f(z) \in \mathcal{A}$ is said to be strongly starlike of order $\beta$ in $\mathbb{U}$ if it satisfies

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \tag{1.4}$$

for some real $\beta$ ($0 < \beta \leqq 1$). We denote this class by $\widetilde{\mathcal{S}}^*(\beta)$. Note that $\widetilde{\mathcal{S}}^*(1) = \mathcal{S}^*$.

Define

$$\mathcal{P}_n(\lambda) = \bigl\{f(z) \in \mathcal{A}_n : \bigl|f''(z)\bigr| \leqq \lambda\ (\lambda > 0; z \in \mathbb{U})\bigr\}. \tag{1.5}$$

Mocanu [2] considered the problem of finding $\lambda$ such that

$$f(z) \in \mathcal{P}_n(\lambda) \quad \text{implies} \quad f(z) \in \mathcal{S}^*.$$

Mocanu [2] has shown that:

**Theorem A** ([2]) *If*

$$\lambda = \frac{n(n+1)}{2n+1} \quad (n \in \mathbb{N}),$$

*then* $\mathcal{P}_n(\lambda) \subset \mathcal{S}^*$.

Ponnusamy and Singh [3] proved the following results.

**Theorem B** *Let*

$$\lambda_n = \frac{n(n+1)}{\sqrt{(n+1)^2+1}} \quad (n \in \mathbb{N}).$$

*If* $0 < \lambda \leqq \lambda_n$, *then* $\mathcal{P}_n(\lambda) \subset \mathcal{S}^*(\beta)$, *where*

$$\beta = \beta_n(\lambda) = \begin{cases} \frac{(n+1)(n-\lambda)}{n(n+1)+\lambda}, & \text{if } 0 < \lambda \leqq \frac{n(n+1)}{n+2}, \\ \frac{n^2(n+1)^2-((n+1)^2+1)\lambda^2}{2(n^2(n+1)^2-\lambda^2)}, & \text{if } \frac{n(n+1)}{n+2} \leq \lambda \leq \lambda_n. \end{cases}$$

**Theorem C** *Let* $0 < \beta \leqq 1$ *and*

$$\lambda_n' = \frac{n(n+1)\sin\frac{\pi\beta}{2}}{\sqrt{1+(n+1)^2+2(n+1)\cos\frac{\pi\beta}{2}}} \quad (n \in \mathbb{N}).$$

*If* $0 < \lambda \leqq \lambda_n'$, *then* $\mathcal{P}_n(\lambda) \subset \widetilde{\mathcal{S}}^*(\beta)$.

It is easy to verify that Theorem B and Theorem C are better than Theorem A in two different ways.

In this paper we generalize and refine the above theorems. Furthermore we find $\lambda$ such that $f(z) \in \mathcal{P}_n(\lambda)$ implies $f(z) \in \mathcal{C}(\alpha)$ ($\alpha < 1$). These results are sharp.

## 2 Main results

To derive our first result, we need the following lemma due to Hallenbeck and Ruscheweyh [4].

**Lemma** *Let* $g(z)$ *be analytic and convex univalent in* $\mathbb{U}$ *and* $f(z) = g(0) + \sum_{k=n}^{\infty} a_k z^k$ $(n \in \mathbb{N})$ *be analytic in* $\mathbb{U}$. *If* $f(z) \prec g(z)$, *then*

$$z^{-c} \int_0^z t^{c-1} f(t)\, dt \prec \frac{1}{n} z^{-\frac{c}{n}} \int_0^z t^{\frac{c}{n}-1} g(t)\, dt,$$

*where* $\operatorname{Re}(c) \geqq 0$ *and* $c \neq 0$.

Now, we derive the following.

