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Some new sharp limit Hardy-type inequalities via convexity

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Abstract

Some new limit cases of Hardy-type inequalities are proved, discussed and compared. In particular, some new refinements of Bennett's inequalities are proved. Each of these refined inequalities contain two constants, and both constants are in fact sharp. The technique in the proofs is new and based on some convexity arguments of independent interest.

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1 Introduction

Hardy's inequality in its original continuous form reads (see [1, 2]): If f is non-negative and p-integrable over $(0, \infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p > 1. \tag{1.1}$$

In 1928 Hardy himself (see [3]) proved the first weighted version of (1.1), namely that the following inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^a\,dx \le \left(\frac{p}{p-1-a}\right)^p \int_0^\infty f^p(x)x^a\,dx \tag{1.2}$$

holds for all measurable and non-negative functions f on $(0, \infty)$, whenever $a < p-1$, $p \ge 1$.

The constant $C = (\frac{p}{p-1-a})^p$ in (1.2) is sharp. There exists nowadays a lot of information about Hardy-type inequalities, see, *e.g.*, the books [4, 5] and the references given there. In particular, in these books it is pointed out that such inequalities are specially important for a great variety of applications, *e.g.*, to the theory of function spaces, interpolation theory, approximation theory, partial differential equations, *etc.*

However, there exist very few Hardy-type inequalities with sharp constant in the limit case and when the interval $(0, \infty)$ is replaced by a finite interval $(0, \ell)$, $\ell < \infty$. We continue by giving two such examples, where the first one (Bennett's inequalities) also has direct applications, *e.g.*, to interpolation theory (see Remark 1.1 below).

Proposition A *Let $\alpha > 0$, $1 \le p \le \infty$ and f be a non-negative and measurable function on* $[0,1]$. *Then*

$$\left(\int_0^1 \bigl[\log(e/x)\bigr]^{\alpha p-1}\left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x}\right)^{1/p} \le \alpha^{-1}\left(\int_0^1 x^p\bigl[\log(e/x)\bigr]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p}, \tag{1.3}$$

and

$$\left(\int_0^1 \bigl[\log(e/x)\bigr]^{-\alpha p-1}\left(\int_x^1 f(y)\,dy\right)^p \frac{dx}{x}\right)^{1/p} \le \alpha^{-1}\left(\int_0^1 x^p\bigl[\log(e/x)\bigr]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}\right)^{1/p} \tag{1.4}$$

with the usual modification if $p = \infty$.

Proof The proof is given in [6] for $1 \le p < \infty$. For completeness we present a short proof for $p = \infty$. We consider first (1.3).

We have that

$$\int_0^x f(t)\,dt \le \left(\sup_{0<t<x} t\log^{1+\alpha}\left(\frac{e}{t}\right)f(t)\right)\int_0^x \log^{-(1+\alpha)}\left(\frac{e}{t}\right)\frac{dt}{t}.$$

After a change of variable and easy calculations, we get that

$$\left(\log\frac{e}{x}\right)^{\alpha}\int_0^x f(t)\,dt \le \alpha^{-1}\sup_{0<t<1} t\log^{1+\alpha}\left(\frac{e}{t}\right)f(t)$$

and, hence,

$$\sup_{0<x<1}\left(\log\frac{e}{x}\right)^{\alpha}\int_0^x f(t)\,dt \le \alpha^{-1}\sup_{0<x<1} x\log^{1+\alpha}\left(\frac{e}{x}\right)f(x). \tag{1.5}$$

In the same way we can prove the inequality corresponding to $p = \infty$ in (1.4),

$$\sup_{0<x<1}\left(\log\frac{e}{x}\right)^{-\alpha}\int_x^1 f(t)\,dt \le \alpha^{-1}\sup_{0<x<1} x\log^{1-\alpha}\left(\frac{e}{x}\right)f(x). \tag{1.6}$$

□

Remark 1.1 This result is due to Bennett [6]. He derived it as an important tool when he described the intermediate spaces between L and $L\log^+ L$ with the Peetre real (K–) method. This result was later on completed and applied in various ways in, *e.g.*, [7–9]. In fact, the constant α^{-1} in both (1.3) and (1.4) is sharp. This was not pointed out in [6], but it is a consequence of our Theorem 2.3.

