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A new approach to (α, ψ)-contractive nonself multivalued mappings

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Abstract
In this paper, we introduce the notions of α-admissible and α-ψ-contractive type condition for nonself multivalued mappings. We establish fixed point theorems using these new notions along with a new condition. Moreover, we have constructed examples to show that our new condition is different from the corresponding existing conditions in the literature.

MSC: 47H10; 54H25

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1 Introduction and preliminaries

In the last decades, metric fixed point theory has been appreciated by a number of authors who have extended the celebrated Banach fixed point theorem for various contractive mapping in the context of different abstract spaces; see, for example, [1–32]. Among them, we mention the interesting fixed point theorems of Samet *et al.* [20]. In this paper [20], the authors introduced the notions of α-ψ-contractive mappings and investigated the existence and uniqueness of a fixed point for such mappings. Further, they showed that several well-known fixed point theorems can be derived from the fixed point theorem of α-ψ-contractive mappings. Following this paper, Karapınar and Samet [21] generalized the notion α-ψ-contractive mappings and obtained a fixed point for this generalized version. On the other hand, Asl *et al.* [22] characterized the notions of α-ψ-contractive mapping and α-admissible mappings with the notions of α_*-ψ-contractive and α_*-admissible mappings to investigate the existence of a fixed point for a multivalued function. Afterward, Ali and Kamran [23] generalized the notion of α_*-ψ-contractive mappings and obtained further fixed point results for multivalued mappings. Some results in this direction in the context of various abstract spaces were also given by the authors [24–28, 33–36]. The purpose of this paper is to prove fixed point theorems for nonself multivalued (α, ψ)-contractive type mappings using a new condition.

Let Ψ be the family of functions $\psi : [0,\infty) \to [0,\infty)$, known in the literature as Bianchini-Grandolfi gauge functions (see, *e.g.*, [30–32]), satisfying the following conditions:

(ψ_1) ψ is nondecreasing;
(ψ_2) $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Notice that such functions are also known as (c)-comparison functions in some sources (see, *e.g.*, [29]).

It is easily proved that if $\psi \in \Psi$, then $\psi(t) < t$ for any $t > 0$ and $\psi(0) = 0$ for $t = 0$ (see, *e.g.*, [20, 29]). Let (X,d) be a metric space. A mapping $G : X \to X$ is called α-ψ-contractive type if there exist two functions $\alpha : X \times X \to [0,\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Gx,Gy) \le \psi\big(d(x,y)\big)$$

for each $x,y \in X$. A mapping $G : X \to X$ is called α-admissible [20] if

$$\alpha(x,y) \ge 1 \quad \Rightarrow \quad \alpha(Gx,Gy) \ge 1.$$

We denote by $N(X)$ the space of all nonempty subsets of X and by $CL(X)$ the space of all nonempty closed subsets of X. For $A \in N(X)$ and $x \in X$, $d(x,A) = \inf\{d(x,a) : a \in A\}$. For every $A,B \in CL(X)$, let

$$H(A,B) = \begin{cases} \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\} & \text{if the maximum exists;} \\ \infty & \text{otherwise.} \end{cases}$$

Such a map H is called a generalized Hausdorff metric induced by d. We use the following lemma in our results.

Lemma 1.1 [23] *Let (X,d) be a metric space and $B \in CL(X)$. Then, for each $x \in X$ with $d(x,B) > 0$ and $q > 1$, there exists an element $b \in B$ such that*

$$d(x,b) < qd(x,B). \tag{1.1}$$

Let $(X,\preceq,d)$ be an ordered metric space and $A,B \subseteq X$. We say that $A \prec_r B$ if for each $a \in A$ and $b \in B$, we have $a \preceq b$.

2 Main results

We begin this section with the following definition which is a modification of the notion of α-admissible.

Definition 2.1 Let (X,d) be a metric space and let D be a nonempty subset of X. A mapping $G : D \to CL(X)$ is called α-admissible if there exists a mapping $\alpha : D \times D \to [0,\infty)$ such that

$$\alpha(x,y) \ge 1 \quad \Rightarrow \quad \alpha(u,v) \ge 1$$

for each $u \in Gx \cap D$ and $v \in Gy \cap D$.

