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Some properties of the interval-valued $\overline{g}$-integrals and a standard interval-valued $\overline{g}$-convolution

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Abstract

Pap and Stajner (Fuzzy Sets Syst. 102:393-415, 1999) investigated a generalized pseudo-convolution of functions based on pseudo-operations. Jang (Fuzzy Sets Syst. 222:45-57, 2013) studied the interval-valued generalized fuzzy integral by using an interval-representable pseudo-multiplication.

In this paper, by using the concepts of interval-representable pseudo-multiplication and g-integral, we define the interval-valued $\overline{g}$-integral represented by its interval-valued generator $\overline{g}$ and a standard interval-valued $\overline{g}$-convolution by means of the corresponding interval-valued $\overline{g}$-integral. We also investigate some characterizations of the interval-valued $\overline{g}$-integral and a standard interval-valued $\overline{g}$-convolution.

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1 Introduction

Benvenuti and Mesiar [1], Daraby [2], Deschrijver [3], Grbic *et al.* [4], Klement *et al.* [5], Mesiar *et al.* [6], Stajner-Papuga *et al.* [7], Sugeno [8], Sugeno and Murofushi [9], Wu *et al.* [10, 11] have been studying pseudo-multiplications and various pseudo-integrals of measurable functions. Markova and Stupnanova [12], Maslov and Samborskij [13], and Pap and Stajner [14] introduced a general notion of pseudo-convolution of functions based on pseudo-mathematical operations and investigated the idempotent with respect to a pseudo-convolution.

Many researchers [3, 4, 15–29] have studied the pseudo-integral of measurable multi-valued function, for example, the Aumann integral, the fuzzy integral, and the Choquet integral of measurable interval-valued functions, in many different mathematical theories and their applications.

Recently, Jang [26] defined the interval-valued generalized fuzzy integral by using an interval-representable pseudo-multiplication and investigated their characterizations. The purpose of this study is to define the interval-valued $\overline{g}$-integral represented by its interval-valued generator $\overline{g}$ and a standard interval-valued $\overline{g}$-convolution by means of the corresponding interval-valued $\overline{g}$-integral, and to investigate an interval-valued idempotent function with respect to a standard interval-valued $\overline{g}$-convolution.

This paper is organized in five sections. In Section 2, we list definitions and some properties of a pseudo-addition, a pseudo-multiplication, a g-integral, and a g-convolution of functions by means of the corresponding g-integral. In Section 3, we define an interval-representable pseudo-addition, an interval-representable pseudo-multiplication, the interval-valued $\overline{g}$-integral represented by its interval-valued generator $\overline{g}$, and investigate some characterizations of the interval-valued $\overline{g}$-integral. In Section 4, we define a standard interval-valued $\overline{g}$-convolution by means of the corresponding interval-valued $\overline{g}$-integral and investigate some basic characterizations of them. In Section 5, we give a brief summary of results and some conclusions.

2 Definitions and preliminaries

Let X be a set, $\mathcal{A}$ be a σ-algebra of X, and $\mathfrak{F}(X)$ be a set of all measurable functions $f : X \longrightarrow [0,\infty)$. We introduce a pseudo-addition and a pseudo-multiplication (see [1–4, 6, 7, 12, 14, 26, 30]).

Definition 2.1 ([12]) (1) A binary operation $\oplus : [0,\infty]^2 \longrightarrow [0,\infty]$ is called a pseudo-addition if it satisfies the following axioms:

(i) $x \oplus y = y \oplus x$ for all $x, y \in [0,\infty]$,
(ii) $x \leq y \Longrightarrow x \oplus z \leq y \oplus z$ for all $x, y, z \in [0,\infty]$,
(iii) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ for all $x, y, z \in [0,\infty]$,
(iv) $\exists \mathbf{0} \in [0,\infty]$ such that $x \oplus \mathbf{0} = x$ for all $x \in [0,\infty]$,
(v) $x_n \longrightarrow x, y_n \longrightarrow y \Longrightarrow x_n \oplus y_n \longrightarrow x \oplus y$.

(2) A binary operation $\odot : [0,\infty]^2 \longrightarrow [0,\infty]$ is called a pseudo-multiplication with respect to $\oplus$ if it satisfies the following axioms:

(i) $x \odot y = y \odot x$ for all $x, y \in [0,\infty]$,
(ii) $x \odot (y \odot z) = x \odot (y \odot z)$ for all $x, y, z \in [0,\infty]$,
(iii) $\exists \mathbf{1} \in [0,\infty]$ such that $x \odot \mathbf{1} = x$ for all $x \in [0,\infty]$,
(iv) $(x \odot y) \oplus z = (x \odot y) \oplus (x \odot z)$ for all $x, y, z \in [0,\infty]$,
(v) $x \odot \mathbf{0} = \mathbf{0}$ for all $x \in [0,\infty]$,
(vi) $x \leq y \Longrightarrow x \odot z \leq y \odot z$ for all $x, y, z \in [0,\infty]$.

