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Some conditions for a class of functions to be completely monotonic

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Abstract
In this article, we present a necessary condition and a necessary and sufficient condition for a class of functions to be completely monotonic.
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1 Introduction and main results

Recall [1] that a function f is said to be completely monotonic on

$$\mathbb{R}^+ := (0, \infty)$$

if f has derivatives of all orders on $\mathbb{R}^+$ and for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in \mathbb{R}^+.$$

Here and throughout the paper, $\mathbb{N}$ is the set of all positive integers. The set of all completely monotonic functions on $\mathbb{R}^+$ is denoted by $CM(\mathbb{R}^+)$.

Bernstein [2] proved that a function f on the interval $\mathbb{R}^+$ is completely monotonic if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t).$$

Also recall [3] that a positive function f is said to be logarithmically completely monotonic on $\mathbb{R}^+$ if f has derivatives of all orders on $\mathbb{R}^+$ and for all $n \in \mathbb{N}$

$$(-1)^n \bigl[\ln f(x)\bigr]^{(n)} \geq 0, \quad x \in \mathbb{R}^+.$$

The class of all logarithmically completely monotonic functions on $\mathbb{R}^+$ is denoted by $LCM(\mathbb{R}^+)$.

It was proved [4] that a logarithmically completely monotonic function is also completely monotonic.

There is a rich literature on completely monotonic, logarithmically completely monotonic functions and their applications. For more recent work, see, for example, [5–28].

The Euler gamma function is defined and denoted for $\operatorname{Re} z > 0$ by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t}\, dt.$$

The logarithmic derivative of $\Gamma(z)$, denoted by

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)},$$

is called the psi or digamma function, and the $\psi^{(k)}$ for $k \in \mathbb{N}$ are called the polygamma functions.

In this article, we give two necessary conditions and a necessary and sufficient condition for a class of functions

$$f_{a,b,c}(x) := (x+a)\ln x - x - \ln\Gamma(x+b) + c, \quad x \in \mathbb{R}^+, \tag{1}$$

where $a, c \in \mathbb{R}$, $b \geq 0$ are parameters, to be completely monotonic. The main results are as follows.

Theorem 1 *A necessary condition for the function $f_{a,b,c}(x)$ to be completely monotonic on the interval $(0, \infty)$ is that*

$$b - a = \frac{1}{2}, \tag{2}$$

$$0 < b \leq \frac{1}{2}, \tag{3}$$

and

$$c \geq \ln\sqrt{2\pi}. \tag{4}$$

Corollary 1 *A necessary condition for the function $f_{a,b,c}(x)$ to be completely monotonic on the interval $(0, \infty)$ is that*

$$-\frac{1}{2} < a \leq 0. \tag{5}$$

Theorem 2 *For*

$$b \in \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right],$$

a necessary and sufficient condition for the function $f_{a,b,c}(x)$ to be completely monotonic on the interval $(0, \infty)$ is that

$$b - a = \frac{1}{2} \tag{6}$$

and

$$c \geq \ln\sqrt{2\pi}. \tag{7}$$

2 Lemmas

We need the following lemmas to prove our main results.

Let the α be real parameters, β a non-negative parameter. Define

$$g_{\alpha,\beta}(x) := \frac{x^{x+\beta-\alpha}}{e^x\Gamma(x+\beta)}, \quad x \in \mathbb{R}^+.$$

Lemma 1 (see [11]) *If*

$$g_{\alpha,\beta} \in LCM(\mathbb{R}^+),$$

then either

$$\beta > 0 \quad \textit{and} \quad \alpha \geq \max\left\{\beta, \frac{1}{2}\right\}$$

or

$$\beta = 0 \quad \textit{and} \quad \alpha \geq 1.$$

Lemma 2 (see [7]) *Let*

$$\beta \in \left[\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2}\right].$$

If

$$\alpha \geq \frac{1}{2},$$

then

$$g_{\alpha,\beta} \in LCM(\mathbb{R}^+).$$

3 Proof of the main results

Proof of Theorem 1 If

$$f_{a,b,c} \in CM(\mathbb{R}^+),$$

then

$$f_{a,b,c}(x) \geq 0, \quad x \in \mathbb{R}^+, \tag{8}$$

and $f_{a,b,c}(x)$ is decreasing on $\mathbb{R}^+$.

