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Two blow-up criteria of solutions to a modified two-component Camassa-Holm system

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Abstract
In this paper, we establish two sufficient conditions on the initial data to guarantee a blow-up phenomenon for the modified two-component Camassa-Holm (MCH2) system.
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1 Introduction

In this paper, we consider the Cauchy problem of the following modified two-component Camassa-Holm (MCH2) system:

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + g\rho\bar{\rho}_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\rho = (1-\partial_x^2)(\bar{\rho}-\hat{\rho})$, u denotes the velocity field, g is the downward constant acceleration of gravity as applied to shallow water waves, $\bar{\rho}$ is the average density, and $\hat{\rho}$ is taken to be a constant. For convenience we let $g = 1$ in this paper. The MCH2 system does admit peaked solutions in the velocity and average density; we refer to Ref. [1] for details. There the authors analytically identified the steepening mechanism that allows the singular solutions to emerge from smooth spatially confined initial data. They found that wave breaking in the fluid velocity does not imply a singularity in the pointwise density ρ at the point of vertical slope. Some other recent works can be found in [2, 3]. We find that the MCH2 system is expressed in terms of an averaged or filtered density $\bar{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho = (1-\partial_x^2)(\bar{\rho}-\hat{\rho})$. Note that the MCH2 system is a version of the CH2 system modified to allow for a dependence on the average density $\bar{\rho}$ (or depth, in the shallow water interpretation) as well as the pointwise density ρ.

Let $\gamma = \bar{\rho}-\hat{\rho}$, then $\gamma = G*\rho$, where the sign $*$ denotes the spatial convolution, $G(x)$ is the associated Green's function of the operator $(1-\partial_x^2)^{-1}$. Therefore system (1.1) is equivalent

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to the following one:

$$\begin{cases} u_t + uu_x + \partial_x(G * (u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2)) = 0, & t > 0, x \in \mathbb{R}, \\ \gamma_t + u\gamma_x + G * ((u_x\gamma_x)_x + u_x\gamma) = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \gamma(0,x) = \gamma_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

The MCH2 may not be integrable unlike CH2. The characteristic is that it will amount to strengthening the norm for $\bar{\rho}$ from L^2 to H^1 in the potential energy term [4]. It means we have the following conserved quantity:

$$E(t) = \int_{\mathbb{R}} u^2 + u_x^2 + \gamma^2 + \gamma_x^2 \, dx.$$

We cannot obtain the conservation of the H^1 norm for the CH2 system, which reads

$$\begin{cases} u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} + g\rho\rho_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases}$$

The CH2 system appeared initially in [5], and recently Constantin and Ivanov in [6] gave a demonstration about its derivation in view of the fluid shallow water theory from the hydrodynamic point of view. This generalization, similar to the Camassa-Holm equation, possesses the peakon, multi-kink solutions and the bi-Hamiltonian structure [7, 8] and is integrable. Well-posedness and the wave breaking mechanism were discussed in [9–11] and the existence of global solutions was analyzed in [6, 10, 12]. The geometric investigation can be found in [13, 14].

Obviously, under the constraint of $\rho(x,t) = 0$, this system reduces to the Camassa-Holm equation, which was derived physically by Camassa and Holm in [15] (found earlier by Fokas and Fuchssteiner [16] as a bi-Hamiltonian generalization of the KdV equation) by directly approximating the Hamiltonian for Euler's equation in the shallow water region with $u(x,t)$ representing the free surface above a flat bottom. Some satisfactory results have been obtained recently, for instance, see Refs. [17–21]. Moreover, wave breaking criteria for a large class of initial data have been established in [18, 20–22]. In [20], McKean established the necessary and sufficient condition of wave breaking, while Zhou and his collaborators gave a new and direct proof in [22] for McKean's theorem. In [23], Xin and Zhang showed global existence of weak solutions but uniqueness was obtained only under a priori assumption that is known to hold only for initial data $u_0(x) \in H^1$ such that $u_0(x) - u_{0xx}(x)$ is a sign-definite random measure. The solitary waves of the Camassa-Holm equation are peaked solutions and are orbitally stable [24]; see also [25] for a very related rod equation. Recently, an asymptotic analysis was given in [26]. If $\rho(x,t) \neq 0$, the CH2 system which includes both velocity and density variables in the dynamics is actually an extension of the CH equation. Although possessing peaked solutions in the velocity, the CH2 system does not admit singular solutions in the density profile. Its mathematical properties have been studied further in many works [6–10, 27, 28].

In Section 2, we recall some preliminary results on well-posedness and blow-up scenario. In Section 3, two detailed blow-up criteria are presented.

2 Preliminaries

In this section, for completeness, we recall some elementary results and skip their proofs. Local well-posedness for the MCH2 system can be obtained by Kato's semi-group theory [29]. In [2], the authors gave a detailed description on the well-posedness theorem.