Next we note that a local variant of (1.2) reads: If $0 < \ell < \infty$, $a < p-1$, $p \ge 1$, then

$$\int_0^{\ell}\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^a\,dx \le \left(\frac{p}{p-1-a}\right)^p\int_0^{\ell} f^p(x)x^a\,dx. \tag{1.7}$$

However, this inequality is not best possible, but also a 'sharp' variant of this inequality is known.

Proposition B *Let $0 < \ell \leq \infty$, $p \geq 1$, $a < p-1$ and f be a non-negative and measurable function on $[0,\ell)$. Then*

$$\int_0^\ell \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^a\,dx \leq \left(\frac{p}{p-1-a}\right)^p \int_0^\ell f^p(x)x^a\left(1-\left(\frac{x}{\ell}\right)^{\frac{p-a-1}{p}}\right)dx. \tag{1.8}$$

The constant $C = (\frac{p}{p-1-a})^p$ is sharp.

Remark 1.2 Inequality (1.8) was proved in [10] (see also [11]) and independently in [12] (for an elementary proof and some further results, see also [13]). Another elementary proof of (1.8) can be found in the recent paper [14]. This proof shows, in particular, that (1.8) holds also for $p < 0$ (here we must assume that $f > 0$ a.e.) and that (1.8) is in fact equivalent to the basic inequality

$$\int_0^\ell \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \leq 1 \cdot \int_0^\ell g^p(x)\left(1-\frac{x}{\ell}\right)\frac{dx}{x}$$

(which holds by Jensen's inequality) and also to a number of other Hardy-type inequalities (see Theorem 1.3 in [14]).

In Section 2 of this paper, we prove a refined version of Proposition A, where all involved constants are sharp (see Theorem 2.3 which, in particular, shows that the constant $C = \alpha^{-1}$ in Proposition A is sharp in both (1.1) and (1.2)). Up to our knowledge, there is not known any other Hardy-type inequalities with this property. The method of proof is completely different from that in [6] and is based on a convexity argument (see Lemma 2.1 and Remark 2.2).

In Section 3 we present some further results and remarks. In particular, we use the idea from the proof of Theorem 2.1 in [14] to derive a sharp inequality of the same type as those in Proposition A (see Proposition 3.3 and Example 3.6) and compare these results. Also in this case the proof is based on a convexity argument. Moreover, we discuss the connections between the results above (*e.g.*, that Proposition A is in a sense almost a limit case of Proposition B when $a = p-1$ and $\alpha = 1/p$). All inequalities we derive are sharp but the optimal test functions are different. Therefore, we can in particular formulate another strict improvement of Proposition A (see, *e.g.*, Remark 3.5 and Example 3.6).

2 A refinement of Proposition A

The following well-known lemma of independent interest will be used in the proof of Theorem 2.3.

Lemma 2.1 *It yields that*

$$h^p - ph + p - 1 \begin{cases} \geq 0 & \text{if } p \geq 1, \\ \leq 0 & \text{if } 0 < p < 1 \end{cases} \tag{2.1}$$

for all $h > 0$. Equality holds if and only if $h = 1$.

Remark 2.2 The crucial inequality (2.1) is called 'a fundamental relationship' in the book [15] (p.12). Two proofs are pointed out in this book. Another proof follows by observing that $y(x) = x^p$ is convex for $p \geq 1$ or $p < 0$ and concave for $0 < p \leq 1$, and that the equation of the tangent at the point $x = 1$ is $y = px - p + 1$. In the general case, the tangent will be $y = \Psi'(1)x + \Psi(1) - \Psi'(1)$, which implies a generalization of (2.1), when $\Psi(x)$ is convex/concave.