It is well known that (see [29, p.47])

$$\ln\Gamma(x+\beta) = \left(x+\beta-\frac{1}{2}\right)\ln x - x + \frac{\ln(2\pi)}{2} + O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty. \tag{9}$$

Hence

$$f_{a,b,c}(x) = \left(\frac{1}{2} - b + a\right)\ln x - \ln\sqrt{2\pi} + c + O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty. \tag{10}$$

From (8) and (10), we get

$$\frac{1}{2} - b + a \geq \frac{\ln\sqrt{2\pi} - c + O(1/x)}{\ln x}, \quad \text{as } x \to \infty. \tag{11}$$

Since

$$\frac{\ln\sqrt{2\pi} - c + O(1/x)}{\ln x} \to 0, \quad \text{as } x \to \infty, \tag{12}$$

from (11) we have

$$b - a \leq \frac{1}{2}. \tag{13}$$

On the other hand, since $f_{a,b,c}(x)$ is decreasing on $\mathbb{R}^+$, from (10), we obtain

$$\left(\frac{1}{2} - b + a\right)\ln x - \ln\sqrt{2\pi} + c + O\left(\frac{1}{x}\right) \leq f_{a,b,c}(\tau), \quad \text{as } x \to \infty, \tag{14}$$

where, in (14), τ is a fixed number in $\mathbb{R}^+$.

Equation (14) is equivalent to

$$\frac{1}{2} - b + a \leq \frac{\ln\sqrt{2\pi} + O(1/x) - c + f_{a,b,c}(\tau)}{\ln x}, \quad \text{as } x \to \infty. \tag{15}$$

It is easy to see that

$$\frac{\ln\sqrt{2\pi} + O(1/x) - c + f_{a,b,c}(\tau)}{\ln x} \to 0, \quad \text{as } x \to \infty. \tag{16}$$

Then from (15) we have

$$b - a \geq \frac{1}{2}. \tag{17}$$

Combining (13) and (17) gives

$$b - a = \frac{1}{2}. \tag{18}$$

From (8), (10), and (18), we obtain

$$c - \ln\sqrt{2\pi} \geq O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty. \tag{19}$$

Since

$$O\left(\frac{1}{x}\right) \to 0, \quad \text{as } x \to \infty, \tag{20}$$

from (19) we have

$$c \geq \ln \sqrt{2\pi}. \tag{21}$$

We note that

$$f_{a,b,c}(x) = \ln g_{b-a,b}(x) + c. \tag{22}$$

If

$$f_{a,b,c} \in CM(\mathbb{R}^+),$$

we can verify that

$$g_{b-a,b} \in LCM(\mathbb{R}^+).$$

By Lemma 1, if

$$b > \frac{1}{2},$$

then

$$b - a \geq b > \frac{1}{2}, \tag{23}$$

which contradicts (18); if

$$b = 0,$$

by Lemma 1, we get

$$b - a \geq 1, \tag{24}$$

which is another contradiction to (18). So we have proved that

$$0 < b \leq \frac{1}{2}. \tag{25}$$

The proof of Theorem 1 is thus completed. □

Proof of Corollary 1 This follows from (2) and (3).
The proof of Corollary 1 is completed. □

Proof of Theorem 2 By Theorem 1, the condition is necessary.
On the other hand, by Lemma 2, we see that

$$g_{b-a,b} \in LCM(\mathbb{R}^+).$$

Then from (22), we have, for $n \in \mathbb{N}$,

$$(-1)^n f_{a,b,c}^{(n)}(x) \geq 0, \quad x \in \mathbb{R}^+. \tag{26}$$

In particular,

$$f'_{a,b,c}(x) \leq 0, \quad x \in \mathbb{R}^+. \tag{27}$$

Hence $f_{a,b,c}(x)$ is decreasing on $\mathbb{R}^+$.

By (9),

$$f_{a,b,c}(x) = \left(\frac{1}{2} - b + a\right) \ln x + c - \ln \sqrt{2\pi} + O\left(\frac{1}{x}\right), \quad \text{as } x \to \infty. \tag{28}$$

If

$$b - a = \frac{1}{2}$$

and

$$c \geq \ln \sqrt{2\pi},$$

from (28), we obtain

$$\lim_{x \to \infty} f_{a,b,c}(x) = c - \ln \sqrt{2\pi} \geq 0. \tag{29}$$

Therefore

$$f_{a,b,c}(x) \geq \lim_{x \to \infty} f_{a,b,c}(x) \geq 0, \quad x \in \mathbb{R}^+, \tag{30}$$

which means that (26) is also valid for $n = 0$. Hence we have proved that

$$f_{a,b,c} \in CM(\mathbb{R}^+).$$

The proof of Theorem 2 is hence completed. □

Competing interests
The author declares that he has no competing interests.

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