Remark 2.1 ([6, 12, 14]) If $g : [0,\infty] \longrightarrow [0,\infty]$ is a generating function for a semigroup $([0,\infty], \oplus, \odot)$, then the pseudo-operations are of the following forms:

$$x \oplus y = g^{-1}\big(g(x) + g(y)\big) \tag{1}$$

and

$$x \odot y = g^{-1}\big(g(x)g(y)\big). \tag{2}$$

Definition 2.2 ([6]) A set function $\mu : \mathcal{A} \longrightarrow [0,\infty]$ is called a $\sigma - \oplus$-measure if it satisfies the following axioms:

(i) $\mu(\emptyset) = 0$,
(ii) $\mu(\bigcup_{i=1}^{\infty} A)_i = \bigoplus_{i=1}^{\infty} \mu(A_i)$ for any sequence $\{A_i\}$ of pairwise disjoint sets from $\mathcal{A}$, where $\bigoplus_{i=1}^{\infty} x_i = \lim_{n\to\infty} \bigoplus_{i=1}^{n} x_i$.

Let $\mathfrak{F}(X)$ be the set of all measurable functions $f : X \longrightarrow [0,\infty)$. We introduce the g-integral with respect to a fuzzy measure induced by a pseudo-addition $\oplus$ and a pseudo-multiplication $\odot$ in Remark 2.1.

Definition 2.3 ([6]) (1) Let $g : [0,\infty] \longrightarrow [0,\infty]$ be a continuous strictly monotone increasing surjection function such that $g(0) = 0$ and $f \in \mathfrak{F}(X)$. The g-integral of f on A is defined by

$$\int_A^{\oplus} f \odot d\mu = g^{-1} \int_A g\bigl(f(x)\bigr)\,dx, \tag{3}$$

where dx is related to the Lebesgue measure and the integral on the right-hand side is the Lebesgue integral.

(2) f is said to be integrable if $\int_A^{\oplus} f \odot d\mu \in [0,\infty)$.

Let $\mathfrak{F}^*(X)$ be the set of all integrable functions. Then we obtain some basic properties of the g-integral with respect to a fuzzy measure.

Theorem 2.2 (1) *If $A \in \mathcal{A}, f, h \in \mathfrak{F}^*(X)$ and $f \le h$, then we have*

$$\int_A^{\oplus} f \odot d\mu \le \int_A^{\oplus} h \odot d\mu. \tag{4}$$

(2) *Let $g : [0,\infty] \longrightarrow [0,\infty]$ be a continuous strictly monotone increasing surjection function such that $g(0) = 0$, $\oplus$, $\odot$ are the same pseudo-operations as in Remark* 2.1. *If $A \in \mathcal{A}$, $f, h \in \mathfrak{F}^*(X)$, then we have*

$$\int_A^{\oplus} (f \oplus h) \odot d\mu = \int_A^{\oplus} f \odot d\mu \oplus \int_A^{\oplus} h \odot d\mu. \tag{5}$$

(3) *Let $g : [0,\infty] \longrightarrow [0,\infty]$ be a continuous strictly monotone increasing surjection function such that $g(0) = 0$, $\oplus$, $\odot$ are the same pseudo-operations as in Remark* 2.1, *and $u \otimes v = g^{-1}(g(u)g(v))$ for $u, v \in [0,\infty)$. If $A \in \mathcal{A}$, $c \in [0,\infty)$, $h \in \mathfrak{F}^*(X)$, then we have*

$$\int_A^{\oplus} (c \otimes h) \odot d\mu = c \otimes \int_A^{\oplus} h \odot d\mu. \tag{6}$$

Proof (1) Note that if $f, h \in \mathfrak{F}^*(X)$ and $f \le h$, then

$$g\bigl(f(x)\bigr) \le g\bigl(h(x)\bigr) \quad \text{and} \quad g^{-1}\bigl(f(x)\bigr) \le g^{-1}\bigl(h(x)\bigr). \tag{7}$$

By Definition 2.3(1), (7), and the monotonicity of the Lebesgue integral,

$$\begin{aligned} \int_A^{\oplus} f \odot d\mu &= g^{-1} \int_A g\bigl(f(x)\bigr)\,dx \\ &\le g^{-1} \int_A g\bigl(h(x)\bigr)\,dx \\ &= \int_A^{\oplus} h \odot d\mu. \end{aligned} \tag{8}$$

(2) By Definition 2.3(1) and the additivity of the Lebesgue integral,

$$\begin{aligned}
\int_A^{\oplus} (f \oplus h) \odot d\mu &= g^{-1} \int_A g\bigl(g^{-1}\bigl(g\bigl(f(x)\bigr) + g\bigl(h(x)\bigr)\bigr)\bigr)\, dx \\
&= g^{-1} \int_A \bigl(g\bigl(f(x)\bigr) + g\bigl(h(x)\bigr)\bigr)\, dx \\
&= g^{-1}\biggl[\int_A g\bigl(f(x)\bigr)\, dx + \int_A g\bigl(h(x)\bigr)\, dx\biggr] \\
&= g^{-1} g g^{-1} \int_A g\bigl(f(x)\bigr)\, dx + g g^{-1} \int_A g\bigl(h(x)\bigr)\, dx \\
&= g^{-1}\biggl(g \int_A^{\oplus} f \odot d\mu + g \int_A^{\oplus} h \odot d\mu\biggr) \\
&= \int_A^{\oplus} f \odot d\mu \oplus \int_A^{\oplus} h \odot d\mu.
\end{aligned} \tag{9}$$

