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The strong convergence theorems for split common fixed point problem of asymptotically nonexpansive mappings in Hilbert spaces

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Dedicated to Professor SS Chang on the occasion of his 80th birthday.

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Abstract

In this paper, an iterative algorithm is introduced to solve the split common fixed point problem for asymptotically nonexpansive mappings in Hilbert spaces. The iterative algorithm presented in this paper is shown to possess strong convergence for the split common fixed point problem of asymptotically nonexpansive mappings although the mappings do not have semi-compactness. Our results improve and develop previous methods for solving the split common fixed point problem.

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1 Introduction and preliminaries

Throughout this paper, let H_1 and H_2 be real Hilbert spaces whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively; let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. A mapping $T: C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for any $x, y \in C$. A mapping $T: C \to C$ is said to be quasi-nonexpansive if $\|Tx - p\| \le \|x - p\|$ for any $x \in C$ and $p \in F(T)$, where $F(T)$ is the set of fixed points of T. A mapping $T: C \to C$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n\|x - y\|$ for any $x, y \in C$. A mapping $T: C \to C$ is semi-compact if, for any bounded sequence $\{x_n\} \subset C$ with $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges strongly to some point $x^* \in C$.

The split feasibility problem (*SFP*) is to find a point $q \in H_1$ with the property

$$q \in C \quad \text{and} \quad Aq \in Q, \tag{1.1}$$

where $A: H_1 \to H_2$ is a bounded linear operator.

Assuming that *SFP* (1.1) is consistent (*i.e.*, (1.1) has a solution), it is not hard to see that $x \in C$ solves (1.1) if and only if it solves the following fixed point equation:

$$x = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x, \quad x \in C, \tag{1.2}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A.

The *SFP* in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. Recently, it has been found that the *SFP* can also be used in various disciplines such as image restoration, computer tomograph, and radiation therapy treatment planning [2–7].

Let $S : H_1 \to H_1$ and $T : H_2 \to H_2$ be two mappings satisfying $F(S) = \{x \in H_1 : Sx = x\} \neq \phi$ and $F(T) = \{x \in H_2 : Tx = x\} \neq \phi$, respectively; let $A : H_1 \to H_2$ be a bounded linear operator. The split common fixed point problem (*SCFP*) for mappings S and T is to find a point $q \in H_1$ with the property

$$q \in F(S) \quad \text{and} \quad Aq \in F(T). \tag{1.3}$$

We use Γ to denote the set of solutions of *SCFP* (1.3), that is, $\Gamma = \{q \in F(S) : Aq \in F(T)\}$.

Since each closed and convex subset may be considered as a fixed point set of a projection on the subset, hence the split common fixed point problem (*SCFP*) is a generalization of the split feasibility problem (*SFP*) and the convex feasibility problem (*CFP*) [5].

Split feasibility problems and split common fixed point problems have been studied by some authors [8–15]. In 2010, Moudafi [10] proposed the following iteration method to approximate a split common fixed point of demi-contractive mappings: for arbitrarily chosen $x_1 \in H_1$,

$$\begin{cases} u_n = x_n + \gamma\beta A^*(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)u_n + \alpha_n U u_n, \quad n \in N, \end{cases}$$

and he proved that $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$, where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two demi-contractive mappings, $A : H_1 \to H_2$ is a bounded linear operator.

Using the iterative algorithm above, in 2011, Moudafi [9] also obtained a weak convergence theorem for the split common fixed point problem of quasi-nonexpansive mappings in Hilbert spaces. After that, some authors also proposed some iterative algorithms to approximate a split common fixed point of other nonlinear mappings, such as nonspreading type mappings [16], asymptotically quasi-nonexpansive mappings [12], κ-asymptotically strictly pseudononspreading mappings [17], asymptotically strictly pseudocontraction mappings [18] *etc.*, but they just obtained weak convergence theorems when those mappings do not have semi-compactness. This naturally brings us to the following question.

Can we construct an iterative scheme which can guarantee the strong convergence for split common fixed point problems without assumption of semi-compactness?