Definition 2.2 Let (X,d) be a metric space and let D be a nonempty subset of X. We say that $G : D \to CL(X)$ is an (α,ψ)-contractive type mapping on D if there exist $\alpha : D \times D \to [0,\infty)$ and $\psi \in \Psi$ satisfying the following conditions:

(i) $Gx \cap D \neq \emptyset$ for all $x \in D$,

(ii) for each $x, y \in D$, we have

$$\alpha(x,y)H(Gx \cap D, Gy \cap D) \leq \psi\big(M(x,y)\big), \tag{2.1}$$

where $M(x,y) = \max\{d(x,y), \frac{d(x,Gx)+d(y,Gy)}{2}, \frac{d(x,Gy)+d(y,Gx)}{2}\}$.

Note that if $\psi \in \Psi$ in the above definition is a strictly increasing function, then $G : D \to CL(X)$ is said to be a strictly (α, ψ)-contractive type mapping on D.

Theorem 2.3 *Let (X, d) be a metric space, let D be a nonempty subset of X which is complete with respect to the metric induced by d, and let G be a strictly (α, ψ)-contractive type mapping on D. Assume that the following conditions hold*:

(i) *G is an α-admissible map*;
(ii) *there exist $x_0 \in D$ and $x_1 \in Gx_0 \cap D$ such that $\alpha(x_0, x_1) \geq 1$*;
(iii) *G is continuous.*

Then G has a fixed point.

Proof By hypothesis, there exist $x_0 \in D$ and $x_1 \in Gx_0 \cap D$ such that $\alpha(x_0, x_1) \geq 1$. If $x_0 = x_1$, then we have nothing to prove. Let $x_0 \neq x_1$. If $x_1 \in Gx_1 \cap D$, then x_1 is a fixed point. Let $x_1 \notin Gx_1 \cap D$. From (2.1) we have

$$\begin{aligned} 0 &< \alpha(x_0,x_1)H(Gx_0 \cap D, Gx_1 \cap D) \\ &\leq \psi\left(\max\left\{d(x_0,x_1), \frac{d(x_0,Gx_0)+d(x_1,Gx_1)}{2}, \frac{d(x_0,Gx_1)+d(x_1,Gx_0)}{2}\right\}\right) \\ &\leq \psi\big(\max\big\{d(x_0,x_1), d(x_1,Gx_1)\big\}\big) \end{aligned} \tag{2.2}$$

since $\frac{d(x_0,Gx_1)}{2} \leq \max\{d(x_0,x_1), d(x_1,Gx_1)\}$ and $\frac{d(x_0,Gx_0)+d(x_1,Gx_1)}{2} \leq \max\{d(x_0,x_1), d(x_1,Gx_1)\}$. Assume that $\max\{d(x_0,x_1), d(x_1,Gx_1)\} = d(x_1,Gx_1)$. Then from (2.2) we have

$$\begin{aligned} 0 &< d(x_1, Gx_1 \cap D) \leq \alpha(x_0,x_1)H(Gx_0 \cap D, Gx_1 \cap D) \\ &\leq \psi\big(d(x_1,Gx_1)\big) \\ &< d(x_1,Gx_1), \end{aligned} \tag{2.3}$$

a contradiction to our assumption. Thus $\max\{d(x_0,x_1), d(x_1,Gx_1)\} = d(x_0,x_1)$. Then from (2.2) we have

$$0 < d(x_1, Gx_1 \cap D) \leq \psi\big(d(x_0,x_1)\big). \tag{2.4}$$

For $q > 1$ by Lemma 1.1, there exists $x_2 \in Gx_1 \cap D$ such that

$$0 < d(x_1,x_2) < qd(x_1, Gx_1 \cap D) \leq q\psi\big(d(x_0,x_1)\big). \tag{2.5}$$

Applying ψ in (2.5), we have

$$0 < \psi\big(d(x_1,x_2)\big) < \psi\big(q\psi\big(d(x_0,x_1)\big)\big). \tag{2.6}$$