(3) By Definition 2.3(1) and the linearity of the Lebesgue integral,

$$\begin{aligned}
\int_A^{\oplus} (c \otimes h) \odot d\mu &= g^{-1} \int_A g\bigl(g^{-1}\bigl(g(c)g(h)\bigr)\bigr)\, dx \\
&= g^{-1}\biggl(\int_A g(c)g(h)\, dx\biggr) \\
&= g^{-1} g(c) \int_A g(h)\, dx \\
&= g^{-1} g(c) g g^{-1}\biggl(\int_A g(h)\, dx\biggr) \\
&= g^{-1} g(c) g\biggl(\int_A^{\oplus} h \odot d\mu\biggr) \\
&= c \otimes \int_A^{\oplus} h \odot d\mu.
\end{aligned} \tag{10}$$

By using the g-integral, we define the g-convolution of functions by means of the corresponding g-integral (see [2, 12–14]). □

Definition 2.4 ([14]) Let g be the same function as in Theorem 2.2, let $\oplus$, $\odot$ be the same pseudo-operations as in Remark 2.1, $u \otimes v = g^{-1}(g(u)g(v))$ for $u, v \in [0, \infty)$, and $f, h \in \mathfrak{F}^*(X)$. The g-convolution of f and h by means of the g-integral is defined by

$$(f * h)(t) = \int_{[0,t]}^{\oplus} \bigl[f(t-u) \otimes h(u)\bigr] \odot d\mu(u) \tag{11}$$

for all $t \in [0, \infty)$.

Finally, we introduce the following basic characterizations of the g-convolution in [14].

Theorem 2.3 ([14]) *If g is the same function as in Theorem* 2.2, $\oplus$, $\odot$ *are the same pseudo-operations as in Remark* 2.1, $u \otimes v = g^{-1}(g(u)g(v))$ *for* $u, v \in [0, \infty)$, *and* $f, h \in \mathfrak{F}^*(X)$, *then* *we have*

$$(f * h)(t) = g^{-1} \int_0^t g\big(f(t-u)\big)g\big(h(u)\big)\,du \tag{12}$$

for all $t \in [0,\infty)$.

Theorem 2.4 ([14]) *If* g *is the same function as in Theorem* 2.2, $\oplus$, $\odot$ *are the same pseudo-operations as in Remark* 2.1, $u \otimes v = g^{-1}(g(u)g(v))$ *for* $u, v \in [0,\infty)$, *and* $f, h, k \in \mathfrak{F}^*(X)$, *then we have*

$$f * h = h * f \tag{13}$$

and

$$(f * h) * k = f * (h * k). \tag{14}$$

3 The interval-valued $\overline{g}$-integrals

In this section, we consider the intervals, a standard interval-valued pseudo-addition, and a standard interval-valued pseudo-multiplication. Let $I(Y)$ be the set of all closed intervals (for short, intervals) in Y as follows:

$$I(Y) = \big\{\overline{a} = [a_l, a_r] \mid a_l, a_r \in Y \text{ and } a_l \le a_r\big\}, \tag{15}$$

where Y is $[0,\infty)$ or $[0,\infty]$. For any $a \in Y$, we define $a = [a,a]$. Obviously, $a \in I(Y)$ (see [1, 21–29]).

Definition 3.1 ([26]) If $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r], \overline{a}_n = [a_{nl}, a_{nr}], \overline{a}_\alpha = [a_{\alpha l}, a_{\alpha r}] \in I(Y)$ for all $n \in \mathbb{N}$ and $\alpha \in [0,\infty)$, and $k \in [0,\infty)$, then we define arithmetic, maximum, minimum, order, inclusion, superior, and inferior operations as follows:

(1) $\overline{a} + \overline{b} = [a_l + b_l, a_r + b_r]$,
(2) $k\overline{a} = [ka_l, ka_r]$,
(3) $\overline{a}\overline{b} = [a_l b_l, a_r b_r]$,
(4) $\overline{a} \vee \overline{b} = [a_l \vee b_l, a_r \vee b_r]$,
(5) $\overline{a} \wedge \overline{b} = [a_l \wedge b_l, a_r \wedge b_r]$,
(6) $\overline{a} \le \overline{b}$ if and only if $a_l \le b_l$ and $a_r \le b_r$,
(7) $\overline{a} < \overline{b}$ if and only if $a_l \le b_l$ and $a_l \ne b_l$,
(8) $\overline{a} \subset \overline{b}$ if and only if $b_l \le a_l$ and $a_r \le b_r$,
(9) $\sup_n \overline{a}_n = [\sup_n a_{nl}, \sup_n a_{nr}]$,
(10) $\inf_n \overline{a}_n = [\inf_n a_{nl}, \inf_n a_{nr}]$,
(11) $\sup_\alpha \overline{a}_\alpha = [\sup_\alpha a_{\alpha l}, \sup_\alpha a_{\alpha r}]$, and
(12) $\inf_\alpha \overline{a}_\alpha = [\inf_\alpha a_{\alpha l}, \inf_\alpha a_{\alpha r}]$.

Definition 3.2 ([26]) (1) A binary operation $\bigoplus : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called a standard interval-valued pseudo-addition if there exist pseudo-additions $\oplus_l$ and $\oplus_r$ such that $x \oplus_l y \le x \oplus_r y$ for all $x, y \in [0,\infty]$, and such that for all $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0,\infty])$,

$$\overline{a} \bigoplus \overline{b} = [a_l \oplus_l b_l, a_r \oplus_r b_r]. \tag{16}$$

Then $\oplus_l$ and $\oplus_r$ are called the representants of $\bigoplus$.

