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A generalization of deferred Cesáro means and some of their applications

Uğur Değer* and Mehmet Küçükaslan

*Correspondence:
degpar@hotmail.com
Department of Mathematics,
Faculty of Science and Literature,
Mersin University, Mersin, 33343,
Turkey

Abstract

The deferred Cesáro transformation, which has useful properties not possessed by the Cesáro transformation, was considered by RP Agnew in 1932. The aim of this paper is to give a generalization of deferred Cesáro transformations by taking account of some well-known transformations and to handle some of their properties as well. On the other hand, we shall consider the approximation by the generalized deferred Cesáro means in a generalized Hölder metric and present some applications of the approach concerning some sequence classes.

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1 Definitions and some notations

Assume that f is a 2π-periodic function and $f \in L_p := L_p[0, 2\pi]$ for $p \geq 1$ where L_p consists of all measurable functions for which the following, denoting the L_p-norm with respect to x, is finite:

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|f(x)\right|^p \, dx \right\}^{\frac{1}{p}} < \infty.$$

Let

$$f \sim \frac{a_o}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x)$$

be the Fourier series of a function $f \in L^1$. The partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L_p$ at a point x is denoted by

$$S_n(f; x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(f; x).$$

Furthermore, a function f is said to belong to the $\operatorname{Lip}(\alpha, p)$ class for $0 < \alpha \leq 1$ and $p \geq 1$ if $\omega_p(\delta, f) = O(\delta^\alpha)$, where

$$\omega_p(\delta, f) = \sup_{|t| \leq \delta} \left\| f(\cdot + t) - f(\cdot) \right\|_p$$

is the integral modulus of continuity of $f \in L_p$.

We shall also use the notations

$$\Delta a_n = a_n - a_{n+1}, \qquad \Delta_m a(n,m) = a(n,m) - a(n,m+1).$$

In this paper we are interested in the following two statements and will proceed in these directions.

1. One of the basic problems in the theory of approximations of functions and the theory of Fourier series is to examine the degree of approximation in given function spaces by certain methods. Naturally, there arises the question how we can generalize these approximation methods. The summability methods used in approximations belong to these methods. In this sense, we will give a generalization of deferred Cesáro means which includes Woronoi-Nörlund and Riesz methods as a summability method in Section 1. We know that the Nörlund and Riesz methods generalize the well-known Cesáro method which has an important place in this theory. In Section 4, we will establish some of summability properties related to this generalization.

2. As an application of these methods in theory of Fourier series, we will consider the degree of approximation in accordance with generalized deferred Cesáro means in a generalized Hölder metric in Section 2 and present some applications of the approach concerned with some sequence classes in Section 3.

Under the outlook given above, let us start with the notation of these generalizations. Accordingly, let $a = (a_n)$ and $b = (b_n)$ be sequences of nonnegative integers with conditions

$$a_n < b_n, \quad n = 1,2,3,\dots \tag{1}$$

and

$$\lim_{n\to\infty} b_n = +\infty. \tag{2}$$

The deferred Cesáro mean, (D) (see [1]), determined by a and b is defined as

$$D_n = D_a^b = \frac{S_{a_n+1} + S_{a_n+2} + \cdots + S_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} S_k,$$

where (S_k) is a sequence of real or complex numbers. Since each D_a^b with conditions (1) and (2) satisfies the Silverman-Toeplitz conditions, every D_a^b is regular. Note that D_a^b involves, except in the case $a_n = 0$ for all n, means of deferred elements of (S_n). It is also well known that D_{n-1}^n is the identity transformation and D_0^n is the $(C,1)$ transformation. The basic properties of D_a^b can be found in [1]. By considering the deferred Cesáro means, we write the following notations with conditions (1) and (2). Let (p_n) be a sequence of positive real numbers. Then we write

$$D_a^b N_n(f;x) = \frac{1}{P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} p_{b_n-m} S_m(f;x)$$

and

$$D_a^b R_n(f;x) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m S_m(f;x),$$

where

$$P_0^{b_n-a_n-1} = \sum_{k=0}^{b_n-a_n-1} p_k \neq 0, \qquad P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0,$$

and

$$S_n(f;x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t)\, dt,$$

in which

$$D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin(\frac{t}{2})}.$$

We will call these two methods the *deferred Woronoi-Nörlund means*, $(D_a^b N, p)$, and the *deferred Riesz means*, $(D_a^b R, p)$, respectively. In the case $b_n = n$ and $a_n = 0$, the methods $D_a^b N_n(f;x)$ and $D_a^b R_n(f;x)$ give us the classically known Woronoi-Nörlund and Riesz means, respectively. Provided that $p_n = 1$ for all n (≥ 0), both of them yield the deferred Cesáro means

$$D_a^b(f;x) = \frac{1}{b_n - a_n} \sum_{m=a_n+1}^{b_n} S_m(f;x)$$

of $S_m(f,x)$. In addition to this, if $b_n = n$, $a_n = 0$, and $p_n = 1$ for these two methods, then they coincide with the Cesáro method C_1. In the event that $a_n = 0$, (b_n) is a strictly increasing sequence of positive integers with $b(0) = 0$ and $p_n = 1$, then they give us the Cesáro submethod which is obtained by deleting a set of rows from the Cesáro matrix (see [2–4]).

