Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Weighted sharp maximal function inequalities and boundedness of multilinear singular integral operator satisfying a variant of Hörmander's condition

Changhong Wu* and Meng Zhang

*Correspondence:
changhongwu_0@163.com
Orient Science and Technology College, Hunan Agriculture University, Changsha, 410128, P.R. China

Abstract
In this paper, we establish the weighted sharp maximal function inequalities for the multilinear operator associated with the singular integral operator satisfying a variant of Hörmander's condition. As an application, we obtain the boundedness of the operator on weighted Lebesgue spaces.
MSC: 42B20; 42B25

Keywords: multilinear operator; singular integral operator; sharp maximal function; weighted *BMO*; weighted Lipschitz function

1 Introduction

As the development of singular integral operators (see [1, 2]), their commutators and multilinear operators have been well studied. In [3–5], the authors proved that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [6]) proved a similar result when singular integral operators are replaced by the fractional integral operators. In [7, 8], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [9, 10], the boundedness for the commutators generated by the singular integral operators and the weighted *BMO* and Lipschitz functions on $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained (also see [11]). In [12, 13], the authors studied some multilinear singular integral operators as follows (also see [14]):

$$T^b(f)(x) = \int \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\,dy,$$

and they obtained some variant sharp function estimates and boundedness of the multilinear operators if $D^\alpha b \in BMO(R^n)$ for all α with $|\alpha| = m$. In [15], some singular integral operators satisfying a variant of Hörmander's condition are introduced, and the boundedness for the operators and their commutators is obtained (see [16, 17]). Motivated by these results, in this paper, we will study the multilinear operator generated by the singular integral operator satisfying a variant of Hörmander's condition and the weighted Lipschitz and *BMO* functions, that is, $D^\alpha b \in BMO(w)$ or $D^\alpha b \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m$.

2 Preliminaries

First, let us introduce some notation. Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a non-negative integrable function ω, let $\omega(Q) = \int_Q \omega(x)\,dx$ and $\omega_Q = |Q|^{-1}\int_Q \omega(x)\,dx$.

For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q \left|f(y) - f_Q\right| dy.$$

It is well known that (see [1])

$$M^{\#}(f)(x) \approx \sup_{Q\ni x} \inf_{c\in C} \frac{1}{|Q|}\int_Q \left|f(y) - c\right| dy.$$

Let

$$M(f)(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q \left|f(y)\right| dy.$$

For $\eta > 0$, let $M_\eta^{\#}(f)(x) = M^{\#}(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and the non-negative weight function ω, set

$$M_{\eta,p,\omega}(f)(x) = \sup_{Q\ni x}\left(\frac{1}{\omega(Q)^{1-p\eta/n}}\int_Q \left|f(y)\right|^p \omega(y)\,dy\right)^{1/p}$$

and

$$M_\omega(f)(x) = \sup_{Q\ni x} \frac{1}{\omega(Q)}\int_Q \left|f(y)\right|\omega(y)\,dy.$$

The A_p weight is defined by (see [1])

$$A_p = \left\{\omega \in L^1_{\mathrm{loc}}(R^n) : \sup_Q\left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty\right\},$$
$$1 < p < \infty$$

and

$$A_1 = \left\{\omega \in L^p_{\mathrm{loc}}(R^n) : M(\omega)(x) \le Cw(x), \text{a.e.}\right\}.$$

Given a non-negative weight function ω. For $1 \le p < \infty$, the weighted Lebesgue space $L^p(R^n,\omega)$ is the space of functions f such that

$$\|f\|_{L^p(\omega)} = \left(\int_{R^n} \left|f(x)\right|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

Given the non-negative weight function ω, the weighted BMO space $BMO(\omega)$ is the space of functions b such that

$$\|b\|_{BMO(\omega)} = \sup_Q \frac{1}{\omega(Q)}\int_Q \left|b(y) - b_Q\right| dy < \infty.$$

For $0 < \beta < 1$, the weighted Lipschitz space $\mathrm{Lip}_\beta(\omega)$ is the space of functions b such that

$$\|b\|_{\mathrm{Lip}_\beta(\omega)} = \sup_Q \frac{1}{\omega(Q)^{\beta/n}} \left(\frac{1}{\omega(Q)} \int_Q \big|b(y) - b_Q\big|^p \omega(x)^{1-p}\, dy \right)^{1/p} < \infty.$$

Remark (1) It has been known that (see [18]), for $b \in \mathrm{Lip}_\beta(\omega)$, $\omega \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)\omega\big(2^kQ\big)^{\beta/n}.$$

(2) Let $b \in \mathrm{Lip}_\beta(\omega)$ and $\omega \in A_1$. By [18], we know that spaces $\mathrm{Lip}_\beta(\omega)$ coincide and the norms $\|b\|_{\mathrm{Lip}_\beta(\omega)}$ are equivalent with respect to different values $1 \le p < \infty$.