(2) A binary operation $\bigodot : I([0,\infty])^2 \longrightarrow I([0,\infty])$ is called a standard interval-valued pseudo-multiplication if there exist pseudo-multiplications $\odot_l$ and $\odot_r$ such that $x \odot_l y \leq x \odot_r y$ for all $x, y \in [0,\infty]$, and such that for all $\overline{a} = [a_l, a_r], \overline{b} = [b_l, b_r] \in I([0,\infty])$,

$$\overline{a} \bigodot \overline{b} = [a_l \odot_l b_l, a_r \odot_r b_r]. \tag{17}$$

Then $\odot_l$ and $\odot_r$ are called the representants of $\bigodot$.

Theorem 3.1 *If two pseudo-additions $\oplus_l$ and $\oplus_r$ are representants of a standard interval-valued pseudo-addition $\bigoplus$, two pseudo-multiplications $\odot_l$ and $\odot_r$ are representants of a standard interval-valued pseudo-multiplication $\bigodot$, then we have*

(1) $\overline{x} \bigoplus \overline{y} = \overline{y} \bigoplus \overline{x}$ *for all* $\overline{x}, \overline{y} \in I([0,\infty])$,
(2) $(\overline{x} \bigoplus \overline{y}) \bigoplus \overline{z} = \overline{x} \bigoplus (\overline{y} \bigoplus \overline{z})$ *for all* $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$,
(3) $\overline{x} \bigodot \overline{y} = \overline{y} \bigodot \overline{x}$ *for all* $\overline{x}, \overline{y} \in I([0,\infty])$,
(4) $(\overline{x} \bigodot \overline{y}) \bigodot \overline{z} = \overline{x} \bigodot (\overline{y} \bigodot \overline{z})$ *for all* $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$,
(5) $\overline{x} \bigodot (\overline{y} \bigoplus \overline{z}) = (\overline{x} \bigodot \overline{y}) \bigoplus (\overline{x} \bigodot \overline{z})$ *for all* $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$.

Proof (1) By the commutativity of $\oplus_l$ and $\oplus_r$, for any $\overline{x}, \overline{y} \in I([0,\infty])$, we have

$$\begin{aligned} \overline{x} \bigoplus \overline{y} &= [x_l \oplus_l y_l, x_r \oplus_r y_r] \\ &= [y_l \oplus_l x_l, y_r \oplus_r x_r] \\ &= \overline{y} \bigoplus \overline{x}. \end{aligned} \tag{18}$$

(2) By the associativity of $\oplus_l$ and $\oplus_r$, for any $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$, we have

$$\begin{aligned} (\overline{x} \bigoplus \overline{y}) \bigoplus \overline{z} &= [x_l \oplus_l y_l, x_r \oplus_r y_r] \bigoplus [z_l, z_r] \\ &= \big[(x_l \oplus_l y_l) \oplus_l z_l, (x_r \oplus_r y_r) \oplus_r z_r\big] \\ &= \big[x_l \oplus_l (y_l \oplus_l z_l), x_r \oplus_r (y_r \oplus_r z_r)\big] \\ &= [x_l, x_r] \bigoplus [y_l \oplus_l x_l, y_r \oplus_r x_r] \\ &= \overline{x} \bigoplus (\overline{y} \bigoplus \overline{z}). \end{aligned} \tag{19}$$

(3) By the commutativity of $\odot_l$ and $\odot_r$, for any $\overline{x}, \overline{y} \in I([0,\infty])$, we have

$$\begin{aligned} \overline{x} \bigodot \overline{y} &= [x_l \odot_l y_l, x_r \odot_r y_r] \\ &= [y_l \odot_l x_l, y_r \odot_r x_r] \\ &= \overline{y} \bigodot \overline{x}. \end{aligned} \tag{20}$$

(4) By the associativity of $\odot_l$ and $\odot_r$ in Definition 2.1(2)(ii), for any $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$, we have

$$\begin{aligned} (\overline{x} \bigodot \overline{y}) \bigodot \overline{z} &= [x_l \odot_l y_l, x_r \odot_r y_r] \bigodot [z_l, z_r] \\ &= \big[(x_l \odot_l y_l) \odot_l z_l, (x_r \odot_r y_r) \odot_r z_r\big] \end{aligned}$$

$$= \big[x_l \odot_l (y_l \odot_l z_l), x_r \odot_r (y_r \odot_r z_r)\big]$$
$$= [x_l, x_r] \bigodot_I [y_l \odot_l x_l, y_r \odot_r x_r]$$
$$= \overline{x} \bigodot (\overline{y} \bigodot \overline{z}). \tag{21}$$

(5) By the distributivity of $\oplus_s$ and $\odot_s$ for $s = l, r$ in Definition 2.1(2)(iv), for any $\overline{x}, \overline{y}, \overline{z} \in I([0,\infty])$, we have

$$\begin{aligned}
\overline{x} \bigodot (\overline{y} \bigoplus \overline{z}) &= [x_l, x_r] \bigodot [y_l \oplus_l z_l, y_r \oplus_r z_r] \\
&= \big[x_l \odot_l (y_l \oplus_l z_l), x_r \odot_r (y_r \oplus_r z_r)\big] \\
&= \big[(x_l \odot_l y_l) \oplus_l (x_l \odot_l z_l), (x_r \odot_r y_r) \oplus_r (x_r \odot_r z_r)\big] \\
&= [x_l \odot_l y_l, x_r \odot_r y_r] \bigoplus [x_l \odot_l z_l, x_r \odot_r z_r] \\
&= (\overline{x} \bigodot \overline{y}) \bigoplus (\overline{x} \bigodot \overline{z}).
\end{aligned} \tag{22}$$