2 Approximation by generalized deferred Cesáro means in generalized Hölder metric

Let

$$H_\alpha = \{ f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha \},$$

where $0 < \alpha \leq 1$ and K is a positive constant, not necessarily the same at each occurrence. It is well known that H_α is a Banach space (see Prösdorff [5]) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_C + \sup_{x\neq y} \Delta^\alpha f(x,y), \tag{3}$$

where

$$\Delta^\alpha f(x,y) = \frac{|f(x) - f(y)|}{|x-y|^\alpha} \quad (x \neq y),$$

by convention $\Delta^0 f(x,y) = 0$, and

$$\|f\|_C = \sup_{x\in[-\pi,\pi]} |f(x)|.$$

The metric generated by the norm (3) on H_α is called the Hölder metric. Prösdorff has studied the degree of approximation in the Hölder metric and proved the following theorem.

Theorem A [5] *Let $f \in H_\alpha$ $(0 < \alpha \leq 1)$ and $0 \leq \beta < \alpha \leq 1$. Then*

$$\left\|\sigma_n(f) - f\right\|_\beta = O(1) \begin{cases} n^{\beta-\alpha}, & 0 < \alpha < 1; \\ n^{\beta-1}\ln n, & \alpha = 1, \end{cases}$$

where $\sigma_n(f)$ is the Cesáro means of the Fourier series of f.

The case $\beta = 0$ in Theorem A is due to Alexits [6]. Chandra obtained a generalization of Theorem A by considering the Woronoi-Nörlund transform [7]. Later, Mohapatra and Chandra considered the problem by a matrix means of the Fourier series of $f \in H_\alpha$ [8].

A generalization of the Hölder metric was given by Das *et al.* (see [9]). Accordingly, let

$$H(\alpha,p) := \left\{f \in L^p, 0 < p \leq \infty : \left\|f(x+h) - f(x)\right\|_p = O\left(|h|^\alpha\right)\right\}$$

for $0 < \alpha \leq 1$. $H(\alpha,p)$ is a Banach space for $p \geq 1$ by the following norm:

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup \frac{\|f(x+h) - f(x)\|_p}{|h|^\alpha},$$

$$\|f\|_{(0,p)} = \|f\|_p.$$

In [9], one studied the results regarding the degree of approximation by an infinite matrix means involved in the deferred Cesaro means in a generalized Hölder metric. In the case $p = \infty$, the space $H(\alpha,\infty)$ coincides with the space H_α given by Prösdorff in [5].

In this section, we shall consider the degree of approximation of $f \in H(\alpha,p)$ with respect to the norm in the space $H(\alpha,p)$ by the deferred Woronoi-Nörlund means and the deferred Riesz means of the Fourier series of the function f by taking into account the method in [9].

Theorem 2.1 *Suppose that (p_n) be a positive sequence with condition*

$$\sum_{k=a_n+1}^{b_n-1} |\Delta p_k| = O\left(|p_{b_n} - p_{a_n+1}|\right). \tag{4}$$

If $f \in H(\alpha,p)$ for $p \geq 1$ and $0 \leq \beta < \alpha \leq 1$, then

$$\left\|f - D_a^b R_n(f)\right\|_{(\beta,p)} = O(1) \begin{cases} \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta/\alpha}}{(b_n-a_n)^{\alpha-\beta}} + \Phi_p(a,b)(b_n-a_n)^{1-\alpha+\beta}, & 0 < \alpha < 1; \\ \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta}}{(b_n-a_n)^{1-\beta}} + \frac{\Phi_p(a,b)(b_n-a_n)^\beta}{(\log(b_n-a_n))^{\beta-1}}, & \alpha = 1, \end{cases} \tag{5}$$

where

$$\Phi_p(a,b) = \frac{1}{P_{a_n+1}^{b_n}}\left\{|p_{b_n} - p_{a_n+1}| + p_{b_n} + p_{a_n+1}\right\}.$$

Proof A standard computation shows that

$$S_n(f,x) - f(x) = \frac{1}{2\pi}\int_0^\pi \Psi_x(t)\left(\frac{\sin(n+\frac{1}{2})t}{\sin(\frac{t}{2})}\right)dt, \tag{6}$$

where $\Psi_x(t) := f(x+t) + f(x-t) - 2f(x)$. Taking into account (6) and the deferred Riesz mean of $S_n(f;x)$, we write

$$\begin{aligned} l_n(x) := D_a^b R_n(f;x) - f(x) &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n} p_m\bigl(S_m(f;x) - f(x)\bigr) \\ &= \int_0^\pi \Psi_x(t)\frac{1}{\pi P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n} p_m D_m(t)\,dt = \int_0^\pi \Psi_x(t)K_n(t)\,dt, \end{aligned}$$

where

$$K_n(t) := \frac{1}{\pi P_{a_n+1}^{b_n}}\sum_{m=a_n+1}^{b_n} p_m D_m(t).$$

We first note that

$$\sum_{m=a_n+1}^{b_n} p_m \sin\left(m+\frac{1}{2}\right)t = O\left(\frac{1}{t}\right)\left\{\sum_{m=a_n+1}^{b_n-1}|\Delta_m p_m| + p_{a_n+1} + p_{b_n}\right\} \tag{7}$$

by using Abel?s transformation and the Jordan inequality. Furthermore, taking into account the definition of $D_a^b R_n(f;x)$, we see that

$$\bigl|K_n(t)\bigr| = O\left(\frac{t^{-1}}{P_{a_n+1}^{b_n}}\right)\sum_{m=a_n+1}^{b_n}|p_m|(m+1)t = O(b_n+1) \tag{8}$$

and

$$K_n(t) = O\bigl(t^{-1}\bigr), \tag{9}$$

for all $0 < t \leq \pi$. An elementary calculation gives

$$\begin{aligned} &l_n(x+y) - l_n(x) \\ &\quad= \int_0^\pi \bigl(\Psi_{x+y}(t) - \Psi_x(t)\bigr)K_n(t)\,dt \\ &\quad= \int_0^\pi \bigl\{f(x+y+t) - f(x+t) + f(x+y-t) - f(x-t) - 2\bigl(f(x+y) - f(x)\bigr)\bigr\}K_n(t)\,dt \\ &\quad= \int_{-\pi}^\pi \bigl(f(x+y+t) - f(x+t)\bigr)K_n(t)\,dt - 2\int_0^\pi \bigl(f(x+y) - f(x)\bigr)K_n(t)\,dt. \end{aligned}$$