**Theorem 1** *Let* $0 < \lambda < n(n+1)$ $(n \in \mathbb{N})$. *If* $f(z) \in \mathcal{P}_n(\lambda)$, *then*

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{n\lambda}{n(n+1) - \lambda} \quad (z \in \mathbb{U}). \tag{2.1}$$

*The bound* $\frac{n\lambda}{n(n+1)-\lambda}$ *in* (2.1) *is sharp.*

*Proof* Let

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in \mathcal{P}_n(\lambda) \quad \text{and} \quad 0 < \lambda < n(n+1) \quad (n \in \mathbb{N}).$$

Then we have

$$zf''(z) = n(n+1) a_{n+1} z^n + \cdots \prec \lambda z. \tag{2.2}$$

Applying the lemma with $c = 1$, it follows from (2.2) that

$$\frac{1}{z} \int_0^z tf''(t)\, dt \prec \frac{\lambda}{n} z^{-\frac{1}{n}} \int_0^z t^{\frac{1}{n}}\, dt,$$

which yields

$$f'(z) - \frac{f(z)}{z} \prec \frac{\lambda z}{n+1}, \tag{2.3}$$

and hence

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{\lambda}{n+1} \quad (z \in \mathbb{U}). \tag{2.4}$$

By (2.3) we can write

$$f'(z) - \frac{f(z)}{z} = \frac{\lambda w(z)}{n+1}, \tag{2.5}$$

where $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in \mathbb{U}$). Since

$$f'(z) - \frac{f(z)}{z} = na_{n+1}z^n + \cdots,$$

the function $w(z)$ in (2.5) satisfies $|w(z)| \leq |z|^n$ ($z \in \mathbb{U}$) by the Schwarz lemma. Also (2.5) leads to

$$\int_0^z \left( \frac{f'(t)}{t} - \frac{f(t)}{t^2} \right) dt = \frac{\lambda}{n+1} \int_0^z \frac{w(t)}{t}\, dt. \tag{2.6}$$

In view of (2.6), we deduce that

$$\begin{aligned} \left| \frac{f(z)}{z} - 1 \right| &= \frac{\lambda}{n+1} \left| \int_0^1 \frac{w(uz)}{u}\, du \right| \leqq \frac{\lambda}{n+1} \int_0^1 \frac{|w(uz)|}{u}\, du \\ &\leqq \frac{\lambda |z|^n}{n+1} \int_0^1 u^{n-1}\, du < \frac{\lambda}{n(n+1)} \end{aligned}$$

and so

$$\left| \frac{f(z)}{z} \right| > 1 - \frac{\lambda}{n(n+1)} > 0 \quad (z \in \mathbb{U}). \tag{2.7}$$

Now, by using (2.4) and (2.7), we find that

$$\begin{aligned} \left| \frac{zf'(z)}{f(z)} - 1 \right| &= \left| \frac{z}{f(z)} \right| \left| f'(z) - \frac{f(z)}{z} \right| \\ &< \frac{\frac{\lambda}{n+1}}{1 - \frac{\lambda}{n(n+1)}} = \frac{n\lambda}{n(n+1) - \lambda} \end{aligned}$$

for $z \in \mathbb{U}$, which shows (2.1).

For sharpness, we consider the function

$$f(z) = z + \frac{\lambda}{n(n+1)} z^{n+1} \quad (z \in \mathbb{U}) \tag{2.8}$$

for $0 < \lambda < n(n+1)$. Obviously $f(z) \in \mathcal{P}_n(\lambda)$. Furthermore we have

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\frac{\lambda}{n+1} z^n}{1 + \frac{\lambda}{n(n+1)} z^n} \right| \to \frac{n\lambda}{n(n+1) - \lambda}$$

as $z \to e^{\frac{\pi i}{n}}$. This completes the proof of Theorem 1. □

Next, we prove the following.

**Theorem 2** *Let* $0 < \lambda < n(n+1)$ $(n \in \mathbb{N})$. *Then*

$$\mathcal{P}_n(\lambda) \subset \mathcal{S}^*(\alpha),$$

*where*

$$\alpha = \alpha_n(\lambda) = \frac{(n+1)(n-\lambda)}{n(n+1)-\lambda}. \tag{2.9}$$

*The result is sharp, that is, the order $\alpha$ is best possible.*

*Proof* If $f(z) \in \mathcal{P}_n(\lambda)$ and $0 < \lambda < n(n+1)$ ($n \in \mathbb{N}$), then an application of Theorem 1 yields

$$1 - \operatorname{Re}\frac{zf'(z)}{f(z)} < \frac{n\lambda}{n(n+1)-\lambda} \quad (z \in \mathbb{U}).$$

Hence $f(z) \in \mathcal{S}^*(\alpha)$ where $\alpha = \alpha_n(\lambda)$ is given by (2.9).