By using a standard interval-valued pseudo-addition and a standard interval-valued pseudo-multiplication, we define the interval-valued $\overline{g}$-integral represented by its interval-valued generator $\overline{g}$. □

Definition 3.3 Let X be a set, two pseudo-additions $\oplus_l$ and $\oplus_r$ be representants of a standard interval-valued pseudo-addition $\bigoplus$, and two pseudo-multiplications $\odot_l$ and $\odot_r$ be representants of a standard interval-valued pseudo-multiplication $\bigodot$.

(1) An interval-valued function $\overline{f} : X \to I([0,\infty)) \setminus \{\emptyset\}$ is said to be measurable if for any open set $O \subset [0,\infty)$,

$$\overline{f}^{-1}(O) = \big\{x \in X \mid \overline{f}(x) \cap O \neq \emptyset\big\} \in \mathcal{A}. \tag{23}$$

(2) Let g_s be a continuous strictly increasing surjective function for $s = l, r$ such that $g_l \le g_r$, $\overline{g} = [g_l, g_r]$, and $g_s(0) = 0$ for $s = l, r$. The interval-valued $\overline{g}$-integral with respect to a fuzzy measure μ of a measurable interval-valued function $\overline{f} = [f_l, f_r]$ is defined by

$$\int_A^{\bigoplus} \overline{f} \bigodot d\mu = \left[\int_A^{\oplus_l} f_l \odot_l d\mu, \int_A^{\oplus_r} f_r \odot_r d\mu\right] \tag{24}$$

for all $A \in \mathcal{A}$.

(3) $\overline{f}$ is said to be integrable on $A \in \mathcal{A}$ if

$$\int_A^{\bigoplus} \overline{f} \bigodot d\mu \in I\big([0,\infty]\big). \tag{25}$$

Let $\mathfrak{IF}(X)$ be the set of all measurable interval-valued functions and $\mathfrak{IF}^*(X)$ be the set of all integrable interval-valued functions. Then, by Definition 3.3, we directly obtain the following theorem.

Theorem 3.2 *If g_s is a continuous strictly increasing surjective function for $s = l, r$ such that $g_l \le g_r$, $\overline{g} = [g_l, g_r]$, and $g_s(0) = 0$ for $s = l, r$, two pseudo-additions $\oplus_l$ and $\oplus_r$ are representants of a standard interval-valued pseudo-addition $\bigoplus$, and two pseudo-multiplications*

$\odot_l$ and $\odot_r$ are representants of a standard interval-valued pseudo-multiplication $\bigodot$, then we have

$$\int_A^{\bigoplus} \overline{f} \bigodot d\mu = \left[g_l^{-1} \int_A g_l\bigl(f_l(x)\bigr)\,dx, g_r^{-1} \int_A g_r\bigl(f_r(x)\bigr)\,dx\right]. \tag{26}$$

Proof By Definition 2.3(1),

$$\int_A^{\oplus_s} f_s \odot_s d\mu = g_s^{-1} \int_A g_s\bigl(f_s(x)\bigr)\,dx \tag{27}$$

for $s = l, r$. By (27) and Definition 3.3, we have

$$\begin{aligned} \int_A^{\bigoplus} \overline{f} \bigodot d\mu &= \left[\int_A^{\oplus_l} f_l \odot_l d\mu, \int_A^{\oplus_r} f_r \odot_r d\mu\right] \\ &= \left[g_l^{-1} \int_A g_l\bigl(f_l(x)\bigr)\,dx, g_r^{-1} \int_A g_r\bigl(f_r(x)\bigr)\,dx\right]. \end{aligned} \tag{28}$$

By the definition of the interval-valued $\overline{g}$-integral, we directly obtain the following basic properties. □

Theorem 3.3 *Let g_s be a continuous strictly increasing surjective function for $s = l, r$ such that $g_l \leq g_r$, $\overline{g} = [g_l, g_r]$, and $g_s(0) = 0$ for $s = l, r$, two pseudo-additions $\oplus_l$ and $\oplus_r$ be representants of a standard interval-valued pseudo-addition $\bigoplus$, two pseudo-multiplications $\odot_l$ and $\odot_r$ be representants of a standard interval-valued pseudo-multiplication $\bigodot$, and two pseudo-multiplications $\otimes_l$ and $\otimes_r$ be representants of a standard interval-valued pseudo-multiplication $\bigotimes$.*

(1) *If $A \in \mathcal{A}$ and $\overline{f}, \overline{h} \in \mathfrak{I}\mathfrak{F}^*(X)$ and $\overline{f} \leq \overline{h}$, then we have*

$$\int_A^{\bigoplus} \overline{f} \bigodot d\mu \leq \int_A^{\bigoplus} \overline{h} \bigodot d\mu. \tag{29}$$

(2) *If $A \in \mathcal{A}$ and $\overline{f}, \overline{h} \in \mathfrak{I}\mathfrak{F}^*(X)$, then we have*

$$\int_A^{\bigoplus} (\overline{f} \bigoplus \overline{h}) \bigodot d\mu = \int_A^{\bigoplus} \overline{f} \bigodot d\mu \bigoplus \int_A^{\bigoplus} \overline{h} \bigodot d\mu. \tag{30}$$