Accordingly, we have

$$\begin{aligned}\|l_n(\cdot+y)-l_n(\cdot)\|_p &\le \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|\int_{-\pi}^{\pi}\big(f(x+t+y)-f(x+t)\big)K_n(t)\,dt\right|^p dx\right\}^{1/p}\\ &\quad+\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|2\int_0^{\pi}\big(f(x+y)-f(x)\big)K_n(t)\,dt\right|^p dx\right\}^{1/p}\\ &\le \int_{-\pi}^{\pi}\left|K_n(t)\right|\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(x+t+y)-f(x+t)\right|^p dx\right\}^{1/p}dt\\ &\quad+2\int_0^{\pi}\left|K_n(t)\right|\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(x+y)-f(x)\right|^p dx\right\}^{1/p}dt\\ &= O\big(|y|^\alpha\big)\int_0^{\pi}\left|K_n(t)\right|dt =: I\end{aligned}$$

by considering $f \in H(\alpha,p)$ and the general form of the Minkowski inequality. Divide the integral I into three parts:

$$I = O\big(|y|^\alpha\big)\left(\int_0^{1/(b_n+1)}+\int_{1/(b_n+1)}^{1/(b_n-a_n)}+\int_{1/(b_n-a_n)}^{\pi}\right)\left|K_n(t)\right|dt =: I_1+I_2+I_3. \tag{10}$$

These integrals, by (8) and (9), can easily be estimated by standard methods:

$$\begin{aligned}I_1 &= O\big(|y|^\alpha(b_n+1)\big)\int_0^{1/(b_n+1)}dt = O\big(|y|^\alpha\big),\\ I_2 &= O\big(|y|^\alpha\big)\int_{1/(b_n+1)}^{1/(b_n-a_n)}\left|K_n(t)\right|dt\\ &= O\big(|y|^\alpha\big)\int_{1/(b_n+1)}^{1/(b_n-a_n)}\frac{dt}{t} = O\left(|y|^\alpha\log\frac{b_n+1}{b_n-a_n}\right)\end{aligned}$$

and finally by (4) and (7) we have

$$\begin{aligned}I_3 &= |y|^\alpha\int_{1/(b_n-a_n)}^{\pi}\left|K_n(t)\right|dt = O\big(|y|^\alpha\Phi_p(a,b)\big)\int_{1/(b_n-a_n)}^{\pi}\frac{dt}{t^2}\\ &= O\big(|y|^\alpha\Phi_p(a,b)(b_n-a_n)\big).\end{aligned}$$

Therefore, collecting our estimates we obtain

$$J_1 := I_1+I_2 = O\left(|y|^\alpha\left(1+\log\frac{b_n+1}{b_n-a_n}\right)\right) \tag{11}$$

and

$$J_2 := I_3 = O\big(|y|^\alpha\Phi_p(a,b)(b_n-a_n)\big). \tag{12}$$

We need another estimation of the order of $\|l_n(\cdot+y)-l_n(\cdot)\|_p$, as different from above, to reach the required result. For this aim, we will proceed in the following way to get another one. We write

$$\begin{aligned}\left\|l_n(\cdot+y)-l_n(\cdot)\right\|_p &\le O(1)\left\{\int_0^{\pi}\left\|f(\cdot+t)-f(\cdot)\right\|_p\left|K_n(t)\right|dt\right.\\&\quad\left.+\int_0^{\pi}\left\|f(\cdot-t)-f(\cdot)\right\|_p\left|K_n(t)\right|dt\right\}\\&=O(1)\int_0^{\pi}t^{\alpha}\left|K_n(t)\right|dt:=J\end{aligned}$$

by using $\|f(\cdot+t)-f(\cdot)\|_p=O(|t|^{\alpha})$ and the generalized Minkowski inequality. Let us split the integral J into two parts:

$$J=\left(\int_0^{1/(b_n-a_n)}+\int_{1/(b_n-a_n)}^{\pi}\right)t^{\alpha}\left|K_n(t)\right|dt=:J_1+J_2.$$

J_1 and J_2 can easily be estimated for any $\alpha\le 1$ as follows. According to (9),

$$J_1=O(1)\int_0^{1/(b_n-a_n)}t^{\alpha-1}\,dt=O\left(\frac{1}{(b_n-a_n)^{\alpha}}\right),\tag{13}$$

and we have, by (4) and (7),

$$\begin{aligned}J_2&=O\big(\Phi_p(a,b)\big)\int_{1/(b_n-a_n)}^{\pi}t^{\alpha-2}\,dt\\&=O\big(\Phi_p(a,b)\big)\begin{cases}(b_n-a_n)^{1-\alpha}, & 0<\alpha<1;\\ \log(b_n-a_n), & \alpha=1.\end{cases}\end{aligned}\tag{14}$$

