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On locally contractive fuzzy set-valued mappings

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Abstract
We prove the existence of common fuzzy fixed points for a sequence of locally contractive fuzzy mappings satisfying generalized Banach type contraction conditions in a complete metric space by using iterations. Our main result generalizes and unifies several well-known fixed-point theorems for multivalued maps. Illustrative examples are also given.

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1 Introduction

The Banach contraction theorem and its subsequent generalizations play a fundamental role in the field of fixed point theory. In particular, Heilpern introduced in [1] the notion of a fuzzy mapping in a metric linear space and proved a Banach type contraction theorem in this framework. Subsequently several other authors [2–10] have studied and established the existence of fixed points of fuzzy mappings. The aim of this paper is to prove a common fixed-point theorem for a sequence of fuzzy mappings in the context of metric spaces without the assumption of linearity. Our results generalize and unify several typical theorems of the literature.

2 Preliminaries

Given a metric space (X,d), denote by $CB(X)$ the family of all nonempty closed bounded subsets of (X,d). As usual, for $\zeta \in X$ and $A \in CB(X)$, we define

$$d(\zeta,A) = \inf_{a\in A} d(\zeta,a).$$

Then the Hausdorff metric H on $CB(X)$ induced by d is defined as

$$H(A,B) = \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\Big\},$$

for all $A,B \in CB(X)$.

A fuzzy set in (X,d) is a function with domain X and values in $I = [0,1]$. I^X denotes the collection of all fuzzy sets in X. If A is a fuzzy set and $\zeta \in X$, then the function value $A(\zeta)$ is called the grade of membership of ζ in A. The α-level set of a fuzzy set A is denoted by

A_α, and it is defined as follows:

$$A_\alpha = \{\zeta : A(\zeta) \geq \alpha\} \quad \text{if } \alpha \in (0,1],$$
$$A_0 = \text{closure of } \{\zeta : A(\zeta) > 0\}.$$

According to Heilpern [1], a fuzzy set A in a metric linear space (X,d) is said to be an approximate quantity if A_α is compact and convex in X, for each $\alpha \in (0,1]$, and $\sup_{\zeta \in X} A(\zeta) = 1$. The family of all approximate quantities of the metric linear space (X,d) is denoted by $W(X)$.

Now, for $A, B \in W(X)$ and $\alpha \in [0,1]$, define

$$D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

and

$$d_\infty(A,B) = \sup_{\alpha \in [0,1]} D_\alpha(A_\alpha, B_\alpha).$$

It is well known that d_∞ is a metric on $W(X)$.

In case that (X,d) is a (non-necessarily linear) metric space, we also define

$$D_\alpha(A,B) = H(A_\alpha, B_\alpha),$$

whenever $A, B \in I^X$ and $A_\alpha, B_\alpha \in CB(X)$, $\alpha \in [0,1]$.

In the sequel the letter $\mathbb{N}$ will denote the set of positive integer numbers.

The following well-known properties on the Hausdorff metric (see *e.g.* [11]) will be useful in the next section.

Lemma 2.1 *Let (X,d) be a metric space and let $A, B \in CB(X)$ with $H(A,B) < r$, $r > 0$. If $a \in A$, then there exists $b \in B$ such that $d(a,b) < r$.*

Lemma 2.2 *Let (X,d) be a metric space and let $\{A_n\}_{n=1}^\infty$ be a sequence in $CB(X)$ such that $\lim_{n\to\infty} H(A_n, A) = 0$, for some $A \in CB(X)$. If $\xi_n \in A_n$, for all $n \in \mathbb{N}$, and $d(\xi_n, \xi) \to 0$, then $\xi \in A$.*

Now, let X be an arbitrary set and let Y be a metric space. A mapping T is called fuzzy mapping if T is a mapping from X into I^Y. In fact, a fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership function $T(\zeta)$. The value $T(\zeta)(\xi)$ is the grade of membership of ξ in $T(\zeta)$.

If (X,d) is a metric space and T is a (fuzzy) mapping from X into I^X, we say that $\xi \in X$ is a fixed point of T if $\xi \in T(\xi)_1$.