(3) *If $A \in \mathcal{A}$ and $\overline{c} = [c_l, c_r] \in I([0, \infty))$, $\overline{h} \in \mathfrak{I}\mathfrak{F}^*(X)$, then we have*

$$\int_A^{\bigoplus} (\overline{c} \bigotimes \overline{h}) \bigodot d\mu = \overline{c} \bigotimes \int_A^{\bigoplus} \overline{h} \bigodot d\mu. \tag{31}$$

Proof (1) Note that if $\overline{f}, \overline{h} \in \mathfrak{I}\mathfrak{F}^*(X)$ and $\overline{f} \leq \overline{h}$, then

$$f_s \leq h_s \tag{32}$$

for $s = l, r$. Since g_l and g_r are strictly monotone increasing,

$$g_s \circ f_s \leq g_s \circ h_s \tag{33}$$

for $s = l, r$. By (33) and Theorem 2.2(1),

$$\int_A^{\oplus_s} f_s \odot_s d\mu \le \int_A^{\oplus_s} h_s \odot_s d\mu \tag{34}$$

for $s = l, r$. By (34) and Theorem 3.2,

$$\begin{aligned}\int_A^{\bigoplus} \overline{f} \bigodot d\mu &= \left[\int_A^{\oplus_l} f_l \odot_l d\mu, \int_A^{\oplus_r} f_r \odot_r d\mu\right] \\ &\le \left[\int_A^{\oplus_l} h_l \odot_l d\mu, \int_A^{\oplus_r} h_r \odot_r d\mu\right] = \int_A^{\bigoplus} \overline{h} \bigodot d\mu.\end{aligned} \tag{35}$$

(2) Note that if $\overline{f}, \overline{h} \in \Im\mathfrak{F}^*(X)$, then

$$\overline{f} \bigoplus \overline{h} = [f_l \oplus_l h_l, f_r \oplus_r h_r]. \tag{36}$$

By Theorem 2.2(2),

$$\int_A^{\oplus_s} (f_s \oplus_s h_s) \odot_s d\mu = \int_A^{\oplus_s} f_s \odot_s d\mu \oplus_s \int_A^{\oplus_s} h_s \odot_s d\mu \tag{37}$$

for $s = l, r$. By (37) and Theorem 3.2,

$$\begin{aligned}&\int_A^{\bigoplus} (\overline{f} \bigoplus \overline{h}) \bigodot d\mu \\ &\quad = \left[\int_A^{\oplus_l} (f_l \oplus_l h_l) \odot_l d\mu, \int_A^{\oplus_r} (f_r \oplus_r h_r) \odot_r d\mu\right] \\ &\quad = \left[\int_A^{\oplus_l} f_l \odot_l d\mu \oplus_l \int_A^{\oplus_l} h_l \odot_l d\mu, \int_A^{\oplus_r} f_r \odot_r d\mu \oplus_r \int_A^{\oplus_r} h_r \odot_r d\mu\right] \\ &\quad = \left[\int_A^{\oplus_l} f_l \odot_l d\mu, \int_A^{\oplus_r} f_r \odot_r d\mu\right] \bigoplus_I \left[\int_A^{\oplus_l} h_l \odot_l d\mu, \int_A^{\oplus_r} h_r \odot_r d\mu\right] \\ &\quad = \int_A^{\bigoplus} \overline{f} \bigodot d\mu \bigoplus \int_A^{\bigoplus} \overline{h} \bigodot d\mu.\end{aligned} \tag{38}$$

(3) Note that if $\overline{f} \in \Im\mathfrak{F}^*(X)$ and $\overline{c} \in I([0, \infty))$, then

$$\overline{c} \bigotimes \overline{f} = [c_l \otimes_l f_l, c_r \otimes_r f_r]. \tag{39}$$

By Theorem 2.2(3),

$$\int_A^{\oplus_s} (c_s \otimes_s f_s) \odot_s d\mu = c_s \otimes_s \int_A^{\oplus_s} f_s \odot_s d\mu \tag{40}$$

for $s = l, r$. By (40) and Definition 3.3(2),

$$\begin{aligned}&\int_A^{\bigoplus} (\overline{c} \bigotimes \overline{f}) \bigodot d\mu \\ &\quad = \left[\int_A^{\oplus_l} (c_l \otimes_l f_l) \odot_l d\mu, \int_A^{\oplus_r} (c_r \otimes_r f_r) \odot_r d\mu\right]\end{aligned}$$

$$= \left[c_l \otimes_l \int_A^{\oplus_l} f_l \odot_l d\mu, c_r \otimes_r \int_A^{\oplus_r} f_r \odot_r d\mu\right]$$
$$= \overline{c} \bigotimes \int_A^{\bigoplus} \overline{f} \bigodot d\mu. \tag{41}$$

□

4 An interval-valued $\overline{g}$-convolution

In this section, by using the interval-valued $\overline{g}$-integral, we define the interval-valued $\overline{g}$-convolution of interval-valued functions in $\mathfrak{I}\mathfrak{F}^*(X)$.