The main result in this section is a refinement and extension of Proposition A.

Theorem 2.3 *Let $\alpha, p > 0$ and f be a non-negative and measurable function on $[0,1]$.*

(a) *If $p > 1$, then*

$$\alpha^{p-1}\left(\int_0^1 f(x)\,dx\right)^p + \alpha^p \int_0^1 \left[\log(e/x)\right]^{\alpha p-1}\left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \leq \int_0^1 x^p\left[\log(e/x)\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \tag{2.2}$$

and

$$\alpha^{p-1}\left(\int_0^1 f(x)\,dx\right)^p + \alpha^p \int_0^1 \left[\log(e/x)\right]^{-\alpha p-1}\left(\int_x^1 f(y)\,dy\right)^p \frac{dx}{x} \leq \int_0^1 x^p\left[\log(e/x)\right]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{2.3}$$

Both constants α^{p-1} and α^p in (2.2) and (2.3) are sharp. Equality is never attained unless f is identically zero.

(b) *If $0 < p < 1$, then both (2.2) and (2.3) hold in the reverse direction and the constants in both inequalities are sharp. Equality is never attained unless f is identically zero.*

(c) *If $p = 1$ we have equality in (2.2) and (2.3) for any measurable function f and any $\alpha > 0$.*

Proof (a) Let $p > 1$. Suppose that f is a continuous function and define for $x \in (0,1]$ the function

$$F(x;\alpha,p) := \int_0^x y^p\left(\log(e/y)\right)^{(1+\alpha)p-1} f^p(y)\frac{dy}{y} - \alpha^p \int_0^x \left(\log(e/y)\right)^{\alpha p-1}\left(\int_0^y f(s)\,ds\right)^p \frac{dy}{y} - \alpha^{p-1}\left(\log(e/x)\right)^{\alpha p}\left(\int_0^x f(y)\,dy\right)^p.$$

Differentiation gives that

$$\frac{d}{dx}F(x;\alpha,p) = \left(\log(e/x)\right)^{\alpha p-1} \cdot \frac{1}{x}\left(\alpha \int_0^x f(s)\,ds\right)^p \left[h^p(x;\alpha) - ph(x;\alpha) + p - 1\right]$$

with

$$h(x;\alpha) := \frac{x(\log(e/x))f(x)}{\alpha \int_0^x f(y)\,dy}.$$

Thus, according to Lemma 2.1, we have that $\frac{d}{dx}F(x;\alpha,p) > 0$, *i.e.*, $F(x;\alpha,p)$ is strictly increasing. Hence, in particular,

$$F(1;\alpha,p) \geq \lim_{x\to 0^+} F(x;\alpha,p).$$

By applying Hölder's inequality, we find that

$$\begin{aligned}\int_0^x f(y)\,dy &= \int_0^x \bigl[y^{1-1/p}\bigl(\log(e/y)\bigr)^{\alpha+1-1/p} f(y)\bigr] y^{-1+1/p}\bigl(\log(e/y)\bigr)^{-\alpha-1+1/p}\,dy \\ &\leq \left(\frac{p-1}{\alpha p}\right)^{(p-1)/p}\bigl(\log(e/x)\bigr)^{-\alpha} I(x)\end{aligned}$$

with

$$\bigl(I(x)\bigr)^p = \int_0^x y^{p-1}\bigl(\log(e/y)\bigr)^{(1+\alpha)p-1} f^p(y)\,dy.$$

Hence, we get that

$$0 < \bigl(\log(e/x)\bigr)^{\alpha p}\left(\int_0^x f(y)\,dy\right)^p \leq \left(\frac{p-1}{\alpha p}\right)^{p-1} I^p(x).$$

Since $I(x) \to 0$ as $x \to 0_+$, we have that $\lim_{x\to 0_+} F(x;\alpha,p) = 0$ and, in particular,

$$F(1;\alpha,p) \geq \lim_{x\to 0_+} F(x;\alpha,p) = 0.$$

Hence, we have proved that (2.2) holds for all continuous functions. By standard approximation arguments, (2.2) holds for all measurable functions.