Definition 1 Let $\Phi = \{\phi_1, \dots, \phi_l\}$ be a finite family of bounded functions in R^n. For any locally integrable function f, the Φ sharp maximal function of f is defined by

$$M_\Phi^\#(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\dots,c_l\}} \frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_i\phi_i(x_Q - y) \right| dy,$$

where the infimum is taken over all m-tuples $\{c_1, \dots, c_l\}$ of complex numbers and x_Q is the center of Q. For $\eta > 0$, let

$$M_{\Phi,\eta}^\#(f)(x) = \sup_{Q \ni x} \inf_{\{c_1,\dots,c_l\}} \left(\frac{1}{|Q|} \int_Q \left| f(y) - \sum_{i=1}^{l} c_j\phi_i(x_Q - y) \right|^\eta dy \right)^{1/\eta}.$$

Remark We note that $M_\Phi^\# \approx f^\#$ if $l = 1$ and $\phi_1 = 1$.

Definition 2 Given a positive and locally integrable function f in R^n, we say that f satisfies the reverse Hölder's condition (write this as $f \in RH_\infty(R^n)$), if for any cube Q centered at the origin we have

$$0 < \sup_{x \in Q} f(x) \le C \frac{1}{|Q|} \int_Q f(y)\, dy.$$

In this paper, we will study some singular integral operators as follows (see [15]).

Definition 3 Let $K \in L^2(R^n)$ and satisfy

$$\|K\|_{L^\infty} \le C,$$
$$\big|K(x)\big| \le C|x|^{-n},$$

there exist functions $B_1, \dots, B_l \in L^1_{\mathrm{loc}}(R^n - \{0\})$ and $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$, and for a fixed $\delta > 0$ and any $|x| > 2|y| > 0$,

$$\left| K(x - y) - \sum_{i=1}^{l} B_i(x)\phi_i(y) \right| \le C \frac{|y|^\delta}{|x - y|^{n+\delta}}.$$

For $f \in C_0^\infty$, we define the singular integral operator related to the kernel K by

$$T(f)(x) = \int_{R^n} K(x-y)f(y)\,dy.$$

Moreover, let m be the positive integer and b be the function on R^n. Set

$$R_{m+1}(b;x,y) = b(x) - \sum_{|\alpha|\le m} \frac{1}{\alpha!} D^\alpha b(y)(x-y)^\alpha.$$

The multilinear operator related to the operator T is defined by

$$T^b(f)(x) = \int_{R^n} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x-y)f(y)\,dy.$$

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the multilinear operator T^b if $m = 0$. The multilinear operator T^b are the non-trivial generalizations of the commutator. It is well known that commutators and multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [12–14]). The main purpose of this paper is to prove the sharp maximal inequalities for the multilinear operator T^b. As the application, we obtain the weighted L^p-boundedness for the multilinear operator T^b.

We give some preliminary lemmas.

Lemma 1 (see [1, p.485]) *Let $0 < p < q < \infty$ and for any function $f \ge 0$. We define, for $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda \bigl|\{x \in R^n : f(x) > \lambda\}\bigr|^{1/q}, \qquad N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p} / \|\chi_Q\|_{L^r},$$

where the sup *is taken for all measurable sets Q with $0 < |Q| < \infty$. Then*

$$\|f\|_{WL^q} \le N_{p,q}(f) \le \bigl(q/(q-p)\bigr)^{1/p} \|f\|_{WL^q}.$$

Lemma 2 (see [15]) *Let T be the singular integral operator as Definition 2. Then T is bounded on $L^p(R^n,\omega)$ for $\omega \in A_p$ with $1 < p < \infty$, and weak (L^1, L^1) bounded.*

Lemma 3 (see [9]) *Let $b \in BMO(\omega)$. Then*

$$|b_Q - b_{2^jQ}| \le Cj\|b\|_{BMO(\omega)} \omega_{Q_j},$$

where $\omega_{Q_j} = \max_{1\le i\le j} |2^iQ|^{-1} \int_{2^iQ} \omega(x)\,dx$.

Lemma 4 (see [9]) *Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $\varepsilon > 0$ such that $\omega^{-r/p} \in A_r$ for any $p' \le r \le p' + \varepsilon$.*

Lemma 5 (see [9]) *Let $b \in BMO(\omega)$, $\omega = (\mu\nu^{-1})^{1/p}$, $\mu, \nu \in A_p$ and $p > 1$. Then there exists $\varepsilon > 0$ such that for $p' \le r \le p' + \varepsilon$,*

$$\int_Q \bigl|b(x) - b_Q\bigr|^r \mu(x)^{-r/p}\,dx \le C\|b\|_{BMO(\omega)}^r \int_Q \nu(x)^{-r/p}\,dx.$$