Definition 4.1 If $\overline{g}$, $\bigoplus$, $\bigodot$, and $\bigotimes$ satisfy the hypotheses of Theorem 3.2, then the interval-valued $\overline{g}$-convolution is defined by

$$(\overline{f} \star \overline{h})(t) = \int_{[0,t]}^{\bigoplus} \left[\overline{f}(t-u) \bigotimes \overline{h}(u)\right] \bigodot d\mu(u) \tag{42}$$

for all $t \in [0, \infty)$.

From Definition 4.1, we directly obtain some characterization of an interval-valued $\overline{g}$-convolution by means of the interval-valued $\overline{g}$-integrals.

Theorem 4.1 *If* $\overline{g}$, $\bigoplus$, $\bigodot$, *and* $\bigotimes$ *satisfy the hypotheses of Theorem* 3.2, *then we have*

$$\overline{f} \star \overline{h} = [f_l *_l h_l, f_r *_r h_r], \tag{43}$$

where $(f_s *_s h_s)(t) = \int_{[0,t]}^{\oplus_s} f_s(t-u) \odot_s d\mu$ *for* $s = l, r$.

Proof By Definition 2.4, we have

$$(f_s *_s h_s)(t) = \int_{[0,t]}^{\oplus_s} f_s(t-u) \otimes_s h_s(u) \odot_s d\mu(u) \tag{44}$$

for $s = l, r$. By Theorem 3.2 and (44),

$$(\overline{f} \star \overline{h})(t) = \int_{[0,t]}^{\bigoplus} \left(\overline{f}(t-u) \bigotimes \overline{h}(u)\right) \bigodot d\mu(u)$$
$$= \int_{[0,t]}^{\bigoplus} \left[f_l(t-u) \otimes_l h_l(u), f_r(t-u) \otimes_r h_r(u)\right] \bigodot d\mu(u)$$
$$= \left[\int_{[0,t]}^{\oplus_l} f_l(t-u) \otimes_l h_l(u) \odot_l d\mu, \int_{[0,t]}^{\oplus_r} f_r(t-u) \otimes_r h_r(u) \odot_r d\mu\right]$$
$$= \left[(f_l *_l h_l)(t), (f_r *_r h_r)(t)\right]. \tag{45}$$

□

From Theorem 4.1, we investigate the commutativity and the associativity of a standard interval-valued $\overline{g}$-convolution.

Theorem 4.2 *If* $\overline{g}$, $\bigoplus$, $\bigodot$, *and* $\bigotimes$ *satisfy the hypotheses of Theorem* 3.2 *and* $\overline{f}$, $\overline{h}$, *and* $\overline{k} \in \mathfrak{I}\mathfrak{F}^*(X)$, *then we have*

(1) $\overline{f} \star \overline{h} = \overline{h} \star \overline{f}$,

(2) $(\overline{f} \star \overline{h}) \star \overline{k} = \overline{f} \star (\overline{h} \star \overline{k})$.

Proof Let $\overline{f} = [f_l, f_r]$, $\overline{h} = [h_l, h_r]$, $\overline{k} = [k_l, k_r] \in \mathfrak{I}\mathfrak{F}^*(X)$. By (16), we have

$$f_l *_l h_l = h_l *_l f_l \quad \text{and} \quad f_r *_r h_r = h_r *_r f_r. \tag{46}$$

By Theorem 4.1 and (46), we have

$$\begin{aligned}\overline{f} \star \overline{h} &= [f_l *_l h_l, f_r *_r h_r] \\ &= [h_l *_l f_l, h_r *_r f_r] \\ &= \overline{h} \star \overline{f}.\end{aligned} \tag{47}$$

By (17), we have

$$(f_l *_l h_l) *_l k_l = f_l *_l (h_l *_l k_l) \quad \text{and} \quad (f_r *_r h_r) *_r k_r = f_r *_r (h_r *_r k_r). \tag{48}$$

By Theorem 4.1 and (48),

$$\begin{aligned}(\overline{f} \star \overline{h}) \star \overline{k} &= \bigl[(f_l *_l h_l) *_l k_l, (f_r *_r h_r) *_r k_r\bigr] \\ &= \bigl[f_l *_l (h_l *_l k_l), f_r *_r (h_r *_r k_r)\bigr] \\ &= \overline{f} \star (\overline{h} \star \overline{k}).\end{aligned} \tag{49}$$

□

Finally, we illustrate the following examples which are related with the interval-valued $\overline{g}$-integral and the interval-valued $\overline{g}$-convolution as follows.

Example 4.1 We give three examples of the interval-valued $\overline{g}$-integral.

(1) If $g_l(x) = g_r(x) = x$ for all $x \in [0, \infty]$ are the generators of $\odot_l$, $\odot_r$, $\oplus_l$, and $\oplus_r$, and $\overline{f}(x) = [\frac{e^{-x}}{2}, e^{-x}]$ for all $x \in [0, \infty)$, and $A = [0, t]$ for all $t \in [0, \infty)$, then we have

$$\begin{aligned}\int_A^{\oplus} \overline{f} \bigodot d\mu &= \left[g_l^{-1} \int_0^t g_l\bigl(f_l(x)\bigr)\,dx, g_r^{-1} \int_0^t g_r\bigl(f_r(x)\bigr)\,dx\right] \\ &= \left[\int_0^t \frac{1}{2} e^{-x}\,dx, \int_0^t e^{-x}\,dx\right] \\ &= \left[\frac{1}{2}\bigl(1 - e^{-t}\bigr), \bigl(1 - e^{-t}\bigr)\right].\end{aligned} \tag{50}$$