Put $q_1 = \frac{\psi(q\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$. Since G is an α-admissible mapping, $\alpha(x_1,x_2) \geq 1$. If $x_2 \in Gx_2 \cap D$, then x_2 is a fixed point. Let $x_2 \notin Gx_2 \cap D$. From (2.1) we have

$$\begin{aligned}
0 &< \alpha(x_1,x_2)H(Gx_1 \cap D, Gx_2 \cap D) \\
&\leq \psi\left(\max\left\{d(x_1,x_2), \frac{d(x_1,Gx_1)+d(x_2,Gx_2)}{2}, \frac{d(x_1,Gx_2)+d(x_2,Gx_1)}{2}\right\}\right) \\
&\leq \psi\left(\max\left\{d(x_1,x_2), d(x_2,Gx_2)\right\}\right) \qquad (2.7)
\end{aligned}$$

since $\frac{d(x_1,Gx_2)}{2} \leq \max\{d(x_1,x_2), d(x_2,Gx_2)\}$ and $\frac{d(x_1,Gx_1)+d(x_2,Gx_2)}{2} \leq \max\{d(x_1,x_2), d(x_2,Gx_2)\}$. Assume that $\max\{d(x_1,x_2), d(x_2,Gx_2)\} = d(x_2,Gx_2)$. Then from (2.7) we have

$$\begin{aligned}
0 &< d(x_2, Gx_2 \cap D) \leq \alpha(x_1,x_2)H(Gx_1 \cap D, Gx_2 \cap D) \\
&\leq \psi\left(d(x_2,Gx_2)\right) \\
&< d(x_2,Gx_2), \qquad (2.8)
\end{aligned}$$

a contradiction to our assumption. Thus $\max\{d(x_1,x_2), d(x_2,Gx_2)\} = d(x_1,x_2)$. Then from (2.7) we have

$$0 < d(x_2, Gx_2 \cap D) \leq \psi\left(d(x_1,x_2)\right). \qquad (2.9)$$

For $q_1 > 1$ by Lemma 1.1, there exists $x_3 \in Gx_2 \cap D$ such that

$$0 < d(x_2,x_3) < q_1 d(x_2, Gx_2 \cap D) \leq q_1\psi\left(d(x_1,x_2)\right) = \psi\left(q\psi\left(d(x_0,x_1)\right)\right). \qquad (2.10)$$

Applying ψ in (2.10), we have

$$0 < \psi\left(d(x_2,x_3)\right) < \psi^2\left(q\psi\left(d(x_0,x_1)\right)\right). \qquad (2.11)$$

Put $q_2 = \frac{\psi^2(q\psi(d(x_0,x_1)))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$. Since G is an α-admissible mapping, $\alpha(x_2,x_3) \geq 1$. If $x_3 \in Gx_3 \cap D$, then x_3 is a fixed point. Let $x_3 \notin Gx_3 \cap D$. From (2.1) we have

$$\begin{aligned}
0 &< \alpha(x_2,x_3)H(Gx_2 \cap D, Gx_3 \cap D) \\
&\leq \psi\left(\max\left\{d(x_2,x_3), \frac{d(x_2,Gx_2)+d(x_3,Gx_3)}{2}, \frac{d(x_2,Gx_3)+d(x_3,Gx_2)}{2}\right\}\right) \\
&\leq \psi\left(\max\left\{d(x_2,x_3), d(x_3,Gx_3)\right\}\right) \qquad (2.12)
\end{aligned}$$

since $\frac{d(x_2,Gx_3)}{2} \leq \max\{d(x_2,x_3), d(x_3,Gx_3)\}$ and $\frac{d(x_2,Gx_2)+d(x_3,Gx_3)}{2} \leq \max\{d(x_2,x_3), d(x_3, Gx_3)\}$. Assume that $\max\{d(x_2,x_3), d(x_3,Gx_3)\} = d(x_3,Gx_3)$. Then from (2.12) we have

$$\begin{aligned}
0 &< d(x_3, Gx_3 \cap D) \leq \alpha(x_2,x_3)H(Gx_2 \cap D, Gx_3 \cap D) \\
&\leq \psi\left(d(x_3,Gx_3)\right) \\
&< d(x_3,Gx_3), \qquad (2.13)
\end{aligned}$$

a contradiction to our assumption. Thus $\max\{d(x_2,x_3), d(x_3,Gx_3)\} = d(x_2,x_3)$. Then from (2.12) we have

$$0 < d(x_3, Gx_3 \cap D) \le \psi\big(d(x_2,x_3)\big). \tag{2.14}$$

For $q_2 > 1$ by Lemma 1.1, there exists $x_4 \in Gx_3 \cap D$ such that

$$0 < d(x_3,x_4) < q_2 d(x_3, Gx_3 \cap D) \le q_2\psi\big(d(x_2,x_3)\big) = \psi^2\big(q\psi\big(d(x_0,x_1)\big)\big). \tag{2.15}$$