Lemma 6 (see [9]) *Let $\omega \in A_p$, $1 < p < \infty$. Then there exists $0 < \delta < 1$ such that $\omega^{1-r'/p} \in A_{p/r'}(d\mu)$ for any $p' < r < p'(1+\delta)$, where $d\mu = \omega^{r'/p}\,dx$.*

Lemma 7 (see [9]) *Let $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$, $1 < p < \infty$. Then there exists $1 < q < p$ such that*

$$\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{|Q|}\int_Q \omega(x)^{-q'}\nu(x)^{-q'/q}\,dx\right)^{1/q'} \le C.$$

Lemma 8 (see [1, 6]) *Let $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ and $\omega \in A_1$. Then*

$$\left\|M_{\eta,s,\omega}(f)\right\|_{L^q(\omega)} \le C\|f\|_{L^p(\omega)}.$$

Lemma 9 (see [15, 17]) *Let $1 < p < \infty$, $0 < \eta < \infty$, $\omega \in A_\infty$ and $\Phi = \{\phi_1, \dots, \phi_l\} \subset L^\infty(R^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(R^{nl})$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M_\eta(f)(x)^p\omega(x)\,dx \le C\int_{R^n} M^{\#}_{\Phi,\eta}(f)(x)^p\omega(x)\,dx.$$

Lemma 10 (see [13]) *Let b be a function on R^n and $D^\alpha A \in L^s(R^n)$ for all α with $|\alpha| = m$ and any $s > n$. Then*

$$\left|R_m(b;x,y)\right| \le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha b(z)\right|^s dz\right)^{1/s},$$

where $\tilde{Q}$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

3 Theorems and proofs

We shall prove the following theorems.

Theorem 1 *Let T be the singular integral operator as Definition 3, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$, $0 < \eta < 1$ and $D^\alpha b \in BMO(\omega)$ for all α with $|\alpha| = m$. Then there exist a constant $C > 0$, $\varepsilon > 0$, $0 < \delta < 1$, $1 < q < p$ and $p' < r < \min(p' + \varepsilon, p'(1+\delta))$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$\begin{aligned} M^{\#}_{\Phi,\eta}\left(T^b(f)\right)(\tilde{x}) \le{}& C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}\left(\left[M_\nu\left(\left|\omega T(f)\right|^q\right)(\tilde{x})\right]^{1/q}\right. \\ &\left. + \left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(\tilde{x})\right]^{1/r'} + \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\right). \end{aligned}$$

Theorem 2 *Let T be the singular integral operator as Definition 3, $\omega \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in \mathrm{Lip}_\beta(\omega)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$M^{\#}_{\Phi,\eta}\left(T^b(f)\right)(\tilde{x}) \le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x}).$$

Theorem 3 *Let T be the singular integral operator as Definition* 3, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$ *and* $D^\alpha b \in BMO(\omega)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^p(R^n, \mu)$ *to* $L^p(R^n, \nu)$.

Theorem 4 *Let T be the singular integral operator as Definition* 3, $\omega \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $D^\alpha b \in \mathrm{Lip}_\beta(\omega)$ *for all* α *with* $|\alpha| = m$. *Then* T^b *is bounded from* $L^p(R^n, \omega)$ *to* $L^q(R^n, \omega^{1-q})$.

Corollary *Let* $[b, T](f) = bT(f) - T(bf)$ *be the commutator generated by the singular integral operator T as Definition* 2 *and* b. *Then Theorems* 1-4 *hold for* $[b, T]$.

Proof of Theorem 1 It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left(\left[M_\nu\left(\left|\omega T(f)\right|^q\right)(\tilde{x})\right]^{1/q} + \left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(\tilde{x})\right]^{1/r'} + \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\right),$$

where Q is any a cube centered at x_0, $C_0 = \sum_{j=1}^l c_j\phi_j(x_0 - x)$ and $c_j = \int_{R^n} \frac{K(x_0,y)}{|x_0-y|^m} B_j(x_0 - y) f_2(y)\, dy$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} \times (D^\alpha b)_{\tilde{Q}} x^\alpha$, then $R_m(b; x, y) = R_m(\tilde{b}; x, y)$ and $D^\alpha \tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} T^b(f)(x) &= \int_{R^n} \frac{R_m(\tilde{b}; x, y)}{|x-y|^m} K(x, x-y) f_1(y)\, dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-y)^\alpha D^\alpha \tilde{b}(y)}{|x-y|^m} K(x, x-y) f_1(y)\, dy \\ &\quad + \int_{R^n} \frac{R_{m+1}(\tilde{b}; x, y)}{|x-y|^m} K(x, x-y) f_2(y)\, dy \\ &= T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right) - T\left(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right) + T^{\tilde{b}}(f_2)(x), \end{aligned}$$

then

$$\begin{aligned} &\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \\ &\quad \le C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m} \frac{(x-\cdot)^\alpha D^\alpha \tilde{b}}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\ &\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - C_0\right|^\eta dx\right)^{1/\eta} \\ &\quad = I_1 + I_2 + I_3. \end{aligned}$$

