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Some inequalities for coneigenvalues

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Abstract

In this manuscript, we present some inequalities for coneigenvalues which extend some classical relations between eigenvalues and singular values.

Keywords: coneigenvalues; conjugate normal matrices; singular values

1 Introduction

Let $M_n(\mathbb{C})$ be the space of complex $n \times n$ matrices and $\mathbb{R}^+$ be nonnegative real numbers. For any $A \in M_n(\mathbb{C})$, the conjugate transpose of A is denoted by A^*, *i.e.*, $A^* = \overline{A}^T = \overline{A^T}$. A^T stands for the transpose of A. The real part of A is denoted by $\Re(A) = \frac{A+A^*}{2}$. If $A^*A = AA^*$, we call A normal. If $A = A^*$, A is Hermitian. Here, for $A \in M_n(\mathbb{C})$, $\lambda^\downarrow(A) = (\lambda_1^\downarrow(A), \ldots, \lambda_n^\downarrow(A))$ ($\sigma^\downarrow(A) = (\sigma_1^\downarrow(A), \ldots, \sigma_n^\downarrow(A))$) represents the eigenvalues (singular values) of A in decreasing order, $\lambda_1^\downarrow(A) \geq \cdots \geq \lambda_n^\downarrow(A)$ ($\sigma_1^\downarrow(A) \geq \cdots \geq \sigma_n^\downarrow(A)$). For $a \in \mathbb{R}^+$, $[a]$ means an integral part of a.

For $A \in M_n(\mathbb{C})$, we define $B = \overline{A}A$. Thus $\lambda(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is symmetric with respect to the real axis and the negative eigenvalues of B (if any) are of even algebraic multiplicity. Therefore, the definition of coneigenvalue is given below.

Definition 1 ([1, p.301]) The coneigenvalues of $A \in M_n(\mathbb{C})$ are n scalars $\mu_1, \mu_2, \ldots, \mu_n$ obtained as follows:

1. If $\lambda_k \in \lambda(B)$ does not lie on the negative real semi-axis, then the corresponding coneigenvalue μ_k is defined as the square root of λ_k with a nonnegative real part. The multiplicity of μ_k is set equal to that of λ_k.
2. With a real negative $\lambda_k \in \lambda(B)$, we associate two conjugate purely imaginary coneigenvalues (*i.e.*, the two square roots of λ_k). The multiplicity of each coneigenvalue is set equal to half the multiplicity of λ_k.

For $A \in M_n(\mathbb{C})$, the vector of its coneigenvalues will be denoted by

$$\mu(A) = \big(\mu_1(A), \mu_2(A), \ldots, \mu_n(A)\big).$$

The definition of conjugate-normal is presented as follows.

Definition 2 ([2]) A matrix $A \in M_n(\mathbb{C})$ is said to be conjugate-normal if

$$AA^* = \overline{A^*A}.$$

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Complex symmetric, skew-symmetric, and unitary matrices are special subclasses of conjugate-normal matrices. For the properties and characterizations of this kind of matrices, readers are referred to [3].

Next, we review two known properties about coneigenvalues.

Define the matrix $\widehat{A} = \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}$.

Proposition 3 ([4]) *If $\mu_1, \mu_2, \ldots, \mu_n$ are the coneigenvalues of the matrix $A \in M_n(\mathbb{C})$, then*

$$\lambda(\widehat{A}) = \bigl(\mu(A), -\mu(A)\bigr).$$

Proposition 4 ([4]) *Let A be a conjugate-normal matrix. Then the coneigenvalues of the matrices $\frac{A+A^T}{2}$ and $\frac{A-A^T}{2}$ are the real and imaginary parts, respectively, of the coneigenvalues of A.*

2 Main results

The purpose of this paper is to extend the property of the relations between eigenvalues and singular values to that of the relations between coneigenvalues and singular values.

A celebrated result due to Fan and Hoffman [5, p.63] is given in the first lemma.

Lemma 1 ([5, p.63]) *Let $A \in M_n(\mathbb{C})$, then*

$$\lambda_j\left(\frac{A + A^*}{2}\right) \le \sigma_j(A), \quad j = 1, \ldots, n.$$

The following lemmas state the relation between eigenvalues and singular values.

