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Inequalities for an n-simplex in spherical space $S_n(1)$

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Abstract

For an n-dimensional simplex Ω_n and any point D in spherical space $S_n(1)$, we establish an inequality for edge lengths of Ω_n and distances from point D to faces of Ω_n, and from this we obtain some inequalities for the edge lengths and the in-radius of the simplex Ω_n. Besides, we establish some inequalities for the edge lengths and altitudes of a spherical simplex, and we establish inequalities for the edge lengths and circumradius of Ω_n.

MSC: 51M10; 52A20; 51M20

Keywords: spherical simplex; edge lengths; in-radius; circumradius; altitudes

1 Introduction

The n-dimensional spherical space of curvature 1 is defined as follows (see [1–4]).

Let $S_n(1) = \{x(x_1, x_2, \dots, x_{n+1}) \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ be the n-dimensional unit sphere in the $(n+1)$-dimensional Euclidean E^{n+1}. For any two points $x(x_1, x_2, \dots, x_{n+1}), y(y_1, y_2, \dots, y_{n+1}) \in S_n(1)$, the spherical distance between points x and y is defined as the smallest non-negative number $\widehat{xy}$ such that

$$\cos \widehat{xy} = x_1y_1 + x_2y_2 + \cdots + x_{n+1}y_{n+1}.$$

The n-dimensional unit sphere $S_n(1)$ with the above spherical distance is called the n-dimensional spherical space of curvature 1. Actually, the spherical space $S_n(1)$ is the boundary of an n-dimensional sphere of radius 1 in the $(n+1)$-dimensional Euclidean space E^{n+1} with geodesic metric (that is, shorter arc).

Let Ω_n be an n-dimensional simplex with vertices P_i $(i = 1, 2, \dots, n+1)$ in the n-dimensional spherical space $S_n(1)$, r and R the in-radius and the circumradius of Ω_n, respectively. Let $\rho_{ij} = \widehat{P_iP_j}$ $(i \neq j,\ i, j = 1, 2, \dots, n+1)$ be the edge lengths of the spherical simplex Ω_n, h_i the altitude of Ω_n from vertex P_i, *i.e.* the spherical distance from point P_i to the face $f_i = \{P_1 \cdots P_{i-1}P_{i+1} \cdots P_{n+1}\}$ ($(n-1)$-dimensional spherical simplex) of Ω_n. Let D be any point inside the simplex Ω_n and r_i be the spherical distance from point D to the face f_i of Ω_n for $i = 1, 2, \dots, n+1$.

For an n-simplex Δ_n in the n-dimensional Euclidean space E^n, some important inequalities for the edge lengths of Δ_n and r_i $(i = 1, 2, \dots, n+1)$, inequalities for edge lengths and in-radius, circumradius, and altitudes of Δ_n were established (see [5–10]). But similar inequalities for an n-simplex in the spherical space $S_n(1)$ have not been established. In this paper, we discuss the problems of inequalities for a spherical simplex and obtain some related inequalities for an n-simplex in the spherical space $S_n(1)$.

2 Inequalities for an n-simplex in the spherical space $S_n(1)$

In this section, we give some inequalities for the distances from an interior point to the faces of spherical simplex Ω_n and inequalities for edge lengths and in-radius, circumradius, and altitudes of Ω_n. Our main results are the following theorems.

Let φ_{ij} $(i \neq j,\ i,j = 1,2,\ldots,n+1)$ be the dihedral angle formed by two faces f_i and f_j of an n-simplex Ω_n in the spherical space $S_n(1)$.

Theorem 1 *Let Ω_n be an n-simplex in the n-dimensional spherical space $S_n(1)$ with dihedral angles φ_{ij} $(i \neq j,\ i,j = 1,2,\ldots,n+1)$, D be any interior point of simplex Ω_n and r_i the distance from the point D to the face f_i of Ω_n for $i = 1,2,\ldots,n+1$. For any real numbers $\lambda_i \neq 0$ $(i = 1,2,\ldots,n+1)$, we have*

$$\begin{aligned}\sum_{i=1}^{n+1} \lambda_i^2 \cos^2 r_i \leq{}& \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1} \lambda_i^2 + 1\right)^2 - \sum_{1\leq i<j\leq n+1} \lambda_i^2\lambda_j^2\right] \\ &+ \sum_{1\leq i<j\leq n+1} \lambda_i^2\lambda_j^2 \cos^2 \varphi_{ij},\end{aligned} \tag{1}$$