Combining the integral estimates (11)-(14) in the form $J_k=J_k^{\beta/\alpha}J_k^{1-\beta/\alpha}$ for $k=1,2$, we observe that

$$J_1=O\big(|y|^{\beta}\big)\left(1+\log\frac{b_n+1}{b_n-a_n}\right)^{\beta/\alpha}(b_n-a_n)^{\beta-\alpha}$$

and

$$\begin{aligned}J_2&=O\big(|y|^{\beta}\big)\Phi_p(a,b)(b_n-a_n)^{\beta/\alpha}\begin{cases}((b_n-a_n)^{1-\alpha})^{1-\beta/\alpha}, & 0<\alpha<1;\\ (\log(b_n-a_n))^{1-\beta}, & \alpha=1,\end{cases}\\&=O\big(|y|^{\beta}\big)\Phi_p(a,b)\begin{cases}(b_n-a_n)^{1+\beta-\alpha}, & 0<\alpha<1;\\ (b_n-a_n)^{\beta}(\log(b_n-a_n))^{1-\beta}, & \alpha=1.\end{cases}\end{aligned}$$

Therefore, by considering the previous evaluations, we have

$$\begin{aligned}\sup_{y\neq 0}\frac{\|l_n(\cdot+y)-l_n(\cdot)\|_p}{|y|^{\beta}}&=O(1)\left(1+\log\frac{b_n+1}{b_n-a_n}\right)^{\beta/\alpha}(b_n-a_n)^{\beta-\alpha}\\&\quad+O(1)\Phi_p(a,b)\begin{cases}(b_n-a_n)^{1+\beta-\alpha}, & 0<\alpha<1;\\ (b_n-a_n)^{\beta}(\log(b_n-a_n))^{1-\beta}, & \alpha=1.\end{cases}\end{aligned}$$

Furthermore, following the same considerations as in the estimation of (13)-(14) we get

$$\left\|D_a^b R_n(f) - f\right\|_p = O\left(\frac{1}{(b_n - a_n)^\alpha}\right) + O\bigl(\Phi_p(a,b)\bigr) \begin{cases} (b_n - a_n)^{1-\alpha}, & 0 < \alpha < 1; \\ \log(b_n - a_n), & \alpha = 1. \end{cases}$$

Collecting our partial results, (5) is obtained and this completes the proof. □

If (p_n) is nonincreasing, then it is obvious that the condition (4) is also true. That is, in the case (p_n) is nonincreasing,

$$\sum_{k=a_n+1}^{b_n-1} |\Delta p_k| = \sum_{k=a_n+1}^{b_n-1} \{p_k - p_{k+1}\} = -p_{b_n} + p_{a_n+1}$$

and

$$\Phi_p(a,b) = \frac{O(p_{a_n+1})}{P_{a_n+1}^{b_n}}.$$

Therefore we can write the following corollary of Theorem 2.1.

Corollary 2.2 *Let (p_n) be a positive and nonincreasing sequence. If $f \in H(\alpha,p)$ for $p \geq 1$ and $0 \leq \beta < \alpha \leq 1$, then*

$$\left\|f - D_a^b R_n(f)\right\|_{(\beta,p)} = O(1) \begin{cases} \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta/\alpha}}{(b_n-a_n)^{\alpha-\beta}} + \frac{p_{a_n+1}}{P_{a_n+1}^{b_n}} (b_n - a_n)^{1-\alpha+\beta}, & 0 < \alpha < 1; \\ \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta}}{(b_n-a_n)^{1-\beta}} + \frac{p_{a_n+1}}{P_{a_n+1}^{b_n}} \frac{(b_n-a_n)^\beta}{(\log(b_n-a_n))^{\beta-1}}, & \alpha = 1. \end{cases}$$

Similarly, if (p_n) is nondecreasing, then

$$\sum_{k=a_n+1}^{b_n-1} |\Delta p_k| = \sum_{k=a_n+1}^{b_n-1} \{p_{k+1} - p_k\} = p_{b_n} - p_{a_n+1}$$

and

$$\Phi_p(a,b) = \frac{O(p_{b_n})}{P_{a_n+1}^{b_n}}.$$

Taking into account this fact, we write the next result as another consequence of Theorem 2.1.

Corollary 2.3 *Let (p_n) be a positive and nondecreasing sequence. If $f \in H(\alpha,p)$ for $p \geq 1$ and $0 \leq \beta < \alpha \leq 1$, then*

$$\left\|f - D_a^b R_n(f)\right\|_{(\beta,p)} = O(1) \begin{cases} \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta/\alpha}}{(b_n-a_n)^{\alpha-\beta}} + \frac{p_{b_n}}{P_{a_n+1}^{b_n}} (b_n - a_n)^{1-\alpha+\beta}, & 0 < \alpha < 1; \\ \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta}}{(b_n-a_n)^{1-\beta}} + \frac{p_{b_n}}{P_{a_n+1}^{b_n}} \frac{(b_n-a_n)^\beta}{(\log(b_n-a_n))^{\beta-1}}, & \alpha = 1. \end{cases}$$

The next result is related to the deferred Woronoi-Nörlund means in a generalized Hölder metric.