Now we prove the sharpness of inequality (2.2). We consider the inequality

$$\begin{aligned}&K_1\left(\int_0^1 f(x)\,dx\right)^p + K_2\int_0^1 \bigl(\log(e/x)\bigr)^{\alpha p-1}\left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \\ &\quad\leq \int_0^1 x^p\bigl(\log(e/x)\bigr)^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}\end{aligned} \tag{2.4}$$

for $\alpha > 0$, $p > 1$ and some constants $0 < K_1, K_2 < \infty$. Assume that (2.4) holds for some constant $K_2 \geq \alpha^p$ and consider the test function

$$f_\varepsilon(x) = \frac{1}{x}\bigl(\log(e/x)\bigr)^{-(1+\varepsilon+\alpha)}, \quad \varepsilon > 0. \tag{2.5}$$

Then we have that

$$\begin{aligned}&\int_0^1 f_\varepsilon(x)\,dx = \frac{1}{\varepsilon+\alpha}, \\ &\int_0^1 \bigl(\log(e/x)\bigr)^{\alpha p-1}\left(\int_0^x f_\varepsilon(y)\,dy\right)^p \frac{dx}{x} = \frac{1}{\varepsilon p(\varepsilon+\alpha)^p}\end{aligned}$$

and

$$\int_0^1 x^p \big(\log(e/x)\big)^{(1+\alpha)p-1} f_\varepsilon^p(x)\frac{dx}{x} = \frac{1}{\varepsilon p}.$$

Hence, by (2.4), we have that

$$\frac{K_1}{(\varepsilon+\alpha)^p} + \frac{K_2}{\varepsilon p(\varepsilon+\alpha)^p} \le \frac{1}{\varepsilon p},$$

i.e., that

$$\varepsilon p K_1 + K_2 \le (\varepsilon+\alpha)^p.$$

By letting $\varepsilon \to 0_+$, we find that $K_2 \le \alpha^p$. This contradiction shows that the sharp constant K_2 in (2.4) is $K_2 = \alpha^p$. We consider now (2.4) with $K_2 = \alpha^p$ and assume that it holds with some $K_1 > \alpha^{p-1}$. By using the same test function f_ε in (2.5), we get that

$$\frac{K_1}{(\varepsilon+\alpha)^p} + \frac{\alpha^p}{\varepsilon p(\varepsilon+\alpha)^p} \le \frac{1}{\varepsilon p},$$

i.e.,

$$K_1 \le \frac{(\varepsilon+\alpha)^p - \alpha^p}{\varepsilon p}.$$

Hence, by letting $\varepsilon \to 0_+$, we obtain that $K_1 \le \alpha^{p-1}$. This shows that $K_1 = \alpha^{p-1}$ is the sharp constant in (2.4) and consequently in (2.2).

The proof of (2.3) is similar. For this we define

$$\begin{aligned} G(x;\alpha,p) := {} & \int_0^x y^{p-1}\big(\log(e/y)\big)^{(1-\alpha)p-1} f^p(y)\,dy \\ & - \alpha^p \int_0^x y^{-1}\big(\log(e/y)\big)^{-\alpha p-1}\left(\int_y^1 f(s)\,ds\right)^p dy \\ & - \alpha^{p-1}\big(\log(e/x)\big)^{-\alpha p}\left(\int_0^x f(s)\,ds\right)^p \end{aligned}$$

and argue similarly as before. The proof of the sharpness of (2.3) consists only of small modifications of the proof above. By Lemma 2.1 it is clear that we cannot have equality neither in (2.2) nor in (2.3) unless f is identically zero. The proof is complete.

(b) Let $0 < p < 1$. In this case the crucial convexity inequality from Lemma 2.1 holds in the reversed direction. Hence, the proofs of the reverse of (2.2) and (2.3) consist only of small modifications of the proofs of (2.2) and (2.3), respectively.