For I_1, noting that $\omega \in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q \omega(x)^{p_0}\,dx\right)^{1/p_0} \le \frac{C}{|Q|}\int_Q \omega(x)\,dx$$

for all cubes Q and some $1 < p_0 < \infty$ (see [1]). We take $s = rp_0/(r + p_0 - 1)$ in Lemma 10 and have $1 < s < r$ and $p_0 = s(r-1)/(r-s)$, then by Lemma 10 and Hölder's inequality, we obtain

$$\begin{aligned}
\left|R_m(b;x,y)\right| &\le C|x-y|^m \sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\left|D^\alpha b(z)\right|^s dz\right)^{1/s} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha b(z)\right|^s \omega(z)^{s(1-r)/r}\omega(z)^{s(r-1)/r}\,dz\right)^{1/s} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\left(\int_{\tilde{Q}(x,y)}\left|D^\alpha b(z)\right|^r \omega(z)^{1-r}\,dz\right)^{1/r} \\
&\quad\times\left(\int_{\tilde{Q}(x,y)}\omega(z)^{s(r-1)/(r-s)}\,dz\right)^{(r-s)/rs} \\
&\le C|x-y|^m \sum_{|\alpha|=m}|\tilde{Q}|^{-1/s}\left\|D^\alpha b\right\|_{BMO(\omega)}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{(r-s)/rs} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)^{p_0}\,dz\right)^{(r-s)/rs} \\
&\le C|x-y|^m \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{1/s-1/r} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)\,dz\right)^{(r-1)/r} \\
&\le C|x-y|^m \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{1/r}|\tilde{Q}|^{1/s-1/r}\omega(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1} \\
&\le C|x-y|^m \sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}\frac{\omega(\tilde{Q})}{|\tilde{Q}|},
\end{aligned}$$

thus, by Lemma 7, we obtain

$$\begin{aligned}
I_1 &\le \frac{C}{|Q|}\int_Q\left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)\right|dx \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}\frac{\omega(\tilde{Q})}{|\tilde{Q}|}\frac{1}{|Q|}\int_Q\left|T(f)(y)\right|\omega(y)\nu(y)^{1/q}\omega(y)^{-1}\nu(y)^{-1/q}\,dy \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}\omega_{\tilde{Q}}\left(\frac{1}{|Q|}\int_Q\left|\omega(y)T(f)(y)\right|^q\nu(y)\,dy\right)^{1/q} \\
&\quad\times\left(\frac{1}{|Q|}\int_Q\omega(y)^{-q'}\nu(y)^{-q'/q}\,dy\right)^{1/q'} \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{BMO(\omega)}\omega_Q(\nu_Q)^{1/q}\left(\frac{1}{\nu(Q)}\int_Q\left|\omega(y)T(f)(y)\right|^q\nu(y)\,dy\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\quad \times \left(\frac{1}{|Q|} \int_Q \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \big[M_\nu \big(|\omega T(f)|^q \big)(\tilde{x}) \big]^{1/q} \\
&\quad \times \omega_Q (\nu_Q)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \big[M_\nu \big(|\omega T(f)|^q \big)(\tilde{x}) \big]^{1/q}.
\end{aligned}
$$

For I_2, we know $\nu^{-r/p} \in A_r$ by Lemma 4, thus

$$
\left(\frac{1}{|Q|} \int_Q \nu(x)^{-r/p}\, dx \right)^{1/r} \leq C \left(\frac{1}{|Q|} \int_Q \nu(x)^{r'/p}\, dx \right)^{-1/r'},
$$

then, by the weak (L^1, L^1) boundedness of T (see Lemma 2) and Kolmogorov's inequality (see Lemma 1), we obtain, by Lemma 5,