with equality if and only if the nonzero eigenvalues of matrix G are all equal. Here

$$G = \begin{bmatrix} & & & \lambda_1 \sin r_1 \\ & \boxed{-\lambda_i\lambda_j \cos\varphi_{ij}} & & \lambda_2 \sin r_2 \\ & & & \vdots \\ & & & \lambda_{n+1} \sin r_{n+1} \\ \lambda_1 \sin r_1 & \lambda_2 \sin r_2 \quad \cdots & \lambda_{n+1} \sin r_{n+1} & 1 \end{bmatrix}, \tag{2}$$

and $\varphi_{ii} = \pi$ $(i = 1,2,\ldots,n+1)$.

Let $M = (\cos\rho_{ij})_{i,j=1}^{n+1}$ be the edge matrix of an n-simplex Ω_n in $S_n(1)$, then M is a positive definite symmetric matrix with diagonal entries equal to 1 (see [3, 11]); by the cosine theorem of a simplex Ω_n in $S_n(1)$ (see [13]), we have

$$\cos\varphi_{ij} = -\frac{M_{ij}}{\sqrt{M_{ii}}\sqrt{M_{jj}}} \quad (i,j = 1,2,\ldots,n+1). \tag{3}$$

Here M_{ij} denotes the cofactor of matrix M corresponding to the (i,j)-entry. From Theorem 1 and (3) we get an inequality for r_i $(i = 1,2,\ldots,n+1)$ and the edge lengths of spherical simplex Ω_n as follows.

Theorem 1′ *For any interior point D of an n-simplex Ω_n in $S_n(1)$ and any real numbers $\lambda_i \neq 0$ $(i = 1,2,\ldots,n+1)$, we have*

$$\sum_{i=1}^{n+1} \lambda_i^2 \cos^2 r_i \leq \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1} \lambda_i^2 + 1\right)^2 - \sum_{1\leq i<j\leq n+1} \lambda_i^2\lambda_j^2\right] + \sum_{1\leq i<j\leq n+1} \lambda_i^2\lambda_j^2 \frac{M_{ij}^2}{M_{ii}M_{jj}}, \tag{4}$$

with equality if and only if the nonzero eigenvalues of matrix G are all equal.

If we take $\lambda_i^2 = M_{ii}$ $(i = 1, 2, \dots, n+1)$ in (4), we get the following corollary.

Corollary 1 *For any interior point D of an n-simplex Ω_n in $S_n(1)$, we have*

$$\sum_{i=1}^{n+1} M_{ii}\cos^2 r_i \leq \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1} M_{ii} + 1\right)^2 - \sum_{1\leq i<j\leq n+1} M_{ii}M_{jj}\right] + \sum_{1\leq i<j\leq n+1} M_{ij}^2. \tag{5}$$

Equality holds if and only if the nonzero eigenvalues of matrix G with $\lambda_i = \sqrt{M_{ii}}$ $(i = 1, 2, \dots, n+1)$ are all equal.

If we take the point D to be the in-center of Ω_n, then $r_i = r$ $(i = 1, 2, \dots, n+1)$ and from Theorem 1 and Theorem 1′, we get an inequality for the simplex Ω_n as follows.

Corollary 2 *For an n-simplex Ω_n in $S_n(1)$ and real numbers $\lambda_i \neq 0$ $(i = 1, 2, \dots, n+1)$, we have*

$$\left(\sum_{i=1}^{n+1}\lambda_i^2\right)\cos^2 r \leq \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1}\lambda_i^2 + 1\right)^2 - \sum_{1\leq i<j\leq n+1}\lambda_i^2\lambda_j^2\right] + \sum_{1\leq i<j\leq n+1}\lambda_i^2\lambda_j^2\cos^2\varphi_{ij}, \tag{6}$$

or

$$\left(\sum_{i=1}^{n+1}\lambda_i^2\right)\cos^2 r \leq \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1}\lambda_i^2 + 1\right)^2 - \sum_{1\leq i<j\leq n+1}\lambda_i^2\lambda_j^2\right] + \sum_{1\leq i<j\leq n+1}\lambda_i^2\lambda_j^2\frac{M_{ij}^2}{M_{ii}M_{jj}}. \tag{7}$$

Equality holds if and only if the nonzero eigenvalues of matrix G with $r_i = r$ $(i = 1, 2, \dots, n+1)$ are all equal.

If we take $\lambda_i^2 = M_{ii}$ $(i = 1, 2, \dots, n+1)$ in (7), we get an inequality for the in-radius and the edge lengths of a simplex as follows.