(c) The equality for $p = 1$ in both (2.2) and (2.3) is just a consequence of partial integration and limiting arguments or of straightforward modifications of the proof above. □

Remark 2.4 Easy calculations show that if $p = \infty$ we get equality in inequality (1.5) for $f(x) = \frac{1}{x}(\log\frac{e}{x})^{-(1+\alpha)}$. Hence, in this case, inequality (1.5) cannot be improved in the same manner as above to a refined inequality of the type (2.2). In the same way, we find that for the case $p = \infty$ inequality (1.6) cannot be improved to some refined inequality of the type (2.3).

3 Further results and remarks

Remark 3.1 We note that, by making the variable transformation $x = lt$, the result in Proposition A can be formulated for the interval $(0, \ell)$ instead of $(0,1)$. Hence, it can be compared with that of Proposition B. The same argument shows that it is no loss of generality to formulate Proposition B with $\ell < \infty$ only for $\ell = 1$, so in the sequel we consider only this case.

Remark 3.2 Inequality (1.7) has no sense if $a = p - 1$, $p > 1$. However, it is reasonable to ask if the following (limit) inequality can hold for some $K > 0$

$$\int_0^1 \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{p-1}\,dx \le K \int_0^1 f^p(x)x^{p-1}\,dx. \tag{3.1}$$

In fact, this is not the case. Assume that (3.1) holds for all measurable functions f and some $K < \infty$. Let $\varepsilon > 0$ and insert the test function

$$f_\varepsilon(x) = \frac{1}{x}\left(\ln\frac{e}{x}\right)^{-(1+\varepsilon+1/p)}$$

into (3.1). A simple calculation shows that

$$\frac{p + p\varepsilon}{\varepsilon p(\varepsilon + 1/p)} \le K$$

and since $(p + p\varepsilon)/(\varepsilon p(\varepsilon + 1/p)) \to \infty$ as $\varepsilon \to 0^+$, we get a contradiction. This shows that (3.1) cannot hold for any $K < \infty$. However, by using Proposition A with $\alpha = 1/p$, we can see that the related inequality

$$\int_0^1 \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{p-1}\,dx \le p^p \int_0^1 f^p(x)x^{p-1}\left(\log\frac{e}{x}\right)^p dx \tag{3.2}$$

holds and that the constant p^p is sharp.

Inspired by Remark 3.2 and the technique used in [14] to prove (1.7), we formulate the following estimate of the same type as that in Proposition A.

Proposition 3.3 *Let $p \in \mathbb{R} \setminus \{0\}$, f be a positive and measurable function on $(0,1)$ and u and v be two weight functions on $(0,1)$ such that*

$$u(x) := x\int_x^1 v(y)\frac{1}{y^2}\,dy.$$

If $p \ge 1$ or $p < 0$, then

$$\int_0^1 \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p v(x)\frac{dx}{x} \le \int_0^1 f^p(x)u(x)\frac{dx}{x} \tag{3.3}$$

(*for the case $p < 0$, we assume that $f > 0$*).

If $0 < p \leq 1$, *then* (3.3) *holds in the reverse direction. Inequality* (3.3) *and the reverse inequality for* $0 < p < 1$ *are sharp and equality holds for* $f(x) \equiv C$, $C > 0$.

Proof By using Jensen's inequality with the convex function $\Psi(x) = x^p$, $p \geq 1$ or $p < 0$, and Fubini's theorem, we obtain that

$$\begin{aligned}\int_0^1 \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p v(x)\frac{dx}{x} &\leq \int_0^1 \left(\int_0^x f^p(y)\,dy\right)\frac{v(x)}{x^2}\,dx \\ &= \int_0^1 f^p(y)\left(\int_y^1 v(x)\frac{1}{x^2}\,dx\right)dy = \int_0^1 f^p(y)u(y)\frac{dy}{y}.\end{aligned}$$

Since for $f \equiv C$ we get equality in our inequality, the sharpness statement is also proved. The proof for the case $0 < p < 1$ follows similarly, since the only inequality above holds in the reverse direction. □