Applying ψ in (2.15), we have

$$0 < \psi\big(d(x_3,x_4)\big) < \psi^3\big(q\psi\big(d(x_0,x_1)\big)\big). \tag{2.16}$$

Continuing the same process, we get a sequence $\{x_n\}$ in D such that $x_{n+1} \in Gx_n \cap D$, $x_{n+1} \ne x_n$, $\alpha(x_n,x_{n+1}) \ge 1$, and

$$d(x_{n+1},x_{n+2}) < \psi^n\big(q\psi\big(d(x_0,x_1)\big)\big) \quad \text{for each } n \in \mathbb{N} \cup \{0\}. \tag{2.17}$$

For $m, n \in \mathbb{N}$, we have

$$d(x_n,x_{n+m}) \le \sum_{i=n}^{n+m-1} d(x_i,x_{i+1}) < \sum_{i=n}^{n+m-1} \psi^{i-1}\big(d(x_0,x_1)\big).$$

Since $\psi \in \Psi$, it follows that $\{x_n\}$ is a Cauchy sequence in D. Since D is complete, there exists $x^* \in D$ such that $x_n \to x^*$ as $n \to \infty$. By the continuity of G, we have

$$d\big(x^*, Gx^*\big) \le \lim_{n\to\infty} H\big(Gx_n, Gx^*\big) = 0. \qquad \square$$

Theorem 2.4 *Let (X,d) be a metric space, D be a nonempty subset of X which is complete with respect to the metric induced by d, and let G be a strictly (α,ψ)-contractive type mapping on D. Assume that the following conditions hold*:

(i) *G is an α-admissible map*;
(ii) *there exist $x_0 \in D$ and $x_1 \in Gx_0 \cap D$ such that $\alpha(x_0,x_1) \ge 1$*;
(iii) *either*
  (a) *for any sequence $\{x_n\}$ in D such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for each $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} \alpha(x_n,x) \ge 1$,*
  or
  (b) *for any sequence $\{x_n\}$ in D such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for each $n \in \mathbb{N} \cup \{0\}$, $\alpha(x_n,x) \ge 1$ for each $n \in \mathbb{N} \cup \{0\}$.*

Then G has a fixed point.

Proof Following the proof of Theorem 2.3, there exists a Cauchy sequence $\{x_n\}$ in D with $x_n \to x^*$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for each $n \in \mathbb{N} \cup \{0\}$. Suppose that $d(x^*,Gx^*) \ne 0$. From (2.1) we have

$$\begin{aligned}\alpha\big(x_n,x^*\big)d\big(x_{n+1},Gx^* \cap D\big) &\le \alpha\big(x_n,x^*\big)H\big(Gx_n \cap D, Gx^* \cap D\big)\\ &\le \psi\left(\max\left\{d\big(x_n,x^*\big), \frac{d(x_n,Gx_n)+d(x^*,Gx^*)}{2},\right.\right.\end{aligned}$$

$$\begin{aligned} &\left.\left.\frac{d(x_n, Gx^*) + d(x^*, Gx_n)}{2}\right\}\right) \\ &< \max\left\{d(x_n, x^*), \frac{d(x_n, Gx_n) + d(x^*, Gx^*)}{2},\right. \\ &\left.\frac{d(x_n, Gx^*) + d(x^*, Gx_n)}{2}\right\}. \end{aligned} \tag{2.18}$$

Letting $n \to \infty$ in (2.18), we have

$$\lim_{n\to\infty} \alpha(x_n, x^*) d(x^*, Gx^* \cap D) \le \frac{d(x^*, Gx^*)}{2}. \tag{2.19}$$

Since $\lim_{n\to\infty} \alpha(x_n, x^*) \ge 1$, by condition (iii)(a), we have

$$d(x^*, Gx^* \cap D) \le \lim_{n\to\infty} \alpha(x_n, x^*) d(x^*, Gx^* \cap D) \le \frac{d(x^*, Gx^*)}{2}. \tag{2.20}$$