Theorem 2.4 *Let* (p_n) *be a positive sequence such that the condition*

$$\sum_{k=a_n+1}^{b_n-1} |\Delta_k p_{b_n-k}| = O\bigl(|p_{b_n-a_n-1} - p_0|\bigr) \tag{15}$$

holds. If $f \in H(\alpha,p)$ *for* $p \geq 1$ *and* $0 \leq \beta < \alpha \leq 1$, *then*

$$\bigl\|f - D_a^b N_n(f)\bigr\|_{(\beta,p)} = O(1) \begin{cases} \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta/\alpha}}{(b_n-a_n)^{\alpha-\beta}} + \Theta_p(a,b)(b_n-a_n)^{1-\alpha+\beta}, & 0 < \alpha < 1; \\ \frac{\{1+\log(\frac{(b_n+1)}{(b_n-a_n)})\}^{\beta}}{(b_n-a_n)^{1-\beta}} + \frac{\Theta_p(a,b)(b_n-a_n)^{\beta}}{(\log(b_n-a_n))^{\beta-1}}, & \alpha = 1, \end{cases} \tag{16}$$

where

$$\Theta_p(a,b) = \frac{1}{P_0^{b_n-a_n-1}}\bigl\{|p_{b_n-a_n-1} - p_0| + p_{b_n-a_n-1} + p_0\bigr\}.$$

Proof First of all we need the following estimations given in Theorem 2.1. By considering (6) and the deferred Woronoi-Nörlund means of $S_n(f;x)$, we write

$$\begin{aligned} D_a^b N_n(f;x) - f(x) &= \frac{1}{P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} p_{b_n-m}\bigl(S_m(f;x) - f(x)\bigr) \\ &= \int_0^\pi \Psi_x(t) \frac{1}{\pi P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} p_{b_n-m} \frac{\sin(m+\frac{1}{2})t}{2\sin(\frac{t}{2})}\,dt \\ &= \int_0^\pi \Psi_x(t) B_n(t)\,dt, \end{aligned}$$

where

$$B_n(t) := \frac{1}{\pi P_0^{b_n-a_n-1}} \sum_{m=a_n+1}^{b_n} p_{b_n-m} \frac{\sin(m+\frac{1}{2})t}{2\sin(\frac{t}{2})}.$$

Moreover, we get

$$\sum_{m=a_n+1}^{b_n} p_{b_n-m} \sin\left(m + \frac{1}{2}\right)t = O\left(\frac{1}{t}\right)\left\{\sum_{m=a_n+1}^{b_n-1} |\Delta_m p_{b_n-m}| + p_{b_n-a_n-1} + p_0\right\}$$

by using Abel?s transformation and the Jordan inequality. Ultimately, an elementary calculation also gives us

$$\bigl|B_n(t)\bigr| = O\left(\frac{t^{-1}}{P_0^{b_n-a_n-1}}\right) \sum_{m=a_n+1}^{b_n} p_{b_n-m}(m+1)t = O(b_n+1)$$

and

$$B_n(t) = O\bigl(t^{-1}\bigr),$$

for all $0 < t \leq \pi$. After this, the proof runs along the same lines as that of Theorem 2.1. □

Analogous results to Corollary 2.2 and Corollary 2.3 can also be given for the deferred Nörlund means. Moreover, since $D_0^n R_n$ and $D_0^n N_n$ in the case $p_n = 1$ (for all n) coincide with the Cesáro method C_1, Theorem 2.1 and Theorem 2.4 are reduced to the result of Prösdorff in $H(\alpha, \infty)$ space. Furthermore we know that if $p_n = 1$ for all n, then these two methods give us the deferred Cesáro means. Therefore our results in Theorem 2.1 and Theorem 2.4 coincide with the results relevant to the deferred Cesáro means of Das *et al.* (see [9]) in some cases.

3 Applications related to some sequence classes

While taking into account these deferred methods, the monotonicity conditions on the sequence (p_n) are important. Especially, the sum of the left side in (4) and (15) in Theorem 2.1 and Theorem 2.4 takes over in the case we have some sequence classes as given the following, respectively. So, let us recall the definitions of some classes of numerical sequences. Some details related to these classes can be found in [10] and [11]. Let $u := (u_n)$ be a nonnegative sequence.

A sequence u is called almost monotone decreasing (briefly $u \in AMDS$) (increasing (briefly $u \in AMIS$)), if there exists a constant $K := K(u)$ which only depends on u such that

$$u_n \le K u_m \qquad (u_m \le K u_n)$$

for all $n \ge m$. This notion is due to Bernstein [12].

A sequence u is called a head bounded variation sequence (briefly $u \in HBVS$), if it has the property

$$\sum_{m=0}^{k-1} |\Delta u_m| \le K(u) u_k$$

for all natural numbers k, or only for all $k \le N$ if the sequence u has only finite nonzero terms and the last nonzero term u_N.

A sequence u tending to zero is called a rest bounded variation sequence (briefly $u \in RBVS$), if it has the property

$$\sum_{m=k}^{\infty} |\Delta u_m| \le K(u) u_k$$

for all natural numbers k (see [10]). It is clear that the following inclusions are true for the above classes of numerical sequences:

$$NIS \subset RBVS \subset AMDS, \qquad NDS \subset HBVS \subset AMIS,$$

where NIS and NDS denote the classes of numerical sequences nonincreasing and nondecreasing, respectively. Taking into account these inclusions, we will revise the results given in Section 2 by weakening the monotonicity conditions.

Theorem 3.1 *Let (p_n) be a positive sequence with condition*

$$\text{(i)}\quad \sum_{k=a_n+1}^{b_n-1} |\Delta p_k| = O(p_{a_n+1}) \quad \textit{or} \quad \text{(ii)}\quad \sum_{k=a_n+1}^{b_n-1} |\Delta p_k| = O(p_{b_n}). \tag{17}$$

If $f \in H(\alpha,p)$ for $p \geq 1$ and $0 \leq \beta < \alpha \leq 1$, then (5) *holds with $\Phi_p(a,b) = O(1)\{p_{b_n} + p_{a_n+1}\}/P_{a_n+1}^{b_n}$.*

We note that if $(p_n) \in NIS$ (or NDS), then (17)(i) (or (ii)) holds. In this case, Theorem 3.1 is reduced to Corollary 2.2 (Corollary 2.3).