We conclude this section with the notion of contractiveness that will be used in our main result.

Definition 2.3 (compare [12]) Let $\varepsilon \in (0,\infty]$. A function $\psi : [0,\varepsilon) \to [0,1)$ is said to be a MT-function if it satisfies Mizoguchi-Takahashi's condition (*i.e.*, $\limsup_{r\to t^+} \psi(r) < 1$, for all $t \in [0,\varepsilon)$).

Clearly, if $\psi : [0,\varepsilon) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then it is a MT-function. So the set of MT-functions is a rich class.

3 Fixed points of fuzzy mappings

Fixed-point theorems for locally contractive mappings were studied, among others, by Edelstein [13], Beg and Azam [14], Holmes [15], Hu [11], Hu and Rosen [16], Ko and Tasi [17], Kuhfitting [18] and Nadler [19].

Heilpern [1] established a fixed-point theorem for fuzzy contraction mappings in metric linear spaces, which is a fuzzy extension of Banach's contraction principle. Afterwards Azam *et al.* [4, 5], and Lee and Cho [10] further extended Banach's contraction principle to fuzzy contractive mappings in Heilpern's sense. In our main result (Theorem 3.1 below) we establish a common fixed-point theorem for a sequence of generalized fuzzy uniformly locally contraction mappings on a complete metric space without the requirement of linearity. This is a generalization of many conventional results of the literature.

Let $\varepsilon \in (0,\infty]$, and $\lambda \in (0,1)$. A metric space (X,d) is said to be ε-chainable if given $\zeta,\xi \in X$, there exists an ε-chain from ζ to ξ (*i.e.*, a finite set of points $\zeta = \zeta_0, \zeta_1, \zeta_2, \dots, \zeta_m = \xi$ such that $d(\zeta_{j-1},\zeta_j) < \varepsilon$, for all $j = 1,2,\dots,m$). A mapping $T : X \to X$ is called an (ε,λ) uniformly locally contractive mapping if $\zeta,\zeta \in X$ and $0 < d(\zeta,\zeta) < \varepsilon$, implies $d(T\zeta,T\xi) \le \lambda d(\zeta,\xi)$. A mapping $T : X \to W(X)$ is called an (ε,λ) uniformly locally contractive fuzzy mapping if $\zeta,\xi \in X$ and $0 < d(\zeta,\xi) < \varepsilon$, imply $d_\infty(T(\zeta),T(\xi)) \le \lambda d(\zeta,\xi)$. We remark that a globally contractive mapping can be regarded as an (∞,λ) uniformly locally contractive mapping and for some special spaces every locally contractive mapping is globally contractive.

Theorem 3.1 *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric space and $\{T_i\}_{i=1}^\infty$ a sequence of fuzzy mappings from X into I^X such that, for each $\zeta \in X$ and $i \in \mathbb{N}$, $T_i(\zeta)_1 \in CB(X)$. If*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad D_1\bigl(T_i(\zeta),T_j(\xi)\bigr) \le \psi\bigl(d(\zeta,\xi)\bigr)d(\zeta,\xi), \tag{1}$$

for all $i,j \in \mathbb{N}$, where $\psi : [0,\varepsilon) \to [0,1)$ is a MT-function, then the sequence $\{T_i\}_{i=1}^\infty$ has a common fixed point, i.e., there is $\xi^* \in X$ such that $\xi^* \in T_i(\xi^*)_1$, for all $i \in \mathbb{N}$.

Proof Let ξ_0 be an arbitrary, but fixed element of X. Find $\xi_1 \in X$ such that $\xi_1 \in T_1(\xi_0)_1$. Let

$$\xi_0 = \zeta_{(1,0)}, \qquad \zeta_{(1,1)}, \zeta_{(1,2)}, \dots, \zeta_{(1,m)} = \xi_1 \in T_1(\xi_0)_1$$

be an arbitrary ε-chain from ξ_0 to ξ_1. (We suppose, without loss of generality, that $\zeta_{(1,i)} \neq \zeta_{(1,j)}$, for each $i,j \in \{0,1,2,\dots,m\}$ with $i \neq j$.)