Example 3.4 By using Proposition 3.3 with $v(x) = x^p(\log\frac{e}{x})^{\alpha p-1}$, we find that

$$I_1 := \int_0^1 \left(\log\frac{e}{x}\right)^{\alpha p-1}\left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \leq \int_0^1 u(x) f^p(x)\frac{dx}{x} =: I_2, \tag{3.4}$$

where $u(x) := x\int_x^1 y^{p-2}(\log(e/y))^{\alpha p-1}\,dy$. Both inequalities (1.3) and (3.4) are sharp but the optimal test functions are completely different so the two results cannot be compared.

Remark 3.5 Example 3.4 shows, in fact, that we have the following strict improvement of inequality (1.3) in Proposition A,

$$I_1 \leq \min(I_2, I_3),$$

where I_1 and I_2 are defined in (3.4) and

$$I_3 = \alpha^{-p}\int_0^1 x^p\left(\log\frac{e}{x}\right)^{(1+\alpha)p-1} f^p(x)\frac{dx}{x}.$$

Moreover, a similar improvement of (1.4) can be stated.

Example 3.6 If $\alpha = 1/p$, $p > 1$, we obtain the following sharp inequality (*cf.* also (3.2))

$$\int_0^1 \left(\int_0^x f(y)\,dy\right)^p \frac{dx}{x} \leq \min\left(p^p\int_0^1 x^{p-1}\left(\log\frac{e}{x}\right)^p f^p(x)\,dx, \frac{1}{p-1}\int_0^1 \left(1-x^{p-1}\right)f^p(x)\,dx\right)$$

as a limit case of the Hardy inequality (corresponding to the case $a = p - 1$).

Remark 3.7 The simple proof of Proposition 3.3 shows that it can be generalized also to multidimensional cases and to integral inequalities with general measures. We can also formulate Proposition 3.3 in terms of a general convex function $\Psi(x)$ instead of the special case $\Psi(x) = x^p$, *cf.* [16] and the references given there.

Remark 3.8 By using suitable variable transformations, all inequalities in this paper can be reformulated over the interval $[1, \infty)$ instead of over the interval $[0, 1]$.

Finally, we present the following variant of Theorem 2.3.

Theorem 3.9 *Let* $\alpha, p > 0$ *and* f *be a non-negative and measurable function on* $[1, \infty)$.

(a) *If* $p > 1$, *then*

$$\alpha^{p-1}\left(\int_1^\infty f(x)\,dx\right)^p + \alpha^p \int_1^\infty \left[\log(ex)\right]^{\alpha p-1}\left(\int_x^\infty f(y)\,dy\right)^p \frac{dx}{x} \leq \int_1^\infty x^p\left[\log(ex)\right]^{(1+\alpha)p-1} f^p(x)\frac{dx}{x} \tag{3.5}$$

and

$$\alpha^{p-1}\left(\int_1^\infty f(x)\,dx\right)^p + \alpha^p \int_1^\infty \left[\log(ex)\right]^{-\alpha p-1}\left(\int_1^x f(y)\,dy\right)^p \frac{dx}{x} \leq \int_1^\infty x^p\left[\log(ex)\right]^{(1-\alpha)p-1} f^p(x)\frac{dx}{x}. \tag{3.6}$$

Both constants α^{p-1} *and* α^p *in* (3.5) *and* (3.6) *are sharp. Equality is never attained unless* f *is identically zero.*

(b) *If* $0 < p < 1$, *then both* (3.5) *and* (3.6) *hold in the reverse direction and the constants in both inequalities are sharp. Equality is never attained unless* f *is identically zero.*

(c) *If* $p = 1$, *we have equality in* (3.5) *and* (3.6) *for any measurable function* f *and any* $\alpha > 0$.

Proof Substitute $f(x)$ by $\frac{1}{x^2}f(\frac{1}{x})$ in Theorem 2.3 and make a change of variables. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have contributed in all parts to equal extent.

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