Lemma 2 ([6, p.175]) *Let $A \in M_n(\mathbb{C})$ have ordered singular values $\sigma_1(A) \ge \cdots \ge \sigma_n(A) \ge 0$ and eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$ ordered so that $|\lambda_1(A)| \ge \cdots \ge |\lambda_n(A)|$. Then, for any real-valued function f such that $\varphi(t) = f(e^t)$ is increasing and convex on the interval $[\sigma_n(A), \sigma_1(A)]$,*

$$\sum_{i=1}^{k} f\bigl(\bigl|\lambda_i(A)\bigr|\bigr) \le \sum_{i=1}^{k} f\bigl(\sigma_i(A)\bigr) \quad \textit{for } k = 1, \ldots, n.$$

Lemma 3 ([6, p.183, Problem 14]) *Let $A \in M_n(\mathbb{C})$. $|\lambda_1(A) \cdots \lambda_k(A)| = \sigma_1(A) \cdots \sigma_k(A)$ for all $k = 1, \ldots, n$ if and only if A is normal.*

Lemma 4 ([6, p.185, Problem 17]) *Let $A \in M_n(\mathbb{C})$ be given,*

$$\bigl|\lambda_1(A) \cdots \lambda_k(A)\bigr| \le \sigma_1\bigl(A^m\bigr)^{\frac{1}{m}} \cdots \sigma_k\bigl(A^m\bigr)^{\frac{1}{m}}, \quad k = 1, \ldots, n, m = 1, 2, \ldots.$$

The following lemma is about the property of conjugate-normal.

Lemma 5 ([3, Theorem 4]) *Let $A \in M_n(\mathbb{C})$. Then A is conjugate-normal if and only if $\widehat{A}$ is normal.*

Now we extend the property of the relations between eigenvalues and singular values in the previous lemmas to that of the relations between coneigenvalues and singular values in the following theorems.

Theorem 6 *Let $A \in M_n(\mathbb{C})$. Then*

$$\mu_j\left(\frac{A+A^T}{2}\right) \le \sigma_{[\frac{j-1}{2}]+1}(A), \quad j = 1, \dots, n.$$

Proof Define

$$\lambda\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix} = \left(\mu\left(\frac{A+A^T}{2}\right), -\mu\left(\frac{A+A^T}{2}\right)\right).$$

Let $\lambda^{\downarrow}\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix}$ be the vector obtained by rearranging the coordinates of $\lambda\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix}$ in decreasing order. That is,

$$\lambda^{\downarrow}\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix} = \left(\lambda_1^{\downarrow}\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix}, \dots, \lambda_{2n}^{\downarrow}\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix}\right).$$

By Proposition 3, $\lambda^{\downarrow}\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix}$ is denoted by

$$\left(\mu_1^{\downarrow}\left(\frac{A+A^T}{2}\right), \dots, \mu_n^{\downarrow}\left(\frac{A+A^T}{2}\right), -\mu_n^{\downarrow}\left(\frac{A+A^T}{2}\right), \dots, -\mu_1^{\downarrow}\left(\frac{A+A^T}{2}\right)\right). \tag{2.1}$$

In the same way, we define the singular value vector of $\sigma\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}$ as

$$\begin{aligned}\sigma\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} &= \left(\sigma_1^{\downarrow}\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}, \dots, \sigma_{2n}^{\downarrow}\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix}\right) \\ &= \left(\sigma_1^{\downarrow}(A), \sigma_1^{\downarrow}(A), \dots, \sigma_n^{\downarrow}(A), \sigma_n^{\downarrow}(A)\right). \end{aligned} \tag{2.2}$$

Therefore,

$$\begin{aligned}\lambda_j\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{A+A^T}{2} & 0 \end{pmatrix} &= \lambda_j\begin{pmatrix} 0 & \frac{A+A^T}{2} \\ \frac{\overline{A+A^*}}{2} & 0 \end{pmatrix} \\ &= \lambda_j\left[\frac{1}{2}\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \overline{A}^* \\ A^* & 0 \end{pmatrix}\right] \\ &\le \sigma_j\begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \quad \text{(by Lemma 1)}, j = 1, \dots, 2n. \end{aligned}$$

So the component of the vector in (2.1) is less than that of the vector in (2.2), respectively.