Since $0 < d(\zeta_{(1,0)},\zeta_{(1,1)}) < \varepsilon$, we deduce that

$$\begin{aligned} D_1\bigl(T_1(\zeta_{(1,0)}),T_2(\zeta_{(1,1)})\bigr) &\le \psi\bigl(d(\zeta_{(1,0)},\zeta_{(1,1)})\bigr)d(\zeta_{(1,0)},\zeta_{(1,1)}) \\ &< \sqrt{\psi\bigl(d(\zeta_{(1,0)},\zeta_{(1,1)})\bigr)}d(\zeta_{(1,0)},\zeta_{(1,1)}) \\ &< d(\zeta_{(1,0)},\zeta_{(1,1)}) < \varepsilon. \end{aligned}$$

Rename ξ_1 as $\zeta_{(2,0)}$. Since $\zeta_{(2,0)} \in T_1(\zeta_{(1,0)})_1$, using Lemma 2.1 we find $\zeta_{(2,1)} \in T_2(\zeta_{(1,1)})_1$ such that

$$\begin{aligned} d(\zeta_{(2,0)}, \zeta_{(2,1)}) &< \sqrt{\psi\bigl(d(\zeta_{(1,0)}, \zeta_{(1,1)})\bigr)}d(\zeta_{(1,0)}, \zeta_{(1,1)}) \\ &< d(\zeta_{(1,0)}, \zeta_{(1,1)}) < \varepsilon. \end{aligned}$$

Similarly we may choose an element $\zeta_{(2,2)} \in T_2(\zeta_{(1,2)})_1$ such that

$$\begin{aligned} d(\zeta_{(2,1)}, \zeta_{(2,2)}) &< \sqrt{\psi\bigl(d(\zeta_{(1,1)}, \zeta_{(1,2)})\bigr)}d(\zeta_{(1,1)}, \zeta_{(1,2)}) \\ &< d(\zeta_{(1,1)}, \zeta_{(1,2)}) < \varepsilon. \end{aligned}$$

Thus we obtain a set $\{\zeta_{(2,0)}, \zeta_{(2,1)}, \zeta_{(2,2)}, \ldots, \zeta_{(2,m)}\}$ of $m+1$ points of X such that $\zeta_{(2,0)} \in T_1(\zeta_{(1,0)})_1$ and $\zeta_{(2,j)} \in T_2(\zeta_{(1,j)})_1$, for $j = 1, 2, \ldots, m$, with

$$\begin{aligned} d(\zeta_{(2,j)}, \zeta_{(2,j+1)}) &< \sqrt{\psi\bigl(d(\zeta_{(1,j)}, \zeta_{(1,j+1)})\bigr)}d(\zeta_{(1,j)}, \zeta_{(1,j+1)}) \\ &< d(\zeta_{(1,j)}, \zeta_{(1,j+1)}) < \varepsilon, \end{aligned}$$

for $j = 0, 1, 2, \ldots, m-1$.

Let $\zeta_{(2,m)} = \xi_2$. Thus the set of points $\xi_1 = \zeta_{(2,0)}, \zeta_{(2,1)}, \zeta_{(2,2)}, \ldots, \zeta_{(2,m)} = \xi_2 \in T_2(\xi_1)_1$ is an ε-chain from ξ_0 to ξ_1. Rename ξ_2 as $\zeta_{(3,0)}$. Then by the same procedure we obtain an ε-chain

$$\xi_2 = \zeta_{(3,0)}, \qquad \zeta_{(3,1)}, \zeta_{(3,2)}, \ldots, \zeta_{(3,m)} = \xi_3 \in T_3(\xi_2)_1$$

from ξ_2 to ξ_3. Inductively, we obtain

$$\xi_n = \zeta_{(n+1,0)}, \qquad \zeta_{(n+1,1)}, \zeta_{(n+1,2)}, \ldots, \zeta_{(n+1,m)} = \xi_{n+1} \in T_{n+1}(\xi_n)_1$$

with

$$\begin{aligned} d(\zeta_{(n+1,j)}, \zeta_{(n+1,j+1)}) &< \sqrt{\psi\bigl(d(\zeta_{(n,j)}, \zeta_{(n,j+1)})\bigr)}d(\zeta_{(n,j)}, \zeta_{(n,j+1)}) \\ &< d(\zeta_{(n,j)}, \zeta_{(n,j+1)}) < \varepsilon, \end{aligned} \tag{2}$$

for $j = 0, 1, 2, \ldots, m-1$.