In this paper, we introduce the following iterative scheme. Let $x_1 \in H_1$, $C_1 = H_1$, the sequence $\{x_n\}$ is defined as follows:

$$\begin{cases} y_n = \alpha_n z_n + (1 - \alpha_n)T_1^n z_n, \\ z_n = x_n + \lambda A^*(T_2^n - I)Ax_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq k_n\|z_n - v\|, \|z_n - v\| \leq k_n\|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{1.4}$$

where $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ are two asymptotically nonexpansive mappings, $A : H_1 \to H_2$ is a bounded linear operator, A^* denotes the adjoint of A. Under some suitable conditions on parameters, the iterative scheme $\{x_n\}$ is shown to converge strongly to a split common fixed point of asymptotically nonexpansive mappings T_1 and T_2 without the assumption of semi-compactness on T_1 and T_2.

The following lemma and results are useful for our proofs.

Lemma 1.1 [19] *Let E be a real uniformly convex Banach space, K be a nonempty closed subset of E, and let $T : K \to K$ be an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at zero, that is, if $\{x_n\} \subset K$ converges weakly to a point $p \in K$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $p = Tp$.*

Let C be a closed convex subset of a real Hilbert space H. P_C denotes the metric projection of H onto C. It is well known that P_C is characterized by the properties: for $x \in H$ and $z \in C$,

$$z = P_C(x) \quad \Leftrightarrow \quad \langle x - z, z - y\rangle \geq 0, \quad \forall y \in C \tag{1.5}$$

and

$$\|y - P_C(x)\|^2 + \|x - P_C(x)\|^2 \leq \|x - y\|^2, \quad \forall y \in C, \forall x \in H. \tag{1.6}$$

In a real Hilbert space H, it is also well known that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall x, y \in H, \forall \lambda \in R \tag{1.7}$$

and

$$2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2, \quad \forall x, y \in H. \tag{1.8}$$

2 Main results

Theorem 2.1 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, $T_1 : H_1 \to H_1$ be an asymptotically nonexpansive mapping with the sequence $\{k_n^{(1)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(1)} = 1$, and $T_2 : H_2 \to H_2$ be an asymptotically nonexpansive mapping with the sequence $\{k_n^{(2)}\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n^{(2)} = 1$, $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$, respectively. Let $x_1 \in H_1$, $C_1 = H_1$, and let the sequence $\{x_n\}$ be defined as follows:*

$$\begin{cases} z_n = x_n + \lambda A^*(T_2^n - I)Ax_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)T_1^n z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq k_n\|z_n - v\|, \|z_n - v\| \leq k_n\|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{2.1}$$

where A^* denotes the adjoint of A, $\lambda \in (0, \frac{1}{\|A^*\|^2})$ and $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, $k_n = \max\{k_n^{(1)}, k_n^{(2)}\}$, $n \geq 1$. If $\Gamma = \{p \in F(T_1) : Ap \in F(T_2)\} \neq \emptyset$, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Proof We will divide the proof into five steps.

Step 1. We first show that C_n is closed and convex for any $n \geq 1$.

Since $C_1 = H_1$, so C_1 is closed and convex. Assume that C_n is closed and convex. For any $v \in C_n$, since

$$\begin{aligned} \|y_n - v\|^2 \leq k_n^2 \|z_n - v\|^2 \quad &\Leftrightarrow \quad \langle 2k_n^2 z_n - 2y_n - k_n^2 v + v, v\rangle \leq k_n^2 \|z_n\|^2 - \|y_n\|^2, \\ \|z_n - v\|^2 \leq k_n^2 \|x_n - v\|^2 \quad &\Leftrightarrow \quad \langle 2k_n^2 x_n - 2z_n - k_n^2 v + v, v\rangle \leq k_n^2 \|x_n\|^2 - \|z_n\|^2, \end{aligned}$$

we know that C_{n+1} is closed and convex. Therefore C_n is closed and convex for any $n \geq 1$.

Step 2. We prove $\Gamma \subset C_n$ for any $n \geq 1$.