That is,

$$\mu_j\left(\frac{A+A^T}{2}\right) \le \sigma_{[\frac{j-1}{2}]+1}(A), \quad j = 1, \dots, n. \qquad \square$$

Theorem 7 *Let $A \in M_n(\mathbb{C})$ and a real-valued function f be such that the function $\varphi(t) = f(e^t)$ is increasing and convex on the interval $[\sigma_n(A), \sigma_1(A)]$ where singular values of A are* *ordered by* $\sigma_1(A) \geq \cdots \geq \sigma_n(A) \geq 0$, *then*

$$\sum_{i=1}^{k} f\bigl(\bigl|\mu_{2i-1}(A)\bigr|\bigr) \leq \sum_{i=1}^{k} f\bigl(\sigma_{2i-1}(A)\bigr), \quad k = 1, \ldots, n.$$

Proof For the vector $\lambda(\widehat{A}) = \lambda\bigl(\begin{smallmatrix} 0 & A \\ \overline{A} & 0 \end{smallmatrix}\bigr) = (\mu(A), -\mu(A))$, let $|\lambda^{\downarrow}(\widehat{A})|$ be the vector obtained by rearranging the coordinates of $|\lambda(\widehat{A})|$ in decreasing order. Thus

$$\bigl|\lambda^{\downarrow}(\widehat{A})\bigr| = \bigl(\bigl|\lambda_1^{\downarrow}(\widehat{A})\bigr|, \ldots, \bigl|\lambda_{2n}^{\downarrow}(\widehat{A})\bigr|\bigr).$$

By Proposition 3, $|\lambda^{\downarrow}(\widehat{A})|$ is denoted by

$$\bigl(\bigl|\mu_1^{\downarrow}(A)\bigr|, \bigl|\mu_1^{\downarrow}(A)\bigr|, \bigl|\mu_2^{\downarrow}(A)\bigr|, \bigl|\mu_2^{\downarrow}(\widehat{A})\bigr|, \ldots, \bigl|\mu_n^{\downarrow}(A)\bigr|, \bigl|\mu_n^{\downarrow}(A)\bigr|\bigr). \tag{2.3}$$

In the same way, we define the singular value vector of $\sigma\bigl(\begin{smallmatrix} 0 & A \\ \overline{A} & 0 \end{smallmatrix}\bigr)$ as

$$\bigl(\sigma_1^{\downarrow}(A), \sigma_1^{\downarrow}(A), \ldots, \sigma_n^{\downarrow}(A), \sigma_n^{\downarrow}(A)\bigr). \tag{2.4}$$

By Lemma 2, we have that

$$\sum_{i=1}^{k} f\bigl(\bigl|\lambda_i^{\downarrow}(\widehat{A})\bigr|\bigr) \leq \sum_{i=1}^{k} f\bigl(\sigma_i^{\downarrow}(\widehat{A})\bigr), \quad k = 1, \ldots, 2n. \tag{2.5}$$

By (2.3) and (2.4), inequality (2.5) is equivalent to the following inequality:

$$\sum_{i=1}^{k} f\bigl(\bigl|\mu_{2i-1}^{\downarrow}(A)\bigr|\bigr) \leq \sum_{i=1}^{k} f\bigl(\sigma_{2i-1}^{\downarrow}(A)\bigr), \quad k = 1, \ldots, n. \qquad \square$$

Theorem 8 *Let* $A \in M_n(\mathbb{C})$. $|\mu_1(A) \cdots \mu_{2i-1}(A)| = \sigma_1(A) \cdots \sigma_{2i-1}(A)$ *for all* $i = 1, 2, \ldots, n$ *if and only if* A *is conjugate-normal* (*i.e.*, $AA^* = \overline{A^*A}$).

Proof $\Longleftarrow$ As A is conjugate-normal, thus by Lemma 5 we know that $\widehat{A}$ is normal: Since $\widehat{A} = \bigl(\begin{smallmatrix} 0 & A \\ \overline{A} & 0 \end{smallmatrix}\bigr)$, it is easy to obtain that

$$\begin{aligned} \widehat{A}\widehat{A}^* &= \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} \begin{pmatrix} 0 & \overline{A}^* \\ A^* & 0 \end{pmatrix} = \begin{pmatrix} AA^* & 0 \\ 0 & \overline{AA}^* \end{pmatrix} \\ &= \begin{pmatrix} \overline{A}^*\overline{A} & 0 \\ 0 & \overline{AA}^* \end{pmatrix} \quad \text{(by } A \text{ is conjugate-normal)} \end{aligned}$$

and

$$\begin{aligned} \widehat{A}^*\widehat{A} &= \begin{pmatrix} 0 & \overline{A}^* \\ A^* & 0 \end{pmatrix} \begin{pmatrix} 0 & A \\ \overline{A} & 0 \end{pmatrix} = \begin{pmatrix} \overline{A}^*\overline{A} & 0 \\ 0 & A^*A \end{pmatrix} \\ &= \begin{pmatrix} \overline{A}^*\overline{A} & 0 \\ 0 & \overline{AA}^* \end{pmatrix} \quad \text{(by } A \text{ is conjugate-normal).} \end{aligned}$$