Consequently, we construct a sequence $\{\xi_n\}_{n=1}^{\infty}$ of points of X with

$$\begin{aligned} &\xi_1 = \zeta_{(1,m)} = \zeta_{(2,0)} \in T_1(\xi_0)_1, \\ &\xi_2 = \zeta_{(2,m)} = \zeta_{(3,0)} \in T_2(\xi_1)_1, \\ &\xi_3 = \zeta_{(3,m)} = \zeta_{(4,0)} \in T_3(\xi_2)_1, \\ &\vdots \\ &\xi_{n+1} = \zeta_{(n+1,m)} = \zeta_{(n+2,0)} \in T_{n+1}(\xi_n)_1, \end{aligned}$$

for all $n \in \mathbb{N}$.

For each $j \in \{0,1,2,\ldots,m-1\}$, we deduce from (2) that $\{d(\zeta_{(n,j)},\zeta_{(n,j+1)})\}_{n=1}^{\infty}$ is a decreasing sequence of non-negative real numbers and therefore there exists $l_j \geq 0$ such that

$$\lim_{n\to\infty} d(\zeta_{(n,j)},\zeta_{(n,j+1)}) = l_j.$$

By assumption, $\limsup_{t\to l_j^+}\psi(t) < 1$, so there exists $n_j \in \mathbb{N}$ such that $\psi(d(\zeta_{(n,j)},\zeta_{(n,j+1)})) < s(l_j)$, for all $n \geq n_j$ where $\limsup_{t\to l_j^+}\psi(t) < s(l_j) < 1$.

Now put

$$M_j = \max\Big\{\max_{i=1,\ldots,n_j}\sqrt{\psi\big(d(\zeta_{(i,j)},\zeta_{(i,j+1)})\big)},\sqrt{s(l_j)}\Big\}.$$

Then, for every $n > n_j$, we obtain

$$\begin{aligned} d(\zeta_{(n,j)},\zeta_{(n,j+1)}) &< \sqrt{\psi\big(d(\zeta_{(n-1,j)},\zeta_{(n-1,j+1)})\big)}d(\zeta_{(n-1,j)},\zeta_{(n-1,j+1)}) \\ &< \sqrt{s(l_j)}d(\zeta_{(n-1,j)},\zeta_{(n-1,j+1)}) \\ &\leq M_j d(\zeta_{(n-1,j)},\zeta_{(n-1,j+1)}) \\ &\leq (M_j)^2 d(\zeta_{(n-2,j)},\zeta_{(n-2,j+1)}) \\ &\leq \cdots \\ &\leq (M_j)^{n-1} d(\zeta_{(1,j)},\zeta_{(1,j+1)}). \end{aligned}$$

Putting $N = \max\{n_j : j = 0,1,2,\ldots,m-1\}$, we have

$$\begin{aligned} d(\xi_{n-1},\xi_n) = d(\zeta_{(n,0)},\zeta_{(n,m)}) &\leq \sum_{j=0}^{m-1} d(\zeta_{(n,j)},\zeta_{(n,j+1)}) \\ &< \sum_{j=0}^{m-1}(M_j)^{n-1} d(\zeta_{(1,j)},\zeta_{(1,j+1)}), \end{aligned}$$

for all $n > N+1$. Hence

$$\begin{aligned} d(\xi_n,\xi_p) &\leq d(\xi_n,\xi_{n+1}) + d(\xi_{n+1},\xi_{n+2}) + \cdots + d(\xi_{p-1},\xi_p) \\ &< \sum_{j=0}^{m-1}(M_j)^n d(\zeta_{(1,j)},\zeta_{(1,j+1)}) + \cdots + \sum_{j=0}^{m-1}(M_j)^{p-1} d(\zeta_{(1,j)},\zeta_{(1,j+1)}), \end{aligned}$$

whenever $p > n > N+1$.