For the function $f(z) \in \mathcal{P}_n(\lambda)$ defined by (2.8), we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} = \operatorname{Re}\left\{\frac{1+\frac{\lambda}{n}z^n}{1+\frac{\lambda}{n(n+1)}z^n}\right\} \to \frac{(n+1)(n-\lambda)}{n(n+1)-\lambda} = \alpha$$

as $z \to e^{\frac{\pi i}{n}}$. Therefore the order $\alpha$ cannot be increased. □

**Remark 1** Let us compare Theorem 2 with Theorem B. Clearly

$$n(n+1) > \lambda_n \quad \text{and} \quad \alpha_n(\lambda) > \beta_n(\lambda) \quad \left(0 < \lambda \leqq \frac{n(n+1)}{n+2}\right).$$

Also, for $\frac{n(n+1)}{n+2} \leqq \lambda \leqq \lambda_n$, we have

$$\begin{aligned}\alpha_n(\lambda) - \beta_n(\lambda) &= \frac{(n+1)(n-\lambda)}{n(n+1)-\lambda} - \frac{n^2(n+1)^2 - ((n+1)^2+1)\lambda^2}{2(n^2(n+1)^2-\lambda^2)} \\ &= \frac{2(n+1)(n-\lambda)(n(n+1)+\lambda) - (n^2(n+1)^2 - ((n+1)^2+1)\lambda^2)}{2(n^2(n+1)^2-\lambda^2)} \\ &= \frac{n^2(n+1-\lambda)^2}{2(n^2(n+1)^2-\lambda^2)} > 0.\end{aligned}$$

Thus we conclude that Theorem 2 extends and improves Theorem B by Ponnusamy and Singh [3].

Taking

$$\lambda = \frac{n(n+1)}{2n+1} \quad \text{and} \quad \lambda = n,$$

Theorem 2 reduces to the following.

**Corollary 1** *For $n \in \mathbb{N}$ we have*

$$\mathcal{P}_n\left(\frac{n(n+1)}{2n+1}\right) \subset \mathcal{S}^*\left(\frac{1}{2}\right) \quad \textit{and} \quad \mathcal{P}_n(n) \subset \mathcal{S}^*. \tag{2.10}$$

*The results are sharp.*

Further, applying Theorem 1, we derive the following.

**Theorem 3** *Let* $0 < \beta \leqq 1$ *and*

$$\widetilde{\lambda}_n = \frac{n(n+1)\sin\frac{\pi\beta}{2}}{n+\sin\frac{\pi\beta}{2}} \quad (n \in \mathbb{N}). \tag{2.11}$$

*If* $0 < \lambda \leqq \widetilde{\lambda}_n$, *then* $\mathcal{P}_n(\lambda) \subset \widetilde{\mathcal{S}}^*(\beta)$ *and the bound* $\widetilde{\lambda}_n$ *cannot be increased.*

*Proof* Let

$$0 < \beta \leqq 1, \qquad f(z) \in \mathcal{P}_n(\lambda) \quad \text{and} \quad 0 < \lambda \leqq \widetilde{\lambda}_n,$$

where $\widetilde{\lambda}_n$ is given by (2.11). Then $\widetilde{\lambda}_n \leqq n$ and it follows from Theorem 1 that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \frac{n\widetilde{\lambda}_n}{n(n+1)-\widetilde{\lambda}_n} = \sin\frac{\pi\beta}{2} \quad (z \in \mathbb{U}).$$

This implies that

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi\beta}{2} \quad (z \in \mathbb{U}).$$

Hence $f(z) \in \widetilde{\mathcal{S}}^*(\beta)$.