If $(p_n) \in RBVS$ (or $HBVS$), then it is obvious that the condition (17)(i) (or (ii)) is also true. Therefore, we write the following result as a consequence of Theorem 3.1.

Corollary 3.2 *Let $p \geq 1$ and $0 \leq \beta < \alpha \leq 1$. If (p_n) belongs to RBVS* (*or HBVS*), *then for $f \in H(\alpha,p)$*, (5) *holds with $\Phi_p(a,b) = O(1)\{p_{b_n} + p_{a_n+1}\}/P_{a_n+1}^{b_n}$.*

Let us consider the family $\{(u_n^{(k)})\}_1^\infty$ of positive sequences such that

$$(u_n^{(k)}) \in AMIS\ (AMDS)$$

for $k = 1, 2, \ldots$. Then from the definition of $AMIS$ ($AMDS$), there exists a constant $C_k := C(u_n^{(k)})$ which only depends on $(u_n^{(k)})$ such that

$$C_k u_n^{(k)} \geq u_m^{(k)} \qquad \bigl(C_k u_m^{(k)} \geq u_n^{(k)}\bigr)$$

for all $n \geq m$ and $k = 1, 2, \ldots$. We want to build a subclass of numerical sequences that belongs to $AMIS$ ($AMDS$) to satisfy our aim. According to this, we define

$$AMIS^+ = \bigl\{u_n^{(k)} : 0 < C_k < \pi, \bigl(u_n^{(k)}\bigr) \in AMIS, k = 1, 2, \ldots\bigr\}$$
$$\bigl(AMDS^+ = \bigl\{u_n^{(k)} : 0 < C_k < \pi, \bigl(u_n^{(k)}\bigr) \in AMDS, k = 1, 2, \ldots\bigr\}\bigr).$$

It is easy to see that $NDS \subset AMIS^+$ ($NIS \subset AMDS^+$) when taking into account $C_k = 1$ for all k. Therefore if $(p_n) \in AMIS^+$ ($AMDS^+$), then there exists a number $\kappa > 0$ ($\kappa_1 > 0$) such that

$$1 = \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \leq \kappa(b_n - a_n)\frac{p_{b_n}}{P_{a_n+1}^{b_n}} \left(\kappa_1(b_n - a_n)\frac{p_{a_n+1}}{P_{a_n+1}^{b_n}}\right)$$

holds. Then if we take $\frac{\kappa p_{b_n}}{P_{a_n+1}^{b_n}}$ $(\frac{\kappa_1 p_{a_n+1}}{P_{a_n+1}^{b_n}})$ instead of $\frac{1}{b_n - a_n}$ in (10), we can write the following result, which weakens the condition of monotonicity on Corollary 2.3 (Corollary 2.2).

Corollary 3.3 *Under the conditions of Theorem* 3.1, *we have*

$$\begin{aligned}
&\bigl\|f - D_a^b R_n(f)\bigr\|_{(\beta,p)} \\
&\quad = O(1)\begin{cases} (\Lambda_{(p_n)})^{\alpha-\beta}\{\{1 + \log((b_n+1)\Lambda_{(p_n)})\}^{\beta/\alpha} + 1\}, & \alpha \in (0,1); \\ (\Lambda_{(p_n)})^{1-\beta}\{\{1 + \log((b_n+1)\Lambda_{(p_n)})\}^{\beta} + \frac{1}{(\log(\frac{1}{\Lambda_{(p_n)}}))^{\beta-1}}\}, & \alpha = 1, \end{cases}
\end{aligned}$$

where

$$\Lambda_{(p_n)} = \begin{cases} \kappa \frac{p_{b_n}}{P_{a_n+1}^{b_n}}, & (p_n) \in AMIS^+, \\ \kappa_1 \frac{p_{a_n+1}}{P_{a_n+1}^{b_n}}, & (p_n) \in AMDS^+. \end{cases}$$

Similarly to the above considerations, the deferred Woronoi-Nörlund means can be written as follows.

Theorem 3.4 *Let (p_n) be a positive sequence with condition*

$$\text{(i)} \quad \sum_{k=a_n+1}^{b_n-1} |\Delta p_{b_n-k}| = O(p_{b_n-a_n-1}) \quad or \quad \text{(ii)} \quad \sum_{k=a_n+1}^{b_n-1} |\Delta p_{b_n-k}| = O(p_0).$$

If $f \in H(\alpha,p)$ for $p \ge 1$ and $0 \le \beta < \alpha \le 1$, then (16) *holds with $\Theta_p(a,b) = O(1)\{p_{b_n-a_n-1} + p_0\}/P_0^{b_n-a_n-1}$.*

Also it is possible to give similar results to the above in the event that (p_n) belongs to *NDS* (*NIS*) and *HBVS* (*RBVS*). But we will only give the next result related to $(p_n) \in AMIS^+$ ($AMDS^+$) without further details. In the case $(p_n) \in AMIS^+$ ($AMDS^+$), there exists a number $\delta > 0$ ($\delta_1 > 0$) such that

$$1 = \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-a_n-1} \le \delta(b_n - a_n)\frac{p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}} \left(\delta_1(b_n - a_n)\frac{p_0}{P_0^{b_n-a_n-1}}\right).$$

Hence, by choosing $\frac{\delta p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}$ $(\frac{\delta_1 p_0}{P_0^{b_n-a_n-1}})$ in replacement of $\frac{1}{b_n-a_n}$ in (10), the subsequent result is as follows.