Since $M_j < 1$, for all $j \in \{0,1,2,\ldots,m-1\}$, it follows that $\{\xi_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Since (X,d) is complete, there is $\xi^* \in X$ such that $\xi_n \to \xi^*$. So for each $\delta \in (0,\varepsilon]$ there is $M_\delta \in \mathbb{N}$ such that $n > M_\delta$ implies $d(\xi_n,\xi^*) < \delta$. This in view of inequality (1) implies $D_1(T_{n+1}(\xi_n),T_i(\xi^*)) < \delta$, for all $i \in \mathbb{N}$. Consequently, $H(T_{n+1}(\xi_n)_1,T_i(\xi^*)_1) \to 0$. Since $\xi_{n+1} \in T_{n+1}(\xi_n)_1$ with $d(\xi_{n+1},\xi^*) \to 0$, we deduce from Lemma 2.2 that $\xi^* \in T_i(\xi^*)_1$, for all $i \in \mathbb{N}$. This completes the proof. □

Corollary 3.2 *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric space and $\{T_i\}_{i=1}^{\infty}$ a sequence of fuzzy mappings from X into I^X such that, for each $\zeta \in X$ and $i \in \mathbb{N}$, $T_i(\zeta)_1 \in CB(X)$. If*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad D_1\big(T_i(\zeta),T_j(\xi)\big) \le \lambda d(\zeta,\xi),$$

for all $i,j \in \mathbb{N}$, where $\lambda \in (0,1)$, then the sequence $\{T_i\}_{i=1}^{\infty}$ has a common fixed point.

Proof Apply Theorem 3.1 when ψ is the MT-function defined as $\psi(t) = \lambda$, for all $t \in [0,\varepsilon)$. □

Corollary 3.3 *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric linear space and $\{T_i\}_{i=1}^{\infty}$ a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition:*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad d_\infty\big(T_i(\zeta),T_j(\xi)\big) \le \psi\big(d(\zeta,\xi)\big)d(\zeta,\xi),$$

for all $i,j \in \mathbb{N}$, where $\psi : [0,\varepsilon) \to [0,1)$ is a MT-function. Then the sequence $\{T_i\}_{i=1}^{\infty}$ has a common fixed point.

Proof Since $W(X) \subseteq CB(X)$ and $D_1(T_i(\zeta),T_j(\xi)) \le d_\infty(T_i(\zeta),T_j(\xi))$, for all $i,j \in \mathbb{N}$, the result follows immediately from Theorem 3.1. □

Corollary 3.4 *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric linear space and $\{T_i\}_{i=1}^{\infty}$ a sequence of fuzzy mappings from X into $W(X)$ satisfying the following condition:*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad d_\infty\big(T_i(\zeta),T_j(\xi)\big) \le \lambda d(\zeta,\xi),$$

for all $i,j \in \mathbb{N}$, where $\lambda \in (0,1)$. Then the sequence $\{T_i\}_{i=1}^{\infty}$ has a common fixed point.

Corollary 3.5 [4] *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric linear space and T_1, T_2, two fuzzy mappings from X into $W(X)$ satisfying the following condition:*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad d_\infty\big(T_i(\zeta),T_j(\xi)\big) \le \psi\big(d(\zeta,\xi)\big)d(\zeta,\xi),$$

for $i,j = 1,2$, where $\psi : [0,\varepsilon) \to [0,1)$ is a MT-function. Then T_1 and T_2 have a common fixed point.

Corollary 3.6 [4, 11] *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric linear space and $T: X \to W(X)$ an (ε,λ) uniformly locally contractive fuzzy mapping. Then T has a fixed point.*

Corollary 3.7 *Let $\varepsilon \in (0,\infty]$, (X,d) a complete ε-chainable metric space and S be a multivalued mapping from X into $CB(X)$ satisfying the following condition:*

$$\zeta,\xi \in X, \quad 0 < d(\zeta,\xi) < \varepsilon \quad \textit{implies} \quad H\big(S(\zeta),S(\xi)\big) \le \psi\big(d(\zeta,\xi)\big)d(\zeta,\xi),$$

where $\psi : [0,\varepsilon) \to [0,1)$ is a MT-function. Then S has a fixed point.