Furthermore, for the function $f \in \mathcal{P}_n(\lambda)$ defined by (2.8) and $\widetilde{\lambda}_n < \lambda < n(n+1)$, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \to \frac{n\lambda}{n(n+1)-\lambda} > \frac{n\widetilde{\lambda}_n}{n(n+1)-\widetilde{\lambda}_n} = \sin\frac{\pi\beta}{2}$$

as $z \to e^{\frac{\pi i}{n}}$. This shows that $f \notin \widetilde{\mathcal{S}}^*(\beta)$ and so the proof of Theorem 3 is completed. □

**Remark 2** Since $\widetilde{\lambda}_n > \lambda'_n$ (*cf.* Theorem C) we see that Theorem 3 is better than Theorem C by Ponnusamy and Singh [3].

Finally we discuss the following.

**Theorem 4** *Let* $0 < \lambda < n$ $(n \in \mathbb{N})$ *and* $0 < \sigma \leqq 1$. *If* $f(z) \in \mathcal{P}_n(\lambda)$, *then*

$$\operatorname{Re}\left\{\sigma\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\sigma)\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (z \in \mathbb{U}), \tag{2.12}$$

*where*

$$\alpha = \alpha_n(\sigma,\lambda) = \sigma\frac{n-(n+1)\lambda}{n-\lambda} + (1-\sigma)\frac{(n+1)(n-\lambda)}{n(n+1)-\lambda}. \tag{2.13}$$

*The result is sharp, that is, the bound* $\alpha_n(\sigma,\lambda)$ *cannot be increased.*

*Proof* Let $f(z) \in \mathcal{P}_n(\lambda)$ and $0 < \lambda < n$. Then, by (2.2) (used in the proof of Theorem 1) and the Schwarz lemma, we can write

$$zf''(z) = \lambda w(z) \quad (z \in \mathbb{U}), \tag{2.14}$$

where $w(z)$ is analytic in $\mathbb{U}$ and $|w(z)| \leq |z|^n$ $(z \in \mathbb{U})$. Further, we deduce from (2.14) that

$$f'(z) - 1 = \int_0^z f''(t)\,dt = \lambda \int_0^z \frac{w(t)}{t}\,dt = \lambda \int_0^1 \frac{w(uz)}{u}\,du,$$

which leads to

$$\begin{aligned} \left|f'(z)\right| &\geqq 1 - \lambda \int_0^1 \frac{|w(uz)|}{u}\,du \\ &> 1 - \lambda |z|^n \int_0^1 u^{n-1}\,du \\ &> 1 - \frac{\lambda}{n} > 0 \quad (z \in \mathbb{U}). \end{aligned} \tag{2.15}$$

Also, by Theorem 2, we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \frac{(n+1)(n-\lambda)}{n(n+1)-\lambda} \quad (z \in \mathbb{U}). \tag{2.16}$$

Let us define the function $g(z)$ by

$$g(z) = \sigma \left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\sigma)\frac{zf'(z)}{f(z)} - \alpha, \tag{2.17}$$

where $0 < \sigma \leqq 1$ and $\alpha$ is given by (2.13). Then $g(z)$ is analytic in $\mathbb{U}$ and

$$\begin{aligned} g(0) = 1 - \alpha &= 1 - \sigma \frac{n-(n+1)\lambda}{n-\lambda} - (1-\sigma)\frac{(n+1)(n-\lambda)}{n(n+1)-\lambda} \\ &= \sigma \frac{n\lambda}{n-\lambda} + (1-\sigma)\frac{n\lambda}{n(n+1)-\lambda} > 0. \end{aligned}$$

We claim that $\operatorname{Re} g(z) > 0$ for $z \in \mathbb{U}$. Otherwise there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re} g(z) > 0 \quad \bigl(|z| < |z_0|\bigr) \quad \text{and} \quad \operatorname{Re} g(z_0) = 0. \tag{2.18}$$