Let $p \in \Gamma$, then from (2.1) we have

$$\begin{aligned} \|z_n - p\|^2 &= \|x_n - p + \lambda A^*(T_2^n - I)Ax_n\|^2 \\ &= \|x_n - p\|^2 + \|\lambda A^*(T_2^n - I)Ax_n\|^2 + 2\lambda\langle x_n - p, A^*(T_2^n - I)Ax_n\rangle, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &2\lambda\langle x_n - p, A^*(T_2^n - I)Ax_n\rangle \\ &\quad = 2\lambda\langle Ax_n - Ap, (T_2^n - I)Ax_n\rangle \\ &\quad = 2\lambda\langle A(x_n - p) + (T_2^n - I)Ax_n - (T_2^n - I)Ax_n, (T_2^n - I)Ax_n\rangle \\ &\quad = 2\lambda\left(\langle T_2^n Ax_n - Ap, (T_2^n - I)Ax_n\rangle - \|(T_2^n - I)Ax_n\|^2\right) \\ &\quad = 2\lambda\left(\frac{1}{2}\|T_2^n Ax_n - Ap\|^2 + \frac{1}{2}\|(T_2^n - I)Ax_n\|^2 \right. \\ &\qquad \left. - \frac{1}{2}\|Ax_n - Ap\|^2 - \|(T_2^n - I)Ax_n\|^2\right) \\ &\quad \leq 2\lambda\left(\frac{1}{2}k_n^2\|Ax_n - Ap\|^2 - \frac{1}{2}\|(T_2^n - I)Ax_n\|^2 - \frac{1}{2}\|Ax_n - Ap\|^2\right) \\ &\quad = -\lambda\|(T_2^n - I)Ax_n\|^2 + \lambda(k_n^2 - 1)\|Ax_n - Ap\|^2. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), we can obtain that

$$\begin{aligned} \|z_n - p\|^2 &\leq \|x_n - p\|^2 + \lambda^2\|A^*\|^2\|(T_2^n - I)Ax_n\|^2 - \lambda\|(T_2^n - I)Ax_n\|^2 \\ &\qquad + \lambda(k_n^2 - 1)\|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \lambda(1 - \lambda\|A^*\|^2)\|(T_2^n - I)Ax_n\|^2 + \lambda\|A\|^2(k_n^2 - 1)\|x_n - p\|^2 \\ &\leq k_n^2\|x_n - p\|^2 - \lambda(1 - \lambda\|A^*\|^2)\|(T_2^n - I)Ax_n\|^2. \end{aligned} \tag{2.4}$$

In addition, it follows from (2.1) that

$$\begin{aligned} \|y_n - p\| &= \|\alpha_n(z_n - p) + (1 - \alpha_n)(T_1^n z_n - p)\| \\ &\leq k_n\|z_n - p\|. \end{aligned} \tag{2.5}$$

Therefore, from (2.4) and (2.5), we know that $p \in C_n$ and $\Gamma \subset C_n$ for any $n \geq 1$.

Step 3. We will show that $\{x_n\}$ is a Cauchy sequence.

Since $\Gamma \subset C_{n+1} \subset C_n$ and $x_{n+1} = P_{C_{n+1}}(x_1) \subset C_n$, then

$$\|x_{n+1} - x_1\| \le \|p - x_1\| \quad \text{for } n \ge 1 \text{ and } p \in \Gamma. \tag{2.6}$$

It means that $\{x_n\}$ is bounded. For any $n \ge 1$, by using (1.6), we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 + \|x_1 - x_n\|^2 &= \|x_{n+1} - P_{C_n}(x_1)\|^2 + \|x_1 - P_{C_n}(x_1)\|^2 \\ &\le \|x_{n+1} - x_1\|^2,\end{aligned}$$

which implies that $0 \le \|x_n - x_{n+1}\|^2 \le \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2$. Thus $\{\|x_n - x_1\|\}$ is nondecreasing. Therefore, by the boundedness of $\{x_n\}$, $\lim_{n\to\infty} \|x_n - x_1\|$ exists. For some positive integers m, n with $m \le n$, from $x_n = P_{C_n}(x_1) \subset C_m$ and (1.6), we have

$$\|x_m - x_n\|^2 + \|x_1 - x_n\|^2 = \|x_m - P_{C_n}(x_1)\|^2 + \|x_1 - P_{C_n}(x_1)\|^2 \le \|x_m - x_1\|^2. \tag{2.7}$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, it follows from (2.7) that $\lim_{n\to\infty} \|x_n - x_m\| = 0$. Therefore $\{x_n\}$ is a Cauchy sequence.