Corollary 3 *For an n-simplex Ω_n in $S_n(1)$, we have*

$$\cos^2 r \leq \frac{1}{\sum_{i=1}^{n+1} M_{ii}}\left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1} M_{ii} + 1\right)^2 - \sum_{1\leq i<j\leq n+1} M_{ii}M_{jj} + \sum_{1\leq i<j\leq n+1} M_{ij}^2\right]. \tag{8}$$

Equality holds if and only if the nonzero eigenvalues of matrix G with $r_i = r$ and $\lambda_i = \sqrt{M_{ii}}$ $(i = 1, 2, \dots, n+1)$ are all equal.

Put $\lambda_i = 1$ $(i = 1, 2, \dots, n+1)$ in (6) and (7), and we get the following corollary.

Corollary 4 *For an n-simplex Ω_n in $S_n(1)$, we have*

$$\cos^2 r \leq \frac{2n^2 + 3n}{2(n+1)^2} + \frac{1}{n+1}\sum_{1\leq i<j\leq n+1}\frac{M_{ij}^2}{M_{ii}M_{jj}}, \tag{9}$$

or

$$\cos^2 r \le \frac{2n^2+3n}{2(n+1)^2} + \frac{1}{n+1}\sum_{1\le i<j\le n+1}\cos^2\varphi_{ij}. \tag{10}$$

Equality holds if and only if the nonzero eigenvalues of matrix G with $r_i = r$ and $\lambda_i = 1$ $(i = 1,2,\dots,n+1)$ are all equal.

Besides, we obtain an inequality for the edge lengths and circumradius of an n-simplex Ω_n in $S_n(1)$ as follows.

Theorem 2 *Let ρ_{ij} $(i,j = 1,2,\dots,n+1)$ and R be the edge lengths and the circumradius of an n-simplex Ω_n in $S_n(1)$,respectively; let $x_i > 0$ $(i = 1,2,\dots,n+1)$ be real numbers, then we have*

$$\sum_{1\le i<j\le n+1} x_i x_j \sin^2\rho_{ij} \le \left[\frac{n}{2(n+1)}\left(\sum_{i=1}^{n+1} x_i + 1\right)^2 - \sum_{i=1}^{n+1} x_i\right] + \left(\sum_{i=1}^{n+1} x_i\right)\cos^2 R. \tag{11}$$

Equality holds if and only if the nonzero eigenvalues of matrix B are all equal. Here

$$B = \begin{bmatrix} & & & \sqrt{x_1}\cos R \\ & \boxed{\sqrt{x_i x_j}\cos\rho_{ij}} & & \vdots \\ & & & \sqrt{x_{n+1}}\cos R \\ \sqrt{x_1}\cos R & \cdots & \sqrt{x_{n+1}}\cos R & 1 \end{bmatrix}. \tag{12}$$

If take $x_1 = x_2 = \cdots = x_{n+1} = 1$ in Theorem 2, we get an inequality as follows.

Corollary 5 *For an n-simplex Ω_n in $S_n(1)$, we have*

$$\sum_{1\le i<j\le n+1}\sin^2\rho_{ij} \le \frac{n^3+2n^2-2}{2(n+1)} + (n+1)\cos^2 R, \tag{13}$$

with equality holding if and only if the nonzero eigenvalues of matrix B with $x_1 = x_2 = \cdots = x_{n+1} = 1$ are all equal.

Finally, we give an inequality for edge lengths and altitudes of an n-simplex in $S_n(1)$ as follows.

Theorem 3 *Let h_i $(i = 1,2,\dots,n+1)$ and M be the altitudes and the edge matrix of an n-simplex Ω_n in $S_n(1)$, respectively; let $x_i > 0$ $(i = 1,2,\dots,n+1)$ be real numbers, then we have*

$$\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\ne i}}^{n+1} x_j\right)\csc^2 h_i \ge (n+1)\left(\prod_{i=1}^{n+1} x_i\right)^{\frac{n}{n+1}} \cdot |M|^{\frac{-1}{n+1}}, \tag{14}$$

with equality holding if and only if the eigenvalues of matrix Q are all equal. Here

$$Q = (\sqrt{x_i x_j}\cos\rho_{ij})_{i,j=1}^{n+1}, \qquad M = (\cos\rho_{ij})_{i,j=1}^{n+1}. \tag{15}$$

If we take $x_i = \csc^2 h_i$ $(i = 1, 2, \ldots, n + 1)$ in (14), we get the following corollary.