Theorem 2.1 [2] *Give $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$, there exist a maximal $T = T(\|X_0\|_{H^s \times H^{s-1}}) > 0$ and a unique solution $X = (u, \gamma)^T$ to system* (1.2) *such that*

$$X = X(\cdot, X_0) \in C\big([0,T], H^s \times H^{s-1}\big) \cap C^1\big([0,T], H^{s-1} \times H^{s-2}\big).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$X_0 \to X(\cdot, X_0) : H^s \times H^s \to C\big([0,T], H^s \times H^{s-1}\big) \cap C^1\big([0,T], H^{s-1} \times H^{s-2}\big)$$

is continuous.

The next result describes the precise blow-up scenario for sufficiently regular solutions to system (1.2).

Theorem 2.2 [2] *Let $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geq 5/2$, and Let T be the maximal existence time of the solution $X = (u, \gamma)^T$ to system* (1.2) *with the initial data X_0. Then the corresponding solution blows up in finite time if and only if*

$$\lim_{t \to T} \inf_{x \in \mathbb{R}} u_x(x,t) = -\infty.$$

We also need to introduce the classical particle trajectory method for later use. Let $q(x,t)$ be the particle line evolved by the solution; that is, it satisfies

$$\frac{dq(x,t)}{dt} = u\big(q(x,t), t\big), \qquad q(x, t = 0) = x.$$

Differentiating the first equation with respect to x, one has

$$\frac{d}{dx} q_t = q_{xt} = u_x(q,t) q_x, \qquad t \in (0,T).$$

Hence

$$q_x(x,t) = e^{\int_0^t u_x(q,s)\,ds}, \qquad q_x(x,0) = 1,$$

which is always positive before the blow-up time. Therefore, the function $q(x,t)$ is an increasing diffeomorphism of the line.

3 Blow-up

Before giving blow-up theorems, we rewrite the system (1.1) by $y = u - u_{xx}$ as follows:

$$\begin{cases} y_t + uy_x + 2yu_x + g\rho\bar{\rho}_x = 0, & t > 0, x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\ y(0,x) = y_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}. \end{cases}$$

As $y(x,t) = (1 - \partial_x^2)u(x,t)$ and as $u(x,t)$ is given by the convolution $u(x,t) = G * y$ with $G = \frac{1}{2}e^{-|x|}$, we have

$$u(x,t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(\xi,t)\,d\xi,$$

from which we get

$$u_x(x,t) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}y(\xi,t)\,d\xi.$$

Thus,

$$(u + u_x)(x,t) = e^{x}\int_{x}^{\infty} e^{-\xi}y(\xi,t)\,d\xi,$$
$$(u - u_x)(x,t) = e^{-x}\int_{-\infty}^{-x} e^{\xi}y(\xi,t)\,d\xi.$$

Now we give our two blow-up theorems.

Theorem 3.1 *Suppose* $X_0 = (u_0, \gamma_0)^T \in H^s \times H^{s-1}$, $s \geq \frac{5}{2}$, *for some point* $x_0 \in \mathbb{R}$, $\rho_0(x_0) = y_0(x_0) = 0$ *and initial data satisfies the following conditions*:
(i) $\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\,d\xi > 0$ *and* $\int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\,d\xi < 0$,
(ii) $\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\,d\xi \int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\,d\xi + E(0) < 0$,
where $E(0) = \|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2$. *Then the solution to system* (1.2) *with the initial value* X_0 *blows up in finite time*.

Remark 3.1 In fact the condition (ii) can be reduced to

$$\int_{-\infty}^{x_0} e^{\xi}y_0(\xi)\,d\xi \int_{x_0}^{\infty} e^{-\xi}y_0(\xi)\,d\xi + \int_{\mathbb{R}} (\gamma^2 + \gamma_\xi^2)(\xi,t)\,d\xi < 0.$$

If $\gamma(\xi,t) \equiv 0$, the theory becomes the blow-up theorem in [21] for the Camassa-Holm. As $\gamma(x,t)$ has nothing to do with the initial data, so we add the initial energy $E(0)$ to condition (ii).