Step 4. We will show that $\lim_{n\to\infty} \|z_n - T_1 z_n\| = \lim_{n\to\infty} \|Ax_n - T_2 Ax_n\| = 0$.

Since $x_{n+1} = P_{C_{n+1}}(x_1) \in C_{n+1} \subset C_n$, we have

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le (1 + k_n)\|x_{n+1} - x_n\| \to 0, \tag{2.8}$$

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \le (1 + k_n^2)\|x_{n+1} - x_n\| \to 0, \tag{2.9}$$

$$\|y_n - z_n\| \le \|y_n - x_n\| + \|x_n - z_n\| \to 0. \tag{2.10}$$

Notice that $\lambda(1 - \lambda\|A^*\|^2) > 0$, it follows from (2.4) that

$$\begin{aligned}\|(T_2^n - I)Ax_n\|^2 &\le \frac{k_n^2\|x_n - p\|^2 - \|z_n - p\|^2}{\lambda(1 - \lambda\|A^*\|^2)} \\ &\le \frac{(k_n^2 - 1)\|x_n - p\|^2 + (\|x_n - p\| + \|z_n - p\|)(\|x_n - p\| - \|z_n - p\|)}{\lambda(1 - \lambda\|A^*\|^2)} \\ &\le \frac{(k_n^2 - 1)\|x_n - p\|^2 + \|x_n - z_n\|(\|x_n - p\| + \|z_n - p\|)}{\lambda(1 - \lambda\|A^*\|^2)},\end{aligned}$$

thus, since $\{x_n\}$ is bounded and $\lim_{n\to\infty} k_n = 1$, from (2.8) we have

$$\lim_{n\to\infty} \|(T_2^n - I)Ax_n\| = 0. \tag{2.11}$$

On the other hand, since

$$\begin{aligned}\|y_n - p\|^2 &= \|\alpha_n(z_n - p) + (1 - \alpha_n)(T_1^n z_n - p)\|^2 \\ &= \alpha_n\|z_n - p\|^2 + (1 - \alpha_n)\|T_1^n z_n - p\|^2 - \alpha_n(1 - \alpha_n)\|T_1^n z_n - z_n\|^2 \\ &\le [1 + (k_n^2 - 1)]\|z_n - p\|^2 - \alpha_n(1 - \alpha_n)\|T_1^n z_n - z_n\|^2,\end{aligned}$$

we have

$$\begin{aligned}\alpha_n(1-\alpha_n)\left\|T_1^n z_n - z_n\right\|^2 &\le \|z_n - p\|^2 - \|y_n - p\|^2 + \left(k_n^2 - 1\right)\|z_n - p\|^2 \\ &\le \left(\|z_n - p\| + \|y_n - p\|\right)\|z_n - y_n\| + \left(k_n^2 - 1\right)\|z_n - p\|^2.\end{aligned}$$

Since $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$ and $\lim_{n\to\infty} k_n = 1$, we know that

$$\lim_{n\to\infty}\left\|\left(T_1^n - I\right)z_n\right\| = 0. \tag{2.12}$$

In addition, since $\|z_{n+1} - z_n\| \le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|$, we know that $\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0$. So from

$$\begin{aligned}\|z_n - T_1 z_n\| &= \left\|z_n - z_{n+1} + z_{n+1} - T_1^{n+1} z_{n+1} + T_1^{n+1} z_{n+1}\right. \\ &\qquad \left. - T_1^{n+1} z_n + T_1^{n+1} z_n - T_1 z_n\right\| \\ &\le (1 + k_{n+1})\|z_n - z_{n+1}\| + \left\|z_{n+1} - T_1^{n+1} z_{n+1}\right\| + k_1\left\|T_1^n z_n - z_n\right\|,\end{aligned}$$

we can obtain that

$$\lim_{n\to\infty}\|z_n - T_1 z_n\| = 0. \tag{2.13}$$

Similarly, we have

$$\lim_{n\to\infty}\|Ax_n - T_2 Ax_n\| = 0. \tag{2.14}$$

Step 5. We will show that $\{x_n\}$ converges strongly to an element of Γ.