(2) If $g_l(x) = \frac{1}{2}x$, $g_r(x) = x$ for all $x \in [0, \infty]$ are the generators of $\odot_l$, $\odot_r$, and $g_l(x) = g_r(x) = x$ for all $x \in [0, \infty]$ are the generators of $\oplus_l$, $\oplus_r$, and $\overline{f}(x) = [\frac{e^{-x}}{2}, e^{-x}]$ for all $x \in [0, \infty)$, and $A = [0, t]$ for all $t \in [0, \infty)$, then we have

$$\begin{aligned}\int_A^{\oplus} \overline{f} \bigodot d\mu &= \left[g_l^{-1} \int_0^t g_l\bigl(f_l(x)\bigr)\,dx, g_r^{-1} \int_0^t g_r\bigl(f_r(x)\bigr)\,dx\right] \\ &= \left[2 \int_0^t \frac{1}{4} e^{-x}\,dx, \int_0^t e^{-x}\,dx\right] \\ &= \left[\frac{1}{2}\bigl(1 - e^{-t}\bigr), \bigl(1 - e^{-t}\bigr)\right].\end{aligned} \tag{51}$$

(3) If $g_l(x) = x^2$, $g_r(x) = 3x^2$ for all $x \in [0,\infty]$ are the generators of $\odot_l$, $\odot_r$, and $g_l(x) = g_r(x) = x$ for all $x \in [0,\infty]$ are the generators of $\oplus_l$, $\oplus_r$, and $\overline{f}(x) = [\frac{e^{-x}}{2}, e^{-x}]$ for all $x \in [0,\infty)$, and $A = [0,t]$ for all $t \in [0,\infty)$, then we have

$$\begin{aligned}\int_A^{\oplus} \overline{f} \bigodot d\mu &= \left[g_l^{-1} \int_0^t g_l\big(f_l(x)\big)\,dx, g_r^{-1} \int_0^t g_r\big(f_r(x)\big)\,dx\right] \\ &= \left[\sqrt{\int_0^t \frac{1}{2} e^{-2x}\,dx}, \sqrt{\frac{1}{3}\int_0^t 3e^{-2x}\,dx}\right] \\ &= \left[\frac{1}{4}\big(1-e^{-2t}\big), \frac{1}{2}\big(1-e^{-t}\big)\right]. \end{aligned} \tag{52}$$

Example 4.2 We give an example of the interval-valued $\overline{g}$-convolution.

If $g_l(x) = x^2$, $g_r(x) = 3x^2$ for all $x \in [0,\infty]$ are the generators of $\odot_l$, $\odot_r$, and $g_l(x) = g_r(x) = x$ for all $x \in [0,\infty]$ are the generators of $\oplus_l$, $\oplus_r$, $\otimes_l$, $\otimes_r$, and $\overline{f}(x) = [\frac{e^{-x}}{2}, e^{-x}]$ for all $x \in [0,\infty)$, $\overline{h}(x) = [\frac{1}{2}x, x]$ for all $x \in [0,\infty)$, and $A = [0,t]$ for all $t \in [0,\infty)$, then we have

$$\begin{aligned}(\overline{f} \star \overline{h})(t) &= \int_A^{\oplus} \big[\overline{f}(t-u) \bigotimes \overline{h}(u)\big] \bigodot d\mu(u) \\ &= \left[\sqrt{\frac{1}{2}\int_0^t e^{-2(t-u)} e^{-2u}\,du}, \sqrt{3\int_0^t \frac{1}{2} e^{-2(x-u)} e^{-2u}\,du}\right] \\ &= \left[\frac{t}{4}e^{-2t}, \frac{3t}{2}e^{-2t}\right]. \end{aligned} \tag{53}$$

5 Conclusions

In this paper, we have considered the g-integral represented by its generating g, the pseudo-addition, the pseudo-multiplication (see Definition 2.3). This study was to define the g-convolution by means of the g-integral (see Definition 2.4) and to investigate some characterizations of the g-integral and the commutativity and the associativity of the g-convolution (see Theorems 2.2, 2.3, and 2.4).

We also defined the interval-valued $\overline{g}$-integral represented by its interval-valued generator $\overline{g}$. By using general notions of an interval-representable pseudo-multiplication (see Definition 3.2), we defined an interval-valued $\overline{g}$-integral (see Definition 3.3) and investigated some basic characterizations of them (see Theorems 3.2, 3.3).

From Definitions 2.3, 2.4, and Theorems 2.2, 2.3, we defined a standard interval-valued $\overline{g}$-convolution (see Definition 4.1). We also investigated some characterizations of a standard interval-valued $\overline{g}$-convolution of interval-valued functions by means of the interval-valued $\overline{g}$-integral including commutativity and associativity of an interval-representable convolution (see Theorems 4.1, 4.2).

In the future, we can study various inequalities of the interval-valued $\overline{g}$-integral and expect that the standard interval-valued $\overline{g}$-convolutions are used (i) to generalize the g-Laplace transform, Hamilton-Jacobi equation on the space of functions, such as in nonlinearity and optimization and such as in information theory (see [1, 14, 29]); (ii) to generalize the Stolasky-type inequality for the pseudo-integral of functions such as in economics, finance, decision making (see [2, 30]), *etc.*

Competing interests
The author declares that they have no competing interests.

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