Corollary 6 *For an n-simplex Ω_n in $S_n(1)$, we have*

$$\prod_{i=1}^{n+1} \sin h_i \le |M|^{\frac{1}{2}} \le \left[\frac{2}{n(n+1)} \sum_{1\le i<j\le n+1} \sin^2 \rho_{ij}\right]^{\frac{n+1}{4}}, \tag{16}$$

with equality holding if Ω_n is regular.

We will prove $|M|^{\frac{1}{2}} \le [\frac{2}{n(n+1)} \sum_{1\le i<j\le n+1} \sin^2 \rho_{ij}]^{\frac{n+1}{4}}$ and we have equality if Ω_n is regular in the next section.

3 Proof of theorems

To prove the theorems in the above section, we need some lemmas.

Lemma 1 *Let $M = (\cos \rho_{ij})_{i,j=1}^{n+1}$ be the edge matrix of an n-simplex Ω_n in $S_n(1)$, then M is a positive definite symmetric matrix with diagonal entries equal to* 1.

For the proof of Lemma 1 one is referred to [3, 11].

Lemma 2 *Let φ_{ij} be the dihedral angle formed by two faces f_i and f_j of an n-simplex Ω_n in $S_n(1)$ for $i \ne j$, $i, j = 1, 2, \ldots, n + 1$, and $\varphi_{ii} = \pi$ $(i = 1, 2, \ldots, n + 1)$, then the Gram matrix $A = (-\cos \varphi_{ij})_{i,j=1}^{n+1}$ is positive definite symmetric matrix with diagonal entries equal to* 1.

For the proof of Lemma 2 one is referred to [1].

Lemma 3 (see [12]) *Let μ be the set of all points and oriented $(n - 1)$-dimensional hyperplanes in the spherical space $S_n(1)$. For arbitrary m elements $e_1, e_2, \ldots, e_m$ of μ, define g_{ij} as follows*:

(i) *if e_i and e_j are two points, then $g_{ij} = \cos \widehat{e_i e_j}$ (where $\widehat{e_i e_j}$ be spherical distance between e_i and e_j);*

(ii) *if e_i and e_j are unit outer normals of two unit outer normal of oriented , then $g_{ij} = \cos \widehat{e_i e_j}$ (where $\widehat{e_i e_j}$ is dihedral angle formed by e_i and e_j);*

(iii) *if either of e_i and e_j is a point, and another is an outer normal, then $g_{ij} = \sin h_{ij}$ (where h_{ij} is the spherical distance with sign based on the direction from the point to the hyperplane).*

If $m > n + 1$, then

$$\det(g_{ij})_{i,j=1}^{m} = 0.$$

Lemma 4 *Let h_i be the altitude from vertex P_i of an n-simplex Ω_n in $S_n(1)$ for $i = 1, 2, \ldots, n + 1$, and $M = (\cos \rho_{ij})_{i,j=1}^{n+1}$ the edge matrix, then we have*

$$\sin^2 h_i = \frac{|M|}{M_{ii}} \quad (i = 1, 2, \ldots, n + 1). \tag{17}$$

For the proof of Lemma 4 one is referred to [13].

Proof of Theorem 1 Let e_i is the unit outer normal of the oriented f_i $(i = 1, 2, \ldots, n+1)$ and the point $e_{n+2} = D$, such that $\widehat{e_ie_j} = \pi - \varphi_{ij}$ $(i, j = 1, 2, \ldots, n+1)$ and the spherical distance with sign based on the direction from the point e_{n+2} to the hyperplane e_i is r_i for $i = 1, 2, \ldots, n+1$.

By Lemma 2 we know that the $(n+1) \times (n+1)$-order matrix $(\cos \widehat{e_ie_j})_{i,j=1}^{n+1} = (-\cos \varphi_{ij})_{i,j=1}^{n+1} = A$ is a positive definite symmetric matrix. Because $\lambda_i \neq 0$ $(i = 1, 2, \ldots, n+1)$, the matrix $T = (-\lambda_i\lambda_j \cos \varphi_{ij})_{i,j=1}^{n+1}$ is also a positive definite symmetric matrix.