$$
\begin{aligned}
I_2 &\leq C \sum_{|\alpha|=m} \left(\frac{1}{|Q|} \int_Q \big| T\big(D^\alpha \tilde{b} f_1 \big)(x) \big|^\eta\, dx \right)^{1/\eta} \\
&\leq C \sum_{|\alpha|=m} \frac{|Q|^{1/\eta - 1}}{|Q|^{1/\eta}} \frac{\| T(D^\alpha \tilde{b} f_1) \chi_Q \|_{L^\eta}}{\| \chi_Q \|_{L^{\eta/(1-\eta)}}} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \big\| T\big(D^\alpha \tilde{b} f_1 \big) \big\|_{WL^1} \\
&\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{R^n} \big| D^\alpha \tilde{b}(x) f_1(x) \big|\, dx \\
&= C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} \big| D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}} \big| \mu(x)^{-1/p} \big| f(x) \big| \omega(x) \nu(x)^{1/p}\, dx \\
&\leq C \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \big| \big(D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}} \big) \big|^r \mu(x)^{-r/p}\, dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)|^{r'} \omega(x)^{r'} \nu(x)^{r'/p}\, dx \right)^{1/r'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \nu(x)^{-r/p}\, dx \right)^{1/r} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)\omega(x)|^{r'} \nu(x)^{r'/p}\, dx \right)^{1/r'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \nu(x)^{r'/p}\, dx \right)^{-1/r'} \\
&\quad \times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} |f(x)\omega(x)|^{r'} \nu(x)^{r'/p}\, dx \right)^{1/r'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \left(\frac{1}{\nu(\tilde{Q})^{r'/p}} \int_{\tilde{Q}} |f(x)\omega(x)|^{r'} \nu(x)^{r'/p}\, dx \right)^{1/r'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{BMO(\omega)} \big[M_{\nu^{r'/p}} \big(|\omega f|^{r'} \big)(\tilde{x}) \big]^{1/r'}.
\end{aligned}
$$

For I_3, note that $|x-y| \approx |x_0-y|$ for $x \in Q$ and $y \in R^n \setminus Q$, we write

$$\begin{aligned}
\left|T^{\tilde{b}}(f_2)(x) - C_0\right| &\le \int_{R^n} \left| \frac{R_m(\tilde{b};x,y)}{|x-y|^m} - \frac{R_m(\tilde{b};x_0,y)}{|x_0-y|^m} \right| \left|K(x-y)\right| \left|f_2(y)\right| dy \\
&\quad + \int_{R^n} \frac{|R_{m+1}(\tilde{b};x_0,y)|}{|x_0-y|^m} \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0-y)\phi_j(x_0-x) \right| \left|f_2(y)\right| dy \\
&\quad + C \sum_{|\alpha|=m} \int_{R^n} \left| \frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_0-y)^\alpha}{|x_0-y|^m} \right| \left|K(x-y)\right| \left|D^\alpha \tilde{b}(y)\right| \left|f_2(y)\right| dy \\
&= I_3^{(1)}(x) + I_3^{(2)}(x) + I_3^{(3)}(x).
\end{aligned}$$

For $I_3^{(1)}$, by the formula (see [13]):

$$R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y) = \sum_{|\gamma|<m} \frac{1}{\gamma!} R_{m-|\gamma|}\left(D^\gamma \tilde{b};x,x_0\right)(x-y)^\gamma$$

and Lemma 10, we have, similar to the proof of I_1 and for $k \ge 0$,

$$\left|R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y)\right| \le C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x-x_0|^{m-|\gamma|} |x-y|^{|\gamma|} \left\|D^\alpha b\right\|_{BMO(\omega)} \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|}$$

and

$$\left|R_m(\tilde{b};x,y)\right| \le C|x-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|},$$

thus

$$\begin{aligned}
I_3^{(1)}(x) &\le \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|R_m(\tilde{b};x,y) - R_m(\tilde{b};x_0,y)\right| \frac{|K(x-y)|}{|x-y|^m} \left|f(y)\right| dy \\
&\quad + \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| \left|R_m(\tilde{b};x_0,y)\right| \left|f(y)\right| dy \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=0}^{\infty} \frac{\omega(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|x-x_0|}{|x_0-y|^{n+1}} \left|f(y)\right| dy \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=1}^{\infty} \omega_{2^k\tilde{Q}} \frac{d}{(2^k d)^{n+1}} \int_{2^k\tilde{Q}} \left|f(y)\right| \omega(y)\nu(y)^{1/q} \omega(y)^{-1}\nu(y)^{-1/q}\, dy \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=1}^{\infty} 2^{-k} \omega_{2^k\tilde{Q}} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|\omega(y)f(y)\right|^q \nu(y)\, dy \right)^{1/q} \\
&\quad \times \left(\frac{1}{|2^k Q|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \right)^{1/q'} \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=1}^{\infty} 2^{-k} \omega_{2^k\tilde{Q}} (\nu_{2^k\tilde{Q}})^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&\times \left(\frac{1}{\nu(2^k\tilde{Q})} \int_{2^k\tilde{Q}} \left|\omega(y)f(y)\right|^q \nu(y)\,dy \right)^{1/q} \\
&\times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\,dy \right)^{1/q'} \\
\le{}& C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q} \sum_{k=1}^{\infty} k2^{-k} \\
&\times \omega_{2^kQ}(\nu_{2^kQ})^{1/q} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\,dy \right)^{1/q'} \\
\le{}& C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q} \sum_{k=1}^{\infty} k2^{-k} \\
\le{}& C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}.
\end{aligned}
$$