Proof Differentiating the first equation in system (1.2) with respect to x, we obtain

$$u_{tx} + uu_{xx} + u_x^2 + \partial_x^2\left(G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right)\right) = 0.$$

Applying $\partial_x^2(G*f) = G*f - f$ to the above equation yields

$$u_{tx} + uu_{xx} = u^2 - \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2 - G*\left(u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right).$$

This equation gives

$$\begin{aligned}\frac{d}{dt}u_x\big(q(x_0,t),t\big) &= (u_{xt} + uu_{xx})\big(q(x_0,t),t\big)\\ &\le \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\ &\quad + \frac{1}{2}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{2}\gamma_x^2\big(q(x_0,t),t\big) - \frac{1}{2}G*\big(\gamma^2 - \gamma_x^2\big)\\ &\le \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\ &\quad + \frac{1}{2}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{2}G*\gamma_x^2,\end{aligned} \tag{3.1}$$

where we used the fact

$$G*\left(u^2 + \frac{1}{2}u_x^2\right) \ge \frac{1}{2}u^2.$$

As regards γ^2 we can deduce that

$$\begin{aligned}\gamma^2 &= \int_{-\infty}^x \gamma\gamma_x\,dx - \int_x^\infty \gamma\gamma_x\,dx \le \int_{-\infty}^x \frac{\gamma^2+\gamma_x^2}{2}\,dx + \int_\infty^x \frac{\gamma^2+\gamma_x^2}{2}\,dx\\ &= \int_{-\infty}^\infty \frac{\gamma^2+\gamma_x^2}{2}\,dx \le \frac{1}{2}E(0).\end{aligned} \tag{3.2}$$

Due to (3.2), we obtain

$$\gamma^2 \le \|\gamma\|_{L^\infty}^2 \le \frac{1}{2}E(0). \tag{3.3}$$

Owing to $G*f = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|}f(\xi)\,d\xi$, we have the following inequality:

$$\begin{aligned}G*\gamma_x^2 &= \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\xi|}\gamma_x^2(\xi)\,d\xi \le \frac{1}{2}\int_{-\infty}^x \gamma_x^2\,d\xi + \frac{1}{2}\int_x^\infty \gamma_x^2\,d\xi\\ &\le \frac{1}{2}\big(\|u_0\|_{H^1}^2 + \|\gamma_0\|_{H^1}^2\big) = \frac{1}{2}E(0).\end{aligned} \tag{3.4}$$

Then using (3.3) and (3.4), we can turn the inequality (3.1) into

$$\begin{aligned}\frac{d}{dt}u_x\big(q(x_0,t),t\big) &\le \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big)\\ &\quad + \frac{1}{2}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{2}G*\gamma_x^2\\ &\le \frac{1}{2}u^2\big(q(x_0,t),t\big) - \frac{1}{2}u_x^2\big(q(x_0,t),t\big) + \frac{1}{2}E(0).\end{aligned} \tag{3.5}$$

In order to reach our result, we need the following claim.

Claim 1 *$u_x(q(x_0,t),t) < 0$ is decreasing, $u^2(q(x_0,t),t) - u_x^2(q(x_0,t),t) + E(0) < 0$ for all $t \in [0,T)$, where T is the maximal existence time of the solution.*

Suppose not, *i.e.*, there exists a t_0 such that $u^2(q(x_0,t),t) - u_x^2(q(x_0,t),t) + E(0) < 0$ on $[0,t_0)$ and $u^2(q(x_0,t_0),t_0) - u_x^2(q(x_0,t_0),t_0) + E(0) = 0$. Now let

$$I(t) := \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi$$

and

$$II(t) := \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi.$$

Firstly for $t \in [0,t_0)$, differentiating $I(t)$, we have

$$\begin{aligned}
\frac{dI(t)}{dt} &= -\frac{1}{2}u\big(q(x_0,t),t\big)e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y_t(\xi,t)\,d\xi \\
&= \frac{1}{2}u(u_x-u)\big(q(x_0,t),t\big) - \frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}(uy_x + 2u_xy + \rho\gamma_x)\,d\xi \\
&\geq \frac{1}{2}u(u_x-u)\big(q(x_0,t),t\big) + \frac{1}{4}\big(u^2+u_x^2-2uu_x\big)\big(q(x_0,t),t\big) \\
&\quad - \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) + \frac{1}{4}G*\big(\gamma^2-\gamma_x^2\big) \\
&\geq \frac{1}{4}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) - \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) \\
&\quad + \frac{1}{4}G*\big(\gamma^2-\gamma_x^2\big) \\
&\geq \frac{1}{4}u_x^2\big(q(x_0,t),t\big) - \frac{1}{4}u^2\big(q(x_0,t),t\big) - \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{4}G*\gamma_x^2 \\
&\geq \frac{1}{4}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big) > 0,
\end{aligned} \tag{3.6}$$

where we used (3.3) and (3.4).