By Lemma 3 we have

$$B = \begin{vmatrix} & & & \sin r_1 \\ & \boxed{-\cos \varphi_{ij}} & & \vdots \\ & & & \sin r_{n+1} \\ \sin r_1 & \cdots & \sin r_{n+1} & 1 \end{vmatrix} = 0. \tag{18}$$

From (18) and $\lambda_i \neq 0$ $(i = 1, 2, \ldots, n+1)$, we get

$$\det G = \begin{vmatrix} & & & \lambda_1 \sin r_1 \\ & \boxed{-\lambda_i\lambda_j \cos \varphi_{ij}} & & \vdots \\ & & & \lambda_{n+1} \sin r_{n+1} \\ \lambda_1 \sin r_1 & \cdots & \lambda_{n+1} \sin r_{n+1} & 1 \end{vmatrix} = 0. \tag{19}$$

Because the matrix $T = (-\lambda_i\lambda_j \cos \varphi_{ij})_{i,j=1}^{n+1}$ is also a positive definite symmetric matrix and $\det G = 0$, the matrix G is a semi-positive definite symmetric matrix and the rank of matrix G is $n+1$. Let $u_i > 0$ $(i = 1, 2, \ldots, n+1)$ and $u_{n+2} = 0$ be the eigenvalues of the matrix G, and

$$\sigma_1 = \sum_{i=1}^{n+2} u_i = \sum_{i=1}^{n+1} u_i, \qquad \sigma_2 = \sum_{1 \leq i<j \leq n+2} u_iu_j = \sum_{1 \leq i<j \leq n+1} u_iu_j.$$

Using Maclaurin's inequality [5], we have

$$\left(\frac{1}{n+1}\sigma_1\right)^2 \geq \frac{2}{n(n+1)}\sigma_2. \tag{20}$$

Equality holds if and only if $u_1 = u_2 = \cdots = u_{n+1}$.

By the relation between the eigenvalues and the principal minors of the matrix G, we have

$$\sigma_1 = \sum_{i=1}^{n+1} \lambda_i^2 + 1, \qquad \sigma_2 = \sum_{1 \leq i<j \leq n+1} \lambda_i^2\lambda_j^2 \sin^2 \varphi_{ij} + \sum_{i=1}^{n+1} \lambda_i^2 \cos^2 r_i. \tag{21}$$

Substituting (21) into (20), we get inequality (1). It is easy to see that equality holds in (1) if and only if the nonzero eigenvalues of matrix G are all equal. □

Proof of Theorem 2 Let C be the circumcenter of Ω_n, then $\widehat{CP_i} = R$ $(i = 1, 2, \ldots, n+1)$. For real numbers $x_i > 0$ $(i = 1, 2, \ldots, n+1)$, by Lemma 1 we know that the matrix Q in (15) is a positive definite symmetric matrix. We take points $e_i = P_i$ $(i = 1, 2, \ldots, n+1)$ and $e_{n+2} = C$, and by Lemma 3 we have

$$\begin{vmatrix} & & & \cos R \\ & \boxed{\cos\rho_{ij}} & & \vdots \\ & & & \cos R \\ \cos R & \cdots & \cos R & 1 \end{vmatrix} = 0.$$

From this and $x_i > 0$ $(i = 1, 2, \ldots, n + 1)$, we get

$$\det B = \begin{vmatrix} & & & \sqrt{x_1}\cos R \\ & \boxed{\sqrt{x_i x_j}\cos\rho_{ij}} & & \vdots \\ & & & \sqrt{x_{n+1}}\cos R \\ \sqrt{x_1}\cos R & \cdots & \sqrt{x_{n+1}}\cos R & 1 \end{vmatrix} = 0. \tag{22}$$

Because the matrix $Q = (\sqrt{x_i x_j}\cos\rho_{ij})_{i,j=1}^{n+1}$ is positive definite symmetric and $\det B = 0$, the matrix B is a semi-positive definite symmetric matrix and its rank is $n + 1$. Let $\nu_i > 0$ $(i = 1, 2, \ldots, n + 1)$, $\nu_{n+2} = 0$ be the eigenvalues of matrix B, and

$$\sigma_1 = \sum_{i=1}^{n+2} \nu_i = \sum_{i=1}^{n+1} \nu_i, \qquad \sigma_2 = \sum_{1\le i<j\le n+2} \nu_i \nu_j = \sum_{1\le i<j\le n+1} \nu_i \nu_j.$$

Using Maclaurin's inequality [5], we have

$$\left(\frac{1}{n+1}\sigma_1\right)^2 \ge \frac{2}{n(n+1)}\sigma_2. \tag{23}$$

Equality holds if and only if $\nu_1 = \nu_2 = \cdots = \nu_{n+1}$.