Since $\{x_n\}$ is a Cauchy sequence, we may assume that $x_n \to x^*$, from (2.8) we have $z_n \to x^*$, which implies that $z_n \rightharpoonup x^*$. So it follows from (2.13) and Lemma 1.1 that $x^* \in F(T_1)$.

In addition, since A is a bounded linear operator, we have that $\lim_{n\to\infty}\|Ax_n - Ax^*\| = 0$. Hence, it follows from (2.14) and Lemma 1.1 that $Ax^* \in F(T_2)$. This means that $x^* \in \Gamma$ and $\{x_n\}$ converges strongly to $x^* \in \Gamma$. The proof is completed. □

In Theorem 2.1, as $T_1 = T_2$ and $H_1 = H_2$, we have the following result.

Corollary 2.2 *Let H_1 be a Hilbert space, $T : H_1 \to H_1$ be an asymptotically nonexpansive mapping with a sequence $\{k_n\} \subset [1,\infty)$ satisfying $\lim_{n\to\infty} k_n = 1$. The sequence $\{x_n\}$ is defined as follows: $x_1 \in H_1$, $C_1 = H_1$*

$$\begin{cases} z_n = x_n + \lambda(T^n - I)x_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)T^n z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \le k_n\|z_n - v\|, \|z_n - v\| \le k_n\|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \ge 1, \end{cases} \tag{2.15}$$

where $\lambda \in (0,1)$ and $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$ satisfies $\liminf_{n\to\infty}\alpha_n(1-\alpha_n) > 0$. If $F(T) = \{p \in H_1 : p = Tp\} \ne \emptyset$, then $\{x_n\}$ converges strongly to a fixed point x^* of T.

In Theorem 2.1, when T_1 and T_2 are two nonexpansive mappings, the following result holds.

Corollary 2.3 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, $T_1 : H_1 \to H_1$ and $T_2 : H_2 \to H_2$ be two nonexpansive mappings such that $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$, respectively. Let $x_1 \in H_1$, $C_1 = H_1$, and let the sequence $\{x_n\}$ be defined as follows*:

$$\begin{cases} z_n = x_n + \lambda A^*(T_2 - I)Ax_n, \\ y_n = \alpha_n z_n + (1 - \alpha_n)T_1 z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{2.16}$$

where A^* denotes the adjoint of A, $\lambda \in (0, \frac{1}{\|A^*\|^2})$ and $\{\alpha_n\} \subset (0, \eta] \subset (0,1)$ satisfies $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. If $\Gamma = \{p \in F(T_1) : Ap \in F(T_2)\} \neq \emptyset$, then $\{x_n\}$ converges strongly to $x^* \in \Gamma$.

Remark 2.4 When T_1 and T_2 are two quasi-nonexpansive mappings and $I - T_1$ and $I - T_2$ are demiclosed at zero, Corollary 2.3 also holds.

Example 2.5 Let C be a unit ball in a real Hilbert space l^2, and let $T_1 : C \to C$ be a mapping defined by

$$T_1 : (x_1, x_2, \ldots) \to \left(0, x_1^2, a_2 x_2, a_3 x_3, \ldots\right).$$

It is proved in Goebel and Kirk [20] that

(i) $\|T_1 x - T_1 y\| \leq 2\|x - y\|$, $\forall x, y \in C$;

(ii) $\|T_1^n x - T_1^n y\| \leq 2\prod_{j=2}^n a_j \|x - y\|$, $\forall x, y \in C$, $\forall n \geq 2$.