For $I_3^{(2)}$, we get

$$
\begin{aligned}
I_3^{(2)}(x) \le{}& C \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|R_m(\tilde{b};x_0,y)|}{|x_0-y|^m} \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0-y)\phi_j(x_0-x) \right| \left|f(y)\right|\,dy \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{|D^\alpha \tilde{b}(y)||(x_0-y)^\alpha|}{|x_0-y|^m} \\
&\times \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0-y)\phi_j(x_0-x) \right| \left|f(y)\right|\,dy \\
\le{}& C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{\omega(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|} \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \left|f(y)\right|\,dy \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|D^\alpha b(y) - \left(D^\alpha b\right)_{2^{k+1}\tilde{Q}}\right| \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \left|f(y)\right|\,dy \\
&+ C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left|\left(D^\alpha b\right)_{2^{k+1}\tilde{Q}} - \left(D^\alpha b\right)_{\tilde{Q}}\right| \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \left|f(y)\right|\,dy \\
\le{}& C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q} \sum_{k=1}^{\infty} k2^{-k\delta} \\
&\times \omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\,dy \right)^{1/q'} \\
&+ C \sum_{k=1}^{\infty} 2^{-k\delta} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|D^\alpha b(y) - \left(D^\alpha b\right)_{2^k\tilde{Q}}\right|^r \mu(y)^{-r/p}\,dy \right)^{1/r} \\
&\times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|f(y)\right|^{r'} \omega(y)^{r'} \nu(y)^{r'/p}\,dy \right)^{1/r'} \\
&+ C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \sum_{k=1}^{\infty} k2^{-k\delta} \omega_{2^k\tilde{Q}} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \left|\omega(y)f(y)\right|^q \nu(y)\,dy \right)^{1/q} \\
&\times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\,dy \right)^{1/q'}
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\sum_{k=1}^\infty 2^{-k\delta}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\nu(y)^{-r/p}\,dy\right)^{1/r}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|f(y)\omega(y)\right|^{r'}\nu(y)^{r'/p}\,dy\right)^{1/r'}\\
&\quad+C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\sum_{k=1}^\infty k2^{-k\delta}\\
&\quad\times\omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\omega(y)^{-q'}\nu(y)^{-q'/q}\,dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(\tilde{x})\right]^{1/r'}+\left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_3^{(3)}(x)&\le C\sum_{|\alpha|=m}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|D^\alpha b(y)-\left(D^\alpha b\right)_{2^{k+1}\tilde{Q}}\right|\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|f(y)\right|dy\\
&\quad+C\sum_{|\alpha|=m}\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\left|\left(D^\alpha b\right)_{2^{k+1}\tilde{Q}}-\left(D^\alpha b\right)_{\tilde{Q}}\right|\frac{|x-x_0|}{|x_0-y|^{n+1}}\left|f(y)\right|dy\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\sum_{k=1}^\infty 2^{-k}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\nu(y)^{-r/p}\,dy\right)^{1/r}\\
&\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\left|f(y)\omega(y)\right|^{r'}\nu(y)^{r'/p}\,dy\right)^{1/r'}\\
&\quad+C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\sum_{k=1}^\infty k2^{-k}\\
&\quad\times\omega_{2^k\tilde{Q}}(\nu_{2^k\tilde{Q}})^{1/q}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}\omega(y)^{-q'}\nu(y)^{-q'/q}\,dy\right)^{1/q'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(\tilde{x})\right]^{1/r'}+\left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\right).
\end{aligned}$$

Thus

$$I_3\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{BMO(\omega)}\left(\left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(\tilde{x})\right]^{1/r'}+\left[M_\nu\left(|\omega f|^q\right)(\tilde{x})\right]^{1/q}\right).$$

These results complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0 that the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q\left|T^b(f)(x)-C_0\right|^\eta dx\right)^{1/\eta}\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}w(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x}),$$

where Q is any cube centered at x_0, $C_0=\sum_{j=1}^m c_j\phi_j(x_0-x)$ and $c_j=\int_{R^n}\frac{K(x_0,y)}{|x_0-y|^m}B_j(x_0-y)f_2(y)\,dy$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n\setminus\tilde{Q}}$,

$$
\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \left|T^b(f)(x) - C_0\right|^\eta dx\right)^{1/\eta} \\
&\quad\le C\left(\frac{1}{|Q|}\int_Q \left|T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T\left(\sum_{|\alpha|=m}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m} f_1\right)\right|^\eta dx\right)^{1/\eta} \\
&\qquad + C\left(\frac{1}{|Q|}\int_Q \left|T^{\tilde{b}}(f_2)(x) - C_0\right|^\eta dx\right)^{1/\eta} \\
&\quad= J_1 + J_2 + J_3.
\end{aligned}
$$