By the relation between the eigenvalues and the principal minors of the matrix B, we have

$$\sigma_1 = \sum_{i=1}^{n+1} x_i + 1, \qquad \sigma_2 = \sum_{1\le i<j\le n+1} x_i x_j \sin^2\rho_{ij} + \sum_{i=1}^{n+1} x_i\bigl(1 - \cos^2 R\bigr). \tag{24}$$

Substituting (24) into (23), we get inequality (11). It is easy to see that equality holds in (11) if and only if the nonzero eigenvalues of matrix B are all equal. □

Proof of Theorem 3 From $x_i > 0$ $(i = 1, 2, \ldots, n + 1)$ and the edge matrix $M = (\cos\rho_{ij})_{i,j=1}^{n+1}$ of Ω_n being a positive definite symmetric matrix, we know that the matrix Q in (15) is also a positive definite symmetric matrix. Let $a_i > 0$ $(i = 1, 2, \ldots, n+1)$ be the eigenvalues of the matrix Q, and

$$\sigma_n = \sum_{i=1}^{n+1} \prod_{\substack{j=1\\ j\ne i}}^{n+1} a_j, \qquad \sigma_{n+1} = \prod_{i=1}^{n+1} a_i.$$

By Maclaurin's inequality [5], we have

$$\left(\frac{1}{n+1}\sigma_n\right)^{\frac{1}{n}} \ge (\sigma_{n+1})^{\frac{1}{n+1}}. \tag{25}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_{n+1}$.

By the relation between the eigenvalues and the principal minors of the matrix Q, we have

$$\sigma_n = \sum_{i=1}^{n+1} Q_{ii} = \sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} x_j\right) M_{ii} \quad (i = 1, 2, \ldots, n+1), \tag{26}$$

$$\sigma_{n+1} = |Q| = \left(\prod_{i=1}^{n+1} x_i\right) \cdot |M|. \tag{27}$$

From (25), (26), and (27), we get

$$\sum_{i=1}^{n+1}\left(\prod_{\substack{j=1\\ j\neq i}}^{n+1} x_j\right) M_{ii} \ge (n+1)\left(\prod_{i=1}^{n+1} x_i\right)^{\frac{n}{n+1}} \cdot |M|^{\frac{n}{n+1}}. \tag{28}$$

By Lemma 4 we have

$$M_{ii} = |M| \csc^2 h_i \quad (i = 1, 2, \ldots, n+1). \tag{29}$$

Substituting (29) into (28), we get inequality (14). It is easy to see that equality holds in (14) if and only if the eigenvalues of matrix Q are all equal.

Finally, we prove that inequality (30) is valid:

$$|M|^{\frac{1}{2}} \le \left[\frac{2}{n(n+1)} \sum_{1\le i<j\le n+1} \sin^2 \rho_{ij}\right]^{\frac{n+1}{4}}. \tag{30}$$

Let b_i $(i = 1, 2, \ldots, n+1)$ be the eigenvalues of the edge matrix $M = (\cos \rho_{ij})_{i,j=1}^{n+1}$. Since the matrix M is a positive definite symmetric matrix, $b_i > 0$. Let

$$\sigma_2 = \sum_{1\le i<j\le n+1} b_i b_j, \qquad \sigma_{n+1} = \prod_{i=1}^{n+1} b_i.$$

By Maclaurin's inequality [5], we have

$$\left(\frac{2}{n(n+1)}\sigma_2\right)^{\frac{1}{2}} \ge (\sigma_{n+1})^{\frac{1}{n+1}}. \tag{31}$$

Equality holds if and only if $b_1 = b_2 = \cdots = b_{n+1}$.

By the relation between the eigenvalues and the principal minors of the matrix M, we have

$$\sigma_2 = \sum_{1\le i<j\le n+1} \sin^2 \rho_{ij}, \qquad \sigma_{n+1} = |M|. \tag{32}$$

From (31) and (32), we get inequality (30). If Ω_n is a regular simplex in $S_n(1)$, then $\rho_{ij} = \frac{\pi}{2}$ $(i \neq j,\ i,j = 1,2,\ldots,n+1)$, $|M| = 1$ and $M_{ii} = 1$ $(i = 1,2,\ldots,n+1)$. By (17) we have $\sin h_i = 1$ $(i = 1,2,\ldots,n+1)$; thus equality holds in (16) if Ω_n is a regular simplex. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors co-authored this paper. All authors read and approved the final manuscript.

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