Taking $a_j = 2^{-\frac{1}{2^{j-1}}}$, $j \geq 2$, it is easy to see that $\prod_{j=2}^\infty a_j = \frac{1}{2}$. So we can take $k_1 = 2$, and $k_n = 2\prod_{j=2}^n a_j$, $n \geq 2$, then

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} 2\prod_{j=2}^{n} 2^{-\frac{1}{2^{j-1}}} = 1.$$

Therefore T_1 is an asymptotically nonexpansive mapping from C into itself with $F(T_1) = \{(0, 0, \ldots, 0, \ldots)\}$.

Let D be an orthogonal subspace of R^n with the norm $\|x\| = \sqrt{\sum_{i=1}^n x_i^2}$ and the inner product $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. For each $x = (x_1, x_2, \ldots, x_n) \in D$, we define a mapping $T_2 : D \to D$ by

$$T_2 x = \begin{cases} (x_1, x_2, \ldots, x_n) & \text{if } \prod_{i=1}^n x_i < 0; \\ (-x_1, -x_2, \ldots, -x_n) & \text{if } \prod_{i=1}^n x_i \geq 0. \end{cases}$$

It is easy to show that $\|T_2^n x - T_2^n y\|^2 = \|x - (-1)^n y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$ or $\|T_2^n x - T_2^n y\|^2 = \|(-1)^n x - y\|^2 = \|x\|^2 + \|y\|^2 = \|x - y\|^2$ for any $x, y \in D$. Therefore T_2 is an asymptotically nonexpansive mapping from D into itself with $F(T_2) = \{(0,0,\dots,0)\} \cup \{(x_1,x_2,\dots,x_n) : \prod_{i=1}^n x_i < 0\}$ since $\|T_2^n x - T_2^n y\|^2 \le k_n\|x-y\|^2$ for any sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$.

Obviously, C and D are closed convex subsets of l^2 and R^N, respectively. Let $A : C \to D$ be defined by $Ax = (x_1,x_2,\dots,x_n)$ for $x = (x_1,x_2,\dots) \in C$. Then A is a bounded linear operator with adjoint operator $A^*z = (x_1,x_2,\dots,x_n,0,0,\dots)$ for $z = (x_1,x_2,\dots,x_n) \in D$. Clearly, $\Gamma = \{(0,0,\dots,0,\dots)\}$, $\|A\| = \|A^*\| = 1$.

Taking $C_1 = C$, $k_1 = 2$, $k_{n+1} = 2\prod_{j=2}^{n+1} 2^{-\frac{1}{2^{j-1}}}$, $n \ge 1$, $\gamma = \frac{1}{2}$ and $\alpha_n = 0.8 - \frac{1}{2n}$, $n \ge 1$. It follows from Theorem 2.1 that $\{x_n\}$ converges strongly to $(0,0,\dots) \in \Gamma$.

3 Applications and examples

Application to the equilibrium problem

Let H be a real Hilbert space, C be a nonempty closed and convex subset of H, and let the bifunction $F : C \times C \to R$ satisfy the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;
(A2) $F(x,y) + F(y,x) \le 0$, $\forall x,y \in C$;
(A3) For all $x,y,z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \le F(x,y)$;
(A4) For each $x \in C$, the function $y \longmapsto F(x,y)$ is convex and lower semi-continuous.

The so-called equilibrium problem for F is to find a point $x^* \in C$ such that $F(x^*,x) \ge 0$ for all $y \in C$. The set of its solutions is denoted by $\mathrm{EP}(F)$.