For J_1 and J_2, by using the same argument as in the proof of Theorem 1, we get

$$
\begin{aligned}
\left|R_m(\tilde{b};x,y)\right| &\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)} \left|D^\alpha\tilde{b}(z)\right|^q \omega(z)^{q(1-r)/r}\omega(z)^{q(r-1)/r}\,dz\right)^{1/q} \\
&\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\left(\int_{\tilde{Q}(x,y)} \left|D^\alpha\tilde{b}(z)\right|^r \omega(z)^{1-r}\,dz\right)^{1/r} \\
&\quad\times\left(\int_{\tilde{Q}(x,y)} \omega(z)^{q(r-1)/(r-q)}\,dz\right)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/q}\left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{(r-q)/rq} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \omega(z)^{p_0}\,dz\right)^{(r-q)/rq} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r} \\
&\quad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} \omega(z)\,dz\right)^{(r-1)/r} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r}\omega(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1} \\
&\le C|x-y|^m \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{Q})^{\beta/n}\omega(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
J_1 &\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{Q})^{\beta/n}\omega(\tilde{x})|Q|^{-1/s}\left(\int_{R^n}\left|f_1(x)\right|^s dx\right)^{1/s} \\
&\le C\sum_{|\alpha|=m}\left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{Q})^{\beta/n}\omega(\tilde{x})|Q|^{-1/s}\left(\int_{\tilde{Q}}\left|f(x)\right|^r\omega(x)\,dx\right)^{1/r} \\
&\quad\times\left(\int_{\tilde{Q}}\omega(x)^{-s/(r-s)}\,dx\right)^{(r-s)/rs}
\end{aligned}
$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) |\tilde{Q}|^{-1/s} \omega(\tilde{Q})^{1/r} \left(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r \omega(x)\, dx \right)^{1/r}$$

$$\times \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \omega(x)^{-s/(r-s)}\, dx \right)^{(r-s)/rs} \left(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \omega(x)\, dx \right)^{1/r} |\tilde{Q}|^{1/s} \omega(\tilde{Q})^{-1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}),$$

$$J_2 \leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| \omega(x)^{-1/r} |f(x)| \omega(x)^{1/r}\, dx$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \left(\int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}|^{r'} \omega(x)^{1-r'}\, dx \right)^{1/r'} \left(\int_{\tilde{Q}} |f(x)|^r \omega(x)\, dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{Q})^{\beta/n+1/r'} \omega(\tilde{Q})^{1/r-\beta/n}$$

$$\times \left(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r \omega(x)\, dx \right)^{1/r}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,\omega}(f)(\tilde{x})$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).$$

For J_3, we have

$$\begin{aligned} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| &\leq C \sum_{|\gamma|<m} \sum_{|\alpha|=m} |x - x_0|^{m-|\gamma|} |x - y|^{|\gamma|} \\ &\quad \times \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \omega(2^k \tilde{Q})^{\beta/n}, \end{aligned}$$

thus

$$\begin{aligned} &|T^{\tilde{b}}(f_2)(x) - C_0| \\ &\quad \leq \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} |R_m(\tilde{b}; x, y) - R_m(\tilde{b}; x_0, y)| \frac{|K(x-y)|}{|x-y|^m} |f(y)|\, dy \\ &\qquad + \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left| \frac{1}{|x-y|^m} - \frac{1}{|x_0-y|^m} \right| |R_m(\tilde{b}; x_0, y)| |K(x-y)| |f(y)|\, dy \\ &\qquad + C \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{|R_m(\tilde{b}; x_0, y)|}{|x_0-y|^m} \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0-y) \phi_j(x_0-x) \right| |f(y)|\, dy \\ &\qquad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \frac{|D^\alpha \tilde{b}(y)| |(x_0-y)^\alpha|}{|x_0-y|^m} \\ &\qquad \times \left| K(x-y) - \sum_{j=1}^{l} B_j(x_0-y) \phi_j(x_0-x) \right| |f(y)|\, dy \\ &\qquad + C \sum_{|\alpha|=m} \sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q} \setminus 2^k \tilde{Q}} \left| \frac{(x-y)^\alpha}{|x-y|^m} - \frac{(x_0-y)^\alpha}{|x_0-y|^m} \right| |K(x-y)| |D^\alpha \tilde{b}(y)| |f(y)|\, dy \end{aligned}$$