That is, $\widehat{A}^*\widehat{A} = \widehat{A}\widehat{A}^*$. Thus, by Lemma 3, we have that

$$\left|\lambda_1(\widehat{A})\cdots\lambda_k(\widehat{A})\right| = \sigma_1(\widehat{A})\cdots\sigma_k(\widehat{A}), \quad k = 1,\ldots,2n.$$

Furthermore,

$$\left|\lambda(\widehat{A})\right| = \left(\left|\mu_1^{\downarrow}(A)\right|, \left|\mu_1^{\downarrow}(A)\right|, \left|\mu_2^{\downarrow}(A)\right|, \left|\mu_2^{\downarrow}(\widehat{A})\right|, \ldots, \left|\mu_n^{\downarrow}(A)\right|\right), \tag{2.6}$$

$$\begin{aligned}\sigma\begin{pmatrix} 0 & A \\ \overline{A} & 0\end{pmatrix} &= \left(\sigma_1^{\downarrow}\begin{pmatrix} 0 & A \\ \overline{A} & 0\end{pmatrix}, \ldots, \sigma_{2n}^{\downarrow}\begin{pmatrix} 0 & A \\ \overline{A} & 0\end{pmatrix}\right) \\ &= \left(\sigma_1^{\downarrow}(A), \sigma_1^{\downarrow}(A), \ldots, \sigma_n^{\downarrow}(A), \sigma_n^{\downarrow}(A)\right). \end{aligned}\tag{2.7}$$

So, by (2.6) and (2.7), we have that

$$\left|\mu_1(A)\cdots\mu_{2i-1}(A)\right| = \sigma_1(A)\cdots\sigma_{2i-1}(A), \quad i = 1,2,\ldots,n.$$

$\Longrightarrow$ By Lemma 3 and Lemma 5, the result is obvious. □

Theorem 9 *Let* $A \in M_n(\mathbb{C})$. $|\mu_1(A)\cdots\mu_{2i-1}(A)| \le \sigma_1(A^m)^{\frac{1}{m}}\cdots\sigma_{2i-1}(A^m)^{\frac{1}{m}}$ *for all* $i = 1,2,\ldots,n$, $m = 1,2,\ldots$.

Proof Let

$$\left|\lambda(\widehat{A})\right| = \left(\left|\mu_1^{\downarrow}(A)\right|, \left|\mu_1^{\downarrow}(A)\right|, \left|\mu_2^{\downarrow}(A)\right|, \left|\mu_2^{\downarrow}(\widehat{A})\right|, \ldots, \left|\mu_n^{\downarrow}(A)\right|\right) \tag{2.8}$$

and

$$\sigma\begin{pmatrix} 0 & A \\ \overline{A} & 0\end{pmatrix} = \left(\sigma_1^{\downarrow}(A), \sigma_1^{\downarrow}(A), \ldots, \sigma_n^{\downarrow}(A), \sigma_n^{\downarrow}(A)\right).$$

By Lemma 4, we have that

$$\left|\lambda_1(\widehat{A})\cdots\lambda_k(\widehat{A})\right| \le \sigma_1\left(\widehat{A}^m\right)^{\frac{1}{m}}\cdots\sigma_k\left(\widehat{A}^m\right)^{\frac{1}{m}} \tag{2.9}$$

for all $k = 1,\ldots,2n$, $m = 1,2\ldots$.

By (2.8) and (2.9), we have that

$$\left|\mu_1(A)\cdots\mu_{2i-1}(A)\right| \le \sigma_1\left(A^m\right)^{\frac{1}{m}}\cdots\sigma_{2i-1}\left(A^m\right)^{\frac{1}{m}}$$

for all $i = 1,2,\ldots,n$; $m = 1,2,\ldots$. □

The relations between eigenvalues or singular values are very active. It is expected that more results on coneigenvalues will be attractive in the future.

Competing interests

The author declares that they have no competing interests.

Author?s contributions
The author has read the manuscript carefully.

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