Lemma 3.1 [21] *Let C be a nonempty closed convex subset of a Hilbert space H, and let $F : C \times C \to R$ be a bifunction satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in K$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \forall y \in C.$$

Lemma 3.2 [21] *Assume that $F : C \times C \to R$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : H \to H$ as follows*:

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C\right\}, \quad \forall x \in H.$$

Then

(1) *T_r is single-valued*;
(2) *T_r is firmly nonexpansive, that is, for all $x,y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = \mathrm{EP}(F)$;
(4) $\mathrm{EP}(F)$ *is nonempty, closed and convex.*

Theorem 3.3 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator, $T : H_2 \to H_2$ be a nonexpansive mapping, $F : H_1 \times H_1 \to R$ be a bifunction satisfying* (A1)-(A4). *Assume that $C := \mathrm{EP}(F) \neq \emptyset$ and $Q := F(T) \neq \emptyset$. Taking $C_1 = H_1$, for arbitrarily* *chosen* $x_1 \in H_1$, *the sequence* $\{x_n\}$ *is defined as follows*:

$$\begin{cases} z_n = x_n + \lambda A^*(T - I)Ax_n, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, \quad \forall y \in H_1, \\ y_n = \alpha_n z_n + (1-\alpha_n)u_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{3.1}$$

where A^* *denotes the adjoint of* A, $\{r_n\} \subset (0,\infty)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$ *and* $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *If* $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \Gamma$.

Proof It follows from Lemma 3.2 that $u_n = T_r(z_n)$, $F(T_r) = \mathrm{EP}(F)$ is nonempty, closed and convex and T_r is a firmly nonexpansive mapping. Hence all conditions in Corollary 2.3 are satisfied. The conclusion of Theorem 3.3 can be directly obtained from Corollary 2.3. □

Let E_1 and E_2 be two real Hilbert spaces. Let C be a closed convex subset of E_1, K be a closed convex subset of E_2, $A : E_1 \to E_2$ be a bounded linear operator. Assume that F is a bi-function from $C \times C$ into R and G is a bi-function from $K \times K$ into R. The split equilibrium problem (*SEP*) is to

$$\text{find an element } p \in C \text{ such that } F(p,y) \geq 0, \quad \forall y \in C \tag{3.2}$$

and

$$\text{such that } u := Ap \in C \text{ solves } G(u,v) \geq 0, \quad \forall v \in K. \tag{3.3}$$

Let $\Omega = \{p \in \mathrm{EP}(F) : Ap \in \mathrm{EP}(G)\}$ denote the solution set of the split equilibrium problem *SEP*.

Example 3.4 [22] Let $E_1 = E_2 = R$, $C := [1,+\infty)$ and $K := (-\infty,-4]$. Let $A(x) = -4x$ for all R, then A is a bounded linear operator. Let $F : C \times C \to R$ and $G : K \times K \to R$ be defined by $F(x,y) = y - x$ and $G(u,v) = 2(u - v)$, respectively. Clearly, $\mathrm{EP}(F) = \{1\}$ and $A(1) = -4 \in \mathrm{EP}(G)$. So $\Omega = \{p \in \mathrm{EP}(F) : Ap \in \mathrm{EP}(G)\} \neq \emptyset$.

Example 3.5 [22] Let $E_2 = R$ with the standard norm $|\cdot|$ and $E_1 = R^2$ with the norm $\|\alpha\| = (a_1^2 + a_2^2)^{\frac{1}{2}}$ for some $\alpha = (a_1,a_2) \in R^2$. $K := [1,+\infty)$ and $C := \{\alpha = (a_1,a_2) \in R^2 | a_2 - a_1 \geq 1\}$. Define a bi-function $F(w,\alpha) = w_1 - w_2 + a_2 - a_1$, where $w = (w_1,w_2)$, $\alpha = (a_1,a_2) \in C$, then F is a bi-function from $C \times C$ into R with $\mathrm{EP}(F) = \{p = (p_1,p_2) | p_2 - p_1 = 1\}$. For each $\alpha = (a_1,a_2) \in E_1$, let $A\alpha = a_2 - a_1$, then A is a bounded linear operator from E_1 into E_2. In fact, it is also easy to verify that $A(a\alpha_1 + b\alpha_2) = aA(\alpha_1) + bA(\alpha_2)$ and $\|A\| = \sqrt{2}$ for some $\alpha_1,\alpha_2 \in E_1$ and $a,b \in R$. Now define another bi-function G as follows: $G(u,v) = v - u$ for all $u,v \in K$. Then G is a bi-function from $K \times K$ into R with $\mathrm{EP}(G) = \{1\}$.