$$
\begin{aligned}
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=0}^{\infty} \omega(2^{k+1}\tilde{Q})^{\beta/n} \\
&\quad \times \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \right) |f(y)|\, dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \right) \\
&\quad \times \int_{2^k\tilde{Q}} \left| D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}} \right| w(y)^{-1/r} |f(y)| \omega(y)^{1/r}\, dy \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{|x-x_0|}{|x_0-y|^{n+1}} + \frac{|x-x_0|^\delta}{|x_0-y|^{n+\delta}} \right) \\
&\quad \times \int_{2^k\tilde{Q}} \left| (D^\alpha b)_{2^k\tilde{Q}} - (D^\alpha b)_{\tilde{Q}} \right| |f(y)| \omega(y)^{1/r} \omega(y)^{-1/r}\, dy \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\delta}{(2^k d)^{n+\delta}} \right) \\
&\quad \times \omega(2^k\tilde{Q})^{\beta/n} \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\, dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-1/(r-1)}\, dy \right)^{(r-1)/r} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)\, dy \right)^{1/r} |2^k\tilde{Q}| \omega(2^k\tilde{Q})^{-1/r} \\
&\quad + C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\delta}{(2^k d)^{n+\delta}} \right) \\
&\quad \times \left(\int_{2^k\tilde{Q}} \left| (D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}}) \right|^{r'} \omega(y)^{1-r'}\, dy \right)^{1/r'} \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\, dy \right)^{1/r} \\
&\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} k\omega(2^k\tilde{Q})^{\beta/n} \left(\frac{d}{(2^k d)^{n+1}} + \frac{d^\delta}{(2^k d)^{n+\delta}} \right) \\
&\quad \times \left(\int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\, dx \right)^{1/r} \\
&\quad \times \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-1/(r-1)}\, dy \right)^{(r-1)/r} \left(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)\, dy \right)^{1/r} |2^k\tilde{Q}| \omega(2^k\tilde{Q})^{-1/r} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} k(2^{-k} + 2^{-k\delta}) \\
&\quad \times \left(\frac{1}{\omega(2^k\tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\, dx \right)^{1/r} \\
&\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\delta}) \\
&\quad \times \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|} \left(\frac{1}{\omega(2^k\tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\, dx \right)^{1/r} \\
&\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 2. □

Proof of Theorem 3 Notice that $\nu^{r'/p} \in A_{r'+1-r'/p} \subset A_p$ and $\nu(x)\,dx \in A_{p/r'}(\nu(x)^{r'/p}\,dx)$ by Lemma 6, thus, by Theorem 1, Lemmas 2 and 9,

$$\begin{aligned}
&\int_{R^n} \left|T^b(f)(x)\right|^p \nu(x)\,dx \\
&\quad\le \int_{R^n} \left|M_\eta\left(T^b(f)\right)(x)\right|^p \nu(x)\,dx \le C \int_{R^n} \left|M^{\#}_{\Phi,\eta}\left(T^b(f)\right)(x)\right|^p \nu(x)\,dx \\
&\quad\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \int_{R^n} \left(\left[M_\nu\left(\left|\omega T(f)\right|^q\right)(x)\right]^{p/q}\right. \\
&\qquad\left. + \left[M_{\nu^{r'/p}}\left(|\omega f|^{r'}\right)(x)\right]^{p/r'} + \left[M_\nu\left(|\omega f|^q\right)(x)\right]^{p/q}\right)\nu(x)\,dx \\
&\quad\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left(\int_{R^n} \left|\omega(x) f(x)\right|^p \nu(x)\,dx + \int_{R^n} \left|\omega(x) T(f)(x)\right|^p \nu(x)\,dx\right) \\
&\quad= C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \left(\int_{R^n} \left|f(x)\right|^p \mu(x)\,dx + \int_{R^n} \left|T(f)(x)\right|^p \mu(x)\,dx\right) \\
&\quad\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{BMO(\omega)} \int_{R^n} \left|f(x)\right|^p \mu(x)\,dx.
\end{aligned}$$

This completes the proof of Theorem 3. □

Proof of Theorem 4 Choose $1 < r < p$ in Theorem 2 and notice $\omega^{1-q} \in A_1$, then we have, by Lemmas 8 and 9,

$$\begin{aligned}
\left\|T^b(f)\right\|_{L^q(\omega^{1-q})} &\le \left\|M_\eta\left(T^b(f)\right)\right\|_{L^q(\omega^{1-q})} \le C\left\|M^{\#}_{\Phi,\eta}\left(T^b(f)\right)\right\|_{L^q(\omega^{1-q})} \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)} \left\|\omega M_{\beta,r,\omega}(f)\right\|_{L^q(\omega^{1-q})} \\
&= C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)} \left\|M_{\beta,r,\omega}(f)\right\|_{L^q(\omega)} \\
&\le C \sum_{|\alpha|=m} \left\|D^\alpha b\right\|_{\mathrm{Lip}_\beta(\omega)} \|f\|_{L^p(\omega)}.
\end{aligned}$$

This completes the proof of Theorem 4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
CW carried out all of the paper, MZ participated in the proof of Theorem 2. All authors read and approved the final manuscript.

Received: 9 August 2013 Accepted: 13 January 2014 Published: 10 Feb 2014

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10.1186/1029-242X-2014-57
Cite this article as: Wu and Zhang: **Weighted sharp maximal function inequalities and boundedness of multilinear singular integral operator satisfying a variant of Hörmander's condition.** *Journal of Inequalities and Applications* 2014, 2014:57