Proof Define a fuzzy mapping T from X into I^X as $T(\xi)(t) = 1$ if $t \in S(\xi)$ and $T(\xi)(t) = 0$, otherwise. Then $T(\xi)_1 = S(\xi)$, for all $\xi \in X$, so $T(\xi)_1 \in CB(X)$, for all $\xi \in X$. Since

$$D_1\big(T(\zeta), T(\xi)\big) = H\big(T(\zeta)_1, T(\xi)_1\big) = H\big(S(\zeta), S(\xi)\big),$$

for all $\zeta, \xi \in X$, we deduce that condition (1) of Theorem 3.1 is satisfied for T. Hence T has a fixed point ξ^*, *i.e.*, $\xi^* \in T(\xi^*)_1$. We conclude that $\xi^* \in S(\xi^*)$. The proof is complete. □

Corollary 3.8 [13] *Let $\varepsilon \in (0, \infty]$, (X, d) a complete ε-chainable metric space and S be a multivalued mapping from X into $CB(X)$ satisfying the following condition*:

$$\zeta, \xi \in X, \quad 0 < d(\zeta, \xi) < \varepsilon \quad \textit{implies} \quad H\big(S(\zeta), S(\xi)\big) \le \lambda d(\zeta, \xi),$$

where $\lambda \in (0, 1)$. Then S has a fixed point.

Corollary 3.9 ([20, 21], see also [9, 13]) *Let (X, d) be a complete metric space, S a multivalued mapping from X into $CB(X)$ and $\psi : [0, \infty) \to [0, 1)$ a MT-function such that*

$$H(S\zeta, S\xi) \le \psi\big(d(\zeta, \xi)\big) d(\zeta, \xi),$$

for all $\zeta, \xi \in X$. Then S has a fixed point in X.

Proof Apply Corollary 3.8 with $\varepsilon = \infty$. □

We conclude the paper with two examples to support Theorem 3.1 and Corollary 3.2.

Example 3.10 Let (X, d) be the compact, and thus complete, metric space such that $X = [0, 1]$, and $d(x, y) = |x - y|$, for all $x, y \in X$. Let λ be a constant such that $\lambda \in [1/14, 1)$ and let $\{T_k\}_{k=1}^{\infty}$ be the sequence of fuzzy mappings defined from X into I^X as follows:

$$\text{if } x = 0, \quad T_k(x)(y) = \begin{cases} 1 & \text{if } y = 0, \\ 1/3k & \text{if } 0 < y \le 1/100, \\ 0 & \text{if } 1/100 < y \le 1, \end{cases} \quad k \in \mathbb{N},$$

$$\text{if } x \ne 0, \quad T_k(x)(y) = \begin{cases} 1 & \text{if } 0 \le y \le x/14, \\ \lambda/2k & \text{if } x/14 < y \le x/12, \\ \lambda/3k & \text{if } x/12 < y < x, \\ 0 & \text{if } x \le y \le 1, \end{cases} \quad k \in \mathbb{N}.$$

For each $x, y \in X$ with $x \ne y$, and $i, j \in \mathbb{N}$ we have

$$D_1\big(T_i(x), T_j(y)\big) = H\big(T_i(x)_1, T_j(y)_1\big) = H\big([0, x/14], [0, y/14]\big) = \frac{1}{14}|x - y|.$$

Hence, for $\psi(t) = \lambda$, the conditions of Corollary 3.2, and hence of Theorem 3.1, are satisfied for any $\varepsilon \in (0, \infty]$, whereas X is not linear. Therefore all previous relevant fixed point results Corollaries 3.3-3.6 on metric linear spaces are not applicable.

Example 3.11 Let (X, d) be the complete metric space such that $X = [0, \infty)$, $d(x, x) = 0$, for all $x \in X$, and $d(x, y) = \max\{x, y\}$ whenever $x \neq y$ (in the sequel we shall write $x \vee y$ instead of $\max\{x, y\}$).