Clearly, when $p \in \mathrm{EP}(F)$, we have $Ap = 1 \in \mathrm{EP}(G)$. So $\Omega = \{p \in \mathrm{EP}(F) : Ap \in \mathrm{EP}(G)\} \neq \emptyset$.

Corollary 3.6 *Let* H_1 *and* H_2 *be two Hilbert spaces*, $A : H_1 \to H_2$ *be a bounded linear operator*, $F : H_1 \times H_1 \to R$ *be a bifunction satisfying* $\mathrm{EP}(F) \neq \emptyset$ *and* $G : H_2 \times H_2 \to R$ *be a* *bifunction satisfying* $\mathrm{EP}(G) \neq \emptyset$. *Taking* $C_1 = H_1$, *for arbitrarily chosen* $x_1 \in H_1$, *the sequence* $\{x_n\}$ *is defined as follows*:

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H_1, \\ G(v_n, z) + \frac{1}{r_n}\langle z - v_n, v_n - Au_n\rangle \geq 0, \quad \forall z \in H_2, \\ z_n = u_n + \lambda A^*(T_{r_n}^G - I)Au_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)T_{r_n}^F x_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{3.4}$$

where A^* *denotes the adjoint of* A, $\{r_n\} \subset (0, \infty)$, $\lambda \in (0, \frac{1}{\|A^*\|^2})$ *and* $\{\alpha_n\} \subset (0, \eta] \subset (0,1)$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *If* $\Omega = \{p \in \mathrm{EP}(F) : Ap \in \mathrm{EP}(G)\} \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges strongly to a point* $x^* \in \Omega$.

Remark 3.7 Since Example 3.4 and Example 3.5 satisfy the conditions of Corollary 2.3, the split equilibrium problems in Example 3.4 and Example 3.5 can be solved by algorithm (3.4).

Application to the hierarchial variational inequality problem

Let H be a real Hilbert space, T_1 and T_2 be two nonexpansive mappings from H to H such that $F(T_1) \neq \emptyset$ and $F(T_2) \neq \emptyset$.

The so-called hierarchical variational inequality problem for nonexpansive mapping T_1 with respect to a nonexpansive mapping $T_2 : H \to H$ is to find a point $x^* \in F(T_1)$ such that

$$\langle x^* - T_2 x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T_1). \tag{3.5}$$

It is easy to see that (3.5) is equivalent to the following fixed point problem:

$$\text{find } x^* \in F(T_1) \text{ such that } x^* = P_{F(T_1)} T_2 x^*, \tag{3.6}$$

where $P_{F(T_1)}$ is the metric projection from H onto $F(T_1)$. Letting $C := F(T_1)$ and $Q := F(P_{F(T_1)}T_2)$ (the fixed point set of the mapping $P_{F(T_1)}T_2$) and $A = I$ (the identity mapping on H), then problem (3.6) is equivalent to the following split feasibility problem:

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{3.7}$$

Hence from Theorem 2.1 we have the following theorem.

Theorem 3.8 *Let* H, T_1, T_2, C *and* Q *be the same as above. Let* $x_1 \in H_1$ *and* $C_1 = H_1$, *and let the sequence* $\{x_n\}$ *be defined as follows*:

$$\begin{cases} z_n = x_n + \lambda(T_2 - I)x_n, \\ y_n = \alpha_n z_n + (1-\alpha_n)T_1 z_n, \\ C_{n+1} = \{v \in C_n : \|y_n - v\| \leq \|z_n - v\| \leq \|x_n - v\|\}, \\ x_{n+1} = P_{C_{n+1}}(x_1), \quad n \geq 1, \end{cases} \tag{3.8}$$

where $\lambda \in (0,1)$ *and* $\{\alpha_n\} \subset (0,\eta] \subset (0,1)$ *satisfies* $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. *If* $C \cap Q \neq \emptyset$, *then the sequence* $\{x_n\}$ *converges strongly to a solution of the hierarchical variational inequality problem* (3.5).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this work. All authors read and approved the final manuscript.

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