Corollary 3.5 *Under the conditions of Theorem* 3.4, *we get*

$$\begin{aligned} &\|f - D_a^b N_n(f)\|_{(\beta,p)} \\ &\quad = O(1) \begin{cases} (\Upsilon_{(p_n)})^{\alpha-\beta}\{\{1 + \log((b_n + 1)\Upsilon_{(p_n)})\}^{\beta/\alpha} + 1\}, & 0 < \alpha < 1; \\ (\Upsilon_{(p_n)})^{1-\beta}\{\{1 + \log((b_n + 1)\Upsilon_{(p_n)})\}^{\beta} + \frac{1}{(\log(\frac{1}{\Upsilon_{(p_n)}}))^{\beta-1}}\}, & \alpha = 1, \end{cases} \end{aligned}$$

where

$$\Upsilon_{(p_n)} = \begin{cases} \frac{\delta p_{b_n-a_n-1}}{P_0^{b_n-a_n-1}}, & (p_n) \in AMIS^+, \\ \frac{\delta_1 p_0}{P_0^{b_n-a_n-1}}, & (p_n) \in AMDS^+. \end{cases}$$

4 Results on generalized deferred Cesáro means

In this section, taking into account the generalized deferred Cesáro means under the perspective of the results of Agnew, we will discuss some results in connection with [1]. Before giving some results, let us recall some notations and definitions. Let $x = (x_k)$ be a sequence of real numbers and $U := (d_{n,k})$ a summability matrix. Then Ux is the sequence whose nth term is given by $\sigma_n = \sum_{k=1}^{\infty} d_{n,k}x_k$. The matrix U is regular if $\lim_n \sigma_n = p$ whenever $\lim_n x_n = p$. The well-known Silverman-Toeplitz, (ST), conditions characterize the regular matrices. According to this, the matrix U is regular if and only if it satisfies the following three conditions:

$$\lim_n d_{n,k} = 0 \quad \text{for each } k = 1,2,3,\dots; \tag{18}$$

$$\sup_n \sum_{k=1}^{\infty} |d_{n,k}| < \infty; \tag{19}$$

$$\lim_n \sum_{k=1}^{\infty} d_{n,k} = 1. \tag{20}$$

Throughout this part we are concerned with the transformations satisfying ST conditions. The Woronoi-Nörlund transformation, (N,p), and the Riesz transformation, (R,p), are obtained by taking $d_{n,k} = p_{n-k}/P_n$ and $d_{n,k} = p_k/P_n$, $n = 0,1,\dots$, $k = 0,1,\dots,n$ in σ_n where (p_n) is a given sequence of positive numbers such that $P_n = \sum_{k=0}^{n} p_k \neq 0$, $P_{-1} = p_{-1} = 0$, respectively. Since the (N,p) and (R,p) transformations satisfy the ST conditions, they are regular. Moreover, if we consider $d_{n,k} = p_{b_n-k}/P_0^{b_n-a_n-1}$ and $d_{n,k} = p_k/P_{a_n+1}^{b_n}$, $a_n < k \leq b_n$, then the deferred Woronoi-Nörlund and the deferred Riesz transformation are obtained, respectively. When considering the $(D_a^b N,p)$ and $(D_a^b R,p)$ transformations, since they satisfy the ST-conditions with (1) and (2), they are also regular.

If (a_n) and (b_n) satisfy, in addition to (1) and (2), the condition

$$\frac{a_n}{b_n - a_n} = O(1) \tag{21}$$

for all n, then (D) is properly deferred; such a transformation is called a proper (D) (see [1]). In this part, we will give a condition similar to (21) for a deferred Riesz transformation. Assume that (a_n) and (b_n) satisfy (1) and (2). Accordingly, if the condition

$$\frac{p_1 + p_2 + \cdots + p_{a_n}}{P_{a_n+1}^{b_n}} = O(1) \tag{22}$$

holds for (p_n), then we shall say that $(D_a^b R,p)$ is properly deferred and such a transformation is called a proper $(D_a^b R,p)$. We see that if $(p_n) = 1$ for all n with the conditions (1) and (2), then $(D_a^b R,p)$ and the condition (22) are reduced to (D) and the condition (21), respectively.

Theorem 4.1 $(R,p) \subset (D_a^b R,p)$ *if and only if* $(D_a^b R,p)$ *is proper.*

Proof Let (S_n) be a sequence of real numbers. Then for any $(D_a^b R,p)$ transformation of (S_n), we have

$$\begin{aligned} D_a^b R_n &= \frac{1}{P_{a_n+1}^{b_n}} \sum_{m=a_n+1}^{b_n} p_m S_m \\ &= \frac{p_1 + p_2 + \cdots + p_{b_n}}{P_{a_n+1}^{b_n}} R_{b_n} - \frac{p_1 + p_2 + \cdots + p_{a_n}}{P_{a_n+1}^{b_n}} R_{a_n}. \end{aligned} \tag{23}$$

Suppose that the relation (23) is considered as a transformation of the form σ_n which carries (R_n) into $(D_a^b R_n)$ and let $(R,p) \subset (D_a^b R,p)$. It is seen that (23) provides the conditions

(18) and (20), since the Riesz transformation is regular. In order that (23) satisfies the condition (19),

$$\frac{p_1 + p_2 + \cdots + p_{b_n}}{P_{a_n+1}^{b_n}} + \frac{p_1 + p_2 + \cdots + p_{a_n}}{P_{a_n+1}^{b_n}} = O(1)$$

is necessary and sufficient, *i.e.* (22) is a necessary and sufficient condition. Therefore, $(D_a^b R, p)$ is proper. On the other hand, assume that $(D_a^b R, p)$ is proper. Then we know that (23) satisfies the ST conditions. Moreover, since (R, p) has an inverse, (23) is regular if and only if $(R, p) \subset (D_a^b R, p)$. Hence the proof is completed. □

The next result is associated with the fact that Riesz transformations contain deferred Riesz transformations in which case we have the following.