Further, it is clear that $d(x^*, Gx^*) \le d(x^*, Gx^* \cap D)$. Then from (2.20) we have

$$d(x^*, Gx^*) \le \frac{d(x^*, Gx^*)}{2},$$

which is impossible. Thus $d(x^*, Gx^*) = 0$. If we use (iii)(b), then from (2.1) we have

$$\begin{aligned} d(x_{n+1}, Gx^* \cap D) &\le \alpha(x_n, x^*) H(Gx_n \cap D, Gx^* \cap D) \\ &\le \psi\left(\max\left\{d(x_n, x^*), \frac{d(x_n, Gx_n) + d(x^*, Gx^*)}{2},\right.\right. \\ &\quad \left.\left.\frac{d(x_n, Gx^*) + d(x^*, Gx_n)}{2}\right\}\right) \\ &< \max\left\{d(x_n, x^*), \frac{d(x_n, Gx_n) + d(x^*, Gx^*)}{2},\right. \\ &\quad \left.\frac{d(x_n, Gx^*) + d(x^*, Gx_n)}{2}\right\}. \end{aligned} \tag{2.21}$$

Letting $n \to \infty$ in (2.21), we have

$$d(x^*, Gx^*) \le d(x^*, Gx^* \cap D) \le \frac{d(x^*, Gx^*)}{2},$$

which is impossible. Thus $d(x^*, Gx^*) = 0$. □

Example 2.5 Let $X = (-\infty, -8) \cup [0, \infty)$ be endowed with the usual metric d, and let $D = [0, \infty)$. Define $G : D \to CL(X)$ by

$$Gx = \begin{cases} [0, \frac{x}{4}] & \text{if } 0 \le x < 4, \\ \{0\} & \text{if } x = 4, \\ (-\infty, -3x] \cup [x, x^2] & \text{if } x > 4 \end{cases}$$

and $\alpha : D \times D \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 4], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $Gx \cap D \neq \emptyset$ for each $x \in D$. Let $\psi(t) = \frac{t}{2}$ for each $t \geq 0$. To see that G is a strictly (α, ψ)-contractive type mapping on D, we consider the following cases.

Case (i) When $x, y \in [0, 4)$, we have

$$\alpha(x,y)H(Gx \cap D, Gy \cap D) = \left|\frac{x}{4} - \frac{y}{4}\right| \leq \frac{|x-y|}{2} = \psi\big(d(x,y)\big) \leq \psi\big(M(x,y)\big).$$

Case (ii) When $x \in [0, 4)$ and $y = 4$, we have

$$\alpha(x,y)H(Gx \cap D, Gy \cap D) = \left|\frac{x}{4}\right| \leq \psi\left(\frac{d(x,Gx) + d(y,Gy)}{2}\right) \leq \psi\big(M(x,y)\big).$$

Case (iii) Otherwise, we have

$$\alpha(x,y)H(Gx \cap D, Gy \cap D) = 0 \leq \psi\big(M(x,y)\big),$$

where $M(x,y) = \max\{d(x,y), \frac{d(x,Gx)+d(y,Gy)}{2}, \frac{d(x,Gy)+d(y,Gx)}{2}\}$.

Thus, G is a strictly (α, ψ)-contractive type mapping on D. For $\alpha(x,y) \geq 1$, we have $x, y \in [0, 4]$, then $Gx \cap D, Gy \cap D \subseteq [0, 1]$, thus $\alpha(u, v) = 1$ for each $u \in Gx \cap D$ and $v \in Gy \cap D$. Further, for any sequence $\{x_n\}$ in D such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) = 1$ for each $n \in \mathbb{N} \cup \{0\}$, $\lim_{n\to\infty} \alpha(x_n, x) = 1$. Therefore, all the conditions of Theorem 2.4 hold and G has a fixed point.