Thus, in view of (2.15)-(2.18) and (2.13), we find that

$$\begin{aligned} \sigma \left|z_0 f''(z_0)\right| &= \left|f'(z_0)\right| \left| g(z_0) + \alpha - \sigma - (1-\sigma)\frac{z_0 f'(z_0)}{f(z_0)} \right| \\ &\geqq \left|f'(z_0)\right| \left| \operatorname{Re} g(z_0) + \alpha - \sigma - (1-\sigma)\operatorname{Re} \frac{z_0 f'(z_0)}{f(z_0)} \right| \\ &> \left(1 - \frac{\lambda}{n}\right)\left(\sigma - \alpha + (1-\sigma)\frac{(n+1)(n-\lambda)}{n(n+1)-\lambda}\right) \\ &= \sigma\lambda > 0. \end{aligned}$$

This contradicts the expression (2.14). Hence, we say that $\operatorname{Re} g(z) > 0$ $(z \in \mathbb{U})$ and (2.12) is proved.

For the function $f(z) \in \mathcal{P}_n(\lambda)$ $(0 < \lambda < n)$ defined by (2.8), we get

$$\begin{aligned}
&\operatorname{Re}\left\{\sigma\left(1+\frac{zf''(z)}{f'(z)}\right)+(1-\sigma)\frac{zf'(z)}{f(z)}\right\}\\
&\quad=\sigma\left(1+\operatorname{Re}\left\{\frac{\lambda z^n}{1+\frac{\lambda}{n}z^n}\right\}\right)+(1-\sigma)\operatorname{Re}\left\{\frac{1+\frac{\lambda}{n}z^n}{1+\frac{\lambda}{n(n+1)}z^n}\right\}\\
&\quad\to\sigma\frac{n-(n+1)\lambda}{n-\lambda}+(1-\sigma)\frac{(n+1)(n-\lambda)}{n(n+1)-\lambda}=\alpha
\end{aligned}$$

as $z \to e^{\frac{\pi i}{n}}$. Therefore the bound $\alpha$ is best possible. □

Making $\sigma = 1$ in Theorem 4, we have the following.

**Corollary 2** *Let* $0 < \lambda < n$ $(n \in \mathbb{N})$. *Then*

$$\mathcal{P}_n(\lambda) \subset \mathcal{C}\left(\frac{n-(n+1)\lambda}{n-\lambda}\right). \tag{2.19}$$

*The result is sharp. In particular, for* $n \in \mathbb{N}$, *we have*

$$\mathcal{P}_n\left(\frac{n}{2n+1}\right) \subset \mathcal{C}\left(\frac{1}{2}\right), \qquad \mathcal{P}_n\left(\frac{n}{n+1}\right) \subset \mathcal{C}, \tag{2.20}$$

*and the results are sharp.*

Taking $\sigma = \frac{1}{2}$ in Theorem 4, we obtain the following.

**Corollary 3** *Let* $0 < \lambda < n$ $(n \in \mathbb{N})$. *If* $f(z) \in \mathcal{P}_n(\lambda)$, *then*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}+\frac{zf'(z)}{f(z)}\right\} > \frac{n-(n+1)\lambda}{n-\lambda}+\frac{(n+1)(n-\lambda)}{n(n+1)-\lambda} \quad (z \in \mathbb{U}). \tag{2.21}$$

*The result is sharp.*

**Competing interests**
The authors declare that there is no conflict of interests regarding the publication of this article.

**Authors' contributions**
The main idea was proposed by NX and D-GY participated in the research. All authors read and approved the final manuscript.

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**References**

1. Duren, PL: Univalent Functions. Springer, New York (1983)
2. Mocanu, PT: Two simple sufficient conditions for starlikeness. Mathematica **34**, 175-181 (1992)
3. Ponnusamy, S, Singh, V: Criteria for strongly starlike functions. Complex Var. Theory Appl. **34**, 267-291 (1997)
4. Hallenbeck, DJ, Ruscheweyh, S: Subordination by convex functions. Proc. Am. Math. Soc. **52**, 191-195 (1975)