Note that a sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d) if and only if $d(x_n, 0) \to 0$. Moreover, $x = 0$ is the only non-isolated point of X for the topology induced by d.

Let $\psi : [0, \infty) \to [0, 1)$ be the MT-function defined as

$$\psi(t) = \begin{cases} 1/2 & \text{if } 0 \leq t \leq 1, \\ t/(t+1) & \text{if } t > 1, \end{cases}$$

and let $\{T_k\}_{k=1}^{\infty}$ be the sequence of fuzzy mappings defined from X into I^X as follows:

$$\text{if } 0 \leq x \leq 1, \quad T_k(x)(y) = \begin{cases} 1 & \text{if } x/4k \leq y \leq x/2k, \\ 0 & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N},$$

$$\text{if } x > 1, \quad T_k(x)(y) = \begin{cases} 1 & \text{if } x/2k \leq y < x^2/k(1+x), \\ 0 & \text{otherwise}, \end{cases} \qquad k \in \mathbb{N}.$$

Observe that, for $0 \leq x \leq 1$,

$$T_k(x)_1 = \left[\frac{x}{4k}, \frac{x}{2k}\right],$$

and, for $x > 1$,

$$T_k(x)_1 = \left[\frac{x}{2k}, \frac{x^2}{k(1+x)}\right).$$

Therefore $T_k(x)_1 \in CB(X)$, for all $x \in X$ and $k \in \mathbb{N}$ (recall that each $x \neq 0$ is an isolated point for the induced topology, so every bounded interval belongs to $CB(X)$).

We show that condition (1) of Theorem 3.1 is satisfied for $\varepsilon = \infty$ and ψ as defined above. Indeed, let $x, y \in X$ with $x \neq y$ and $j, k \in \mathbb{N}$. Assume without loss of generality that $x > y$.

If $x, y > 1$, for each $b \in T_j(y)_1$, we obtain

$$d\big(T_k(x)_1, b\big) = \inf_{a \in T_k(x)_1} (a \vee b) \leq \frac{x^2}{k(1+x)} \vee b \leq \frac{x^2}{k(1+x)} \vee \frac{y^2}{j(1+y)}.$$

Similarly, for each $a \in T_k(x)_1$, we obtain

$$d\big(a, T_j(y)_1\big) \leq \frac{x^2}{k(1+x)} \vee \frac{y^2}{j(1+y)}.$$

Consequently

$$\begin{aligned} D_1\big(T_k(x), T_j(y)\big) &= H\big(T_k(x)_1, T_j(y)_1\big) \leq \frac{x^2}{k(1+x)} \vee \frac{y^2}{j(1+y)} \\ &\leq \frac{(x \vee y)^2}{1 + (x \vee y)} = \frac{d(x, y)}{1 + d(x, y)} d(x, y) \\ &= \psi\big(d(x, y)\big) d(x, y). \end{aligned}$$

If $x > 1$ and $y \leq 1$, we deduce, in a similar way, that

$$\begin{aligned} D_1(T_k(x), T_j(y)) &= H(T_k(x)_1, T_j(y)_1) \leq \frac{x^2}{k(1+x)} \vee \frac{y}{2j} \\ &\leq \frac{x^2}{1+x} \vee \frac{y}{2} \leq \frac{x^2}{1+x} \vee \frac{x}{2} = \frac{x^2}{1+x} \\ &= \frac{(x \vee y)^2}{1+(x \vee y)} = \frac{d(x,y)}{1+d(x,y)} d(x,y) \\ &= \psi(d(x,y)) d(x,y). \end{aligned}$$

Finally, if $x, y \leq 1$, we deduce

$$\begin{aligned} D_1(T_k(x), T_j(y)) &= H(T_k(x)_1, T_j(y)_1) \leq \frac{x}{2k} \vee \frac{y}{2j} \\ &\leq \frac{x \vee y}{2} = \psi(d(x,y)) d(x,y). \end{aligned}$$

We have shown that all conditions of Theorem 3.1 are satisfied (in fact $x = 0$ is the only fixed point of T).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The three authors contributed equally in writing this article. They read and approved the final manuscript.

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