Theorem 4.2 $(D_a^n R, p) \subset (R, p)$.

Proof Assume that the sequence (S_n) is $(D_a^n R, p)$-summable to the sum u, *i.e.*

$$\lim_{n\to\infty} D_a^b R_n = u.$$

We write

$$\begin{aligned}
R_n &= \frac{1}{P_n}\{p_1 S_1 + \cdots + p_{a_n} S_{a_n} + p_{a_n+1} S_{a_n+1} + \cdots + p_n S_n\} \\
&= \frac{1}{P_n}\{p_1 S_1 + \cdots + p_{n^{(1)}} S_{n^{(1)}}\} + \frac{P_{n^{(1)}+1}^n}{P_n} D_{n^{(1)}}^n R_n \quad \text{whenever } a_n = n^{(1)}, \\
R_n &= \frac{1}{P_n}\{p_1 S_1 + \cdots + p_{n^{(2)}} S_{n^{(2)}} + p_{n^{(2)}+1} S_{n^{(2)}+1} + \cdots + p_{n^{(1)}} S_{n^{(1)}}\} + \frac{P_{n^{(1)}+1}^n}{P_n} D_{n^{(1)}}^n R_n \\
&= \frac{1}{P_n}\{p_1 S_1 + \cdots + p_{n^{(2)}} S_{n^{(2)}}\} + \frac{P_{n^{(2)}+1}^{n^{(1)}}}{P_n} D_{n^{(2)}}^{n^{(1)}} R_{n^{(1)}} \\
&\quad + \frac{P_{n^{(1)}+1}^n}{P_n} D_{n^{(1)}}^n R_n \quad \text{whenever } a_{n^{(1)}} = n^{(2)}.
\end{aligned}$$

If we continue this until we come to some positive integer N that depends on n, then we observe that

$$\begin{aligned}
R_n = {}& \frac{P_{n^{(N+1)}+1}^{n^{(N)}}}{P_n} D_{n^{(N+1)}}^{n^{(N)}} R_{n^{(N)}} + \frac{P_{n^{(N)}+1}^{n^{(N-1)}}}{P_n} D_{n^{(N)}}^{n^{(N-1)}} R_{n^{(N-1)}} + \cdots \\
& + \frac{P_{n^{(2)}+1}^{n^{(1)}}}{P_n} D_{n^{(2)}}^{n^{(1)}} R_{n^{(1)}} + \frac{P_{n^{(1)}+1}^n}{P_n} D_{n^{(1)}}^n R_n
\end{aligned}$$

for each n whenever $n^{(N)} \geq 1$ and $n^{(N+1)} = 0$. The above relation may be considered as a transformation of the form σ_n which carries $(D_a^n R_n)$ into (R_n). This transformation provides the conditions (19) and (20) by virtue of $n^{(r)} > n^{(r+1)}$, $r = 1, 2, \ldots, N$, and $n^{(N+1)} = 0$. Furthermore, for a fixed k, the coefficient of $(D_{a_k}^k R_k)$ is either zero or a fractional expression of which the denominator is P_n and the numerator is $\leq P_k$. Therefore (18) is satisfied.

According to this, we get $\lim_{n\to\infty} R_n = u$ in view of $\lim_{n\to\infty} D_a^b R_n = u$ by considering the Silverman-Toeplitz theorem [13] and the proof is completed. □

Let $d_{n,k}$ and $c_{n,k}$ be different from each other for at most a finite set of values of n. We know that since the two transformations $\sigma_n = \sum_{k=1}^{\infty} d_{n,k}x_k$ and $\sigma_n^1 = \sum_{k=1}^{\infty} c_{n,k}x_k$ are equivalent, we write the following corollary as a result of Theorem 4.2.

Corollary 4.3 $(D_a^b R, p) \subset (R, p)$ *whenever* $b_n = n$ *for almost all* n.

Taking into account Theorem 4.1 and Theorem 4.2, we can write the next result.

Theorem 4.4 $(D_a^n R, p) \sim (R, p)$ *if and only if* $(D_a^n R, p)$ *is proper where the symbol* $\sim$ *denotes equivalence between transformations*.

In the case $p_n = 1$ for all n Theorem 4.1, Theorem 4.2, and Theorem 4.4 give us Theorem 4.1, Theorem 6.1, and Theorem 6.2 in [1], respectively.

Theorem 4.5 $(D_a^b R, p) \subset (R, p)$ *whenever* (b_n) *includes almost all positive integers*.

Proof Assume that the sequence (S_n) is $(D_a^n R, p)$-summable to the sum l, *i.e.*

$$\lim_{n\to\infty} D_a^b R_n = l$$

and we choose an integer N so large that (b_n) includes all integers greater than N. Accordingly, we put $i_1 = i_2 = \cdots = i_N = 1$ such that $b_{i_n} = n$ for each $n > N$. By virtue of $\lim_{n\to\infty} i_n = +\infty$ and $\lim_{n\to\infty} D_a^b R_n = l$ it is easily seen that $\lim_{n\to\infty} D_a^b R_{i_n} = l$. Therefore we see that (S_n) is $D_{a_{i_n}}^{b_{i_n}}$-summable to l. Hence, we obtain the required result by considering Corollary 4.3. □

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
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