Corollary 2.6 *Let $(X, \preceq, d)$ be an ordered metric space, let $(D, \preceq)$ be a nonempty subset of X which is complete with respect to the metric induced by d. Let $G : D \to CL(X)$ be a mapping such that $Gx \cap D \neq \emptyset$ for each $x \in D$ and for each $x, y \in D$ with $x \preceq y$, we have*

$$H(Gx \cap D, Gy \cap D) \leq \psi\big(M(x,y)\big),$$

where $M(x,y) = \max\{d(x,y), \frac{d(x,Gx)+d(y,Gy)}{2}, \frac{d(x,Gy)+d(y,Gx)}{2}\}$ and ψ is an increasing function in Ψ. Also, assume that the following conditions hold:

(i) *there exist $x_0 \in D$ and $x_1 \in Gx_0 \cap D$ such that $x_0 \preceq x_1$;*
(ii) *if $x \preceq y$ then $Gx \cap D \prec_r Gy \cap D$;*
(iii) *either*
  (a) *G is continuous,*
  or
  (b) *for any sequence $\{x_n\}$ in D such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, $x_n \preceq x$ as $n \to \infty$,*
  or
  (c) *for any sequence $\{x_n\}$ in D such that $x_n \to x$ as $n \to \infty$ and $x_n \preceq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, $x_n \preceq x$ for each $n \in \mathbb{N} \cup \{0\}$.*

Then G has a fixed point.

Proof Define $\alpha : D \times D \to [0, \infty)$ by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

By using condition (i) and the definition of α, we have $\alpha(x_0, x_1) = 1$. Also, from condition (ii), we have that $x \preceq y$ implies $Gx \cap D \prec_r Gy \cap D$; by using the definitions of α and $\prec_r$, we have that $\alpha(x, y) = 1$ implies $\alpha(u, v) = 1$ for each $u \in Gx \cap D$ and $v \in Gy \cap D$. Moreover, it is easy to check that G is a strictly (α, ψ)-contractive type mapping on D. Therefore, all the conditions of Theorem 2.3 (or Theorem 2.4 for (iii)(b), (iii)(c)) hold, hence G has a fixed point. □

Remark 2.7 Condition (a), in the statement of Theorem 2.4, was introduced by Samet *et al.* [20]. In Theorem 2.4 we introduce a new condition (b). The following examples show that (a) and (b) are independent conditions.

Example 2.8 Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Consider $x_n = \frac{1}{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then $x_n \to 0 = x^*$ as $n \to \infty$. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} \max\{\frac{1}{x}, \frac{1}{y}\} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{1}{x+y} & \text{if either } x = 0 \text{ or } y = 0, \\ 1 & \text{if } x = y = 0. \end{cases}$$

Now, we have $\alpha(x_n, x_{n+1}) = \alpha(\frac{1}{n+1}, \frac{1}{n+2}) = n + 2 > 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\alpha(x_n, x^*) = \alpha(\frac{1}{n+1}, 0) = n + 1 \geq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Thus condition (a) holds but $\lim_{n\to\infty} \alpha(x_n, x^*) = \lim_{n\to\infty}(n + 1) = \infty$. Thus condition (b) does not hold.

Example 2.9 Let $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Consider $x_n = \frac{1}{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$, then $x_n \to 0 = x^*$ as $n \to \infty$. Define $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} \max\{\frac{1}{x}, \frac{1}{y}\} & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \frac{1}{1+(x+y)/2} & \text{if either } x = 0 \text{ or } y = 0, \\ 1 & \text{if } x = y = 0. \end{cases}$$

Now, we have $\alpha(x_n, x_{n+1}) = \alpha(\frac{1}{n+1}, \frac{1}{n+2}) = n + 2 > 1$ for each $n \in \mathbb{N} \cup \{0\}$ and $\alpha(x_n, x^*) = \alpha(\frac{1}{n+1}, 0) = \frac{2n+2}{2n+3}$. Then $\lim_{n\to\infty} \alpha(x_n, x^*) = \lim_{n\to\infty} \frac{2n+2}{2n+3} = 1$. Thus condition (b) holds but for $n = 0$, we have $\alpha(x_n, x^*) = \frac{2}{3}$; for $n = 1$, we have $\alpha(x_n, x^*) = \frac{4}{5}$; for $n = 2$, we have $\alpha(x_n, x^*) = \frac{6}{7}$, which implies that $\alpha(x_n, x) \ngeq 1$ for each $n \in \mathbb{N} \cup \{0\}$. Thus condition (a) does not hold.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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