Secondly, by the same argument, we get

$$\begin{aligned}
\frac{dII(t)}{dt} &\leq -\frac{1}{4}\big(u_x^2-u^2\big)\big(q(x_0,t),t\big) + \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) - \frac{1}{4}\gamma_x^2\big(q(x_0,t),t\big) \\
&\quad - \frac{1}{4}G*\big(\gamma^2-\gamma_x^2\big) \\
&\leq -\frac{1}{4}u_x^2\big(q(x_0,t),t\big) + \frac{1}{4}u^2\big(q(x_0,t),t\big) + \frac{1}{4}\gamma^2\big(q(x_0,t),t\big) + \frac{1}{4}G*\gamma_x^2 \\
&\leq -\frac{1}{4}\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big) < 0.
\end{aligned} \tag{3.7}$$

Hence, it follows from (3.6) and (3.7) and the continuity property of the ODEs that

$$\begin{aligned}
u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0) &= -4I(t)II(t) - E(0) \\
&> -4I(0)II(0) - E(0) > 0,
\end{aligned}$$

for all $t \in [0, t_0)$, where we have used the condition (i) and (ii). The continuity property implies that, when $t = t_0$, we have

$$u_x^2\big(q(x_0,t_0),t_0\big) - u^2\big(q(x_0,t_0),t_0\big) - E(0) > 0.$$

This is an obvious contradiction. Then t_0 can be extended to T. On the other hand

$$\begin{aligned} u_x\big(q(x_0,t),t\big) &= -\frac{1}{2}e^{-q(x_0,t)}\int_{-\infty}^{q(x_0,t)} e^{\xi}y(\xi,t)\,d\xi + \frac{1}{2}e^{q(x_0,t)}\int_{q(x_0,t)}^{\infty} e^{-\xi}y(\xi,t)\,d\xi \\ &< -\frac{1}{2}e^{-x_0}\int_{-\infty}^{x_0} e^{\xi}y_0(\xi,t)\,d\xi + \frac{1}{2}e^{x_0}\int_{x_0}^{\infty} e^{-\xi}y_0(\xi,t)\,d\xi. \end{aligned}$$

Then the initial assumption makes $u_x(q(x_0,t),t) < 0$ obvious. So our claim is proved.

Using (3.6) and (3.7) again, we have the following equation for $(u_x^2 - u^2)(q(x_0,t),t)$:

$$\begin{aligned} \frac{d}{dt}\big(u_x^2 - u^2\big)\big(q(x_0,t),t\big) &= -4II(t)\frac{d}{dt}I(t) - 4I(t)\frac{d}{dt}II(t) \\ &\geq -II(t)\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big) \\ &\quad + I(t)\big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big) \\ &= -u_x\big(q(x_0,t),t\big)\big(u_x^2 - u^2 - E(0)\big)\big(q(x_0,t),t\big), \end{aligned} \tag{3.8}$$

where we used $u_x(q(x_0,t),t) = -I(t) + II(t)$. Due to (3.5), we can obtain

$$u_x\big(q(x_0,t),t\big) \leq \int_0^t \frac{1}{2}\big(u^2 - u_x^2 + E(0)\big)\big(q(x_0,\tau),\tau\big)\,d\tau + u_{0x}(x_0). \tag{3.9}$$

Now, substituting (3.9) into (3.8), it yields

$$\begin{aligned} &\frac{d}{dt}\big(u_x^2 - u^2\big)\big(q(x_0,t),t\big) \\ &\quad \geq \left(-\int_0^t \frac{1}{2}\big(u^2 - u_x^2 + E(0)\big)\big(q(x_0,\tau),\tau\big)\,d\tau - u_{0x}(x_0)\right) \\ &\qquad \times \big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big) \\ &\quad = \frac{1}{2}\left(\int_0^t \big(u_x^2 - u^2 - E(0)\big)\big(q(x_0,\tau),\tau\big)\,d\tau - 2u_{0x}(x_0)\right) \\ &\qquad \times \big(u_x^2\big(q(x_0,t),t\big) - u^2\big(q(x_0,t),t\big) - E(0)\big). \end{aligned} \tag{3.10}$$

Before completing the proof, we need the following technical lemma.

Lemma 3.1 [30] *Suppose that $\Psi(t)$ is a twice continuously differential satisfying*

$$\begin{cases} \Psi''(t) \geq C_0\Psi'(t)\Psi(t), & t > 0, C_0 > 0, \\ \Psi(0) > 0, & \psi'(0) > 0. \end{cases} \tag{3.11}$$

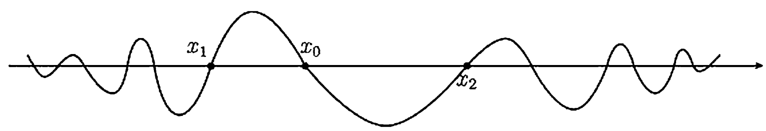

Figure 1 y_0.

Then $\Psi(t)$ blows up in finite time. Moreover the blow-up time T can be estimated in terms of the initial datum as

$$T \leq \max\left\{\frac{2}{C_0\Psi(0)}, \frac{\Psi(0)}{\Psi'(0)}\right\}.$$

Let $\Psi(t) = \int_0^t (u_x^2 - u^2 - E(0))(q(x_0,\tau),\tau)\,d\tau - 2u_{0x}(x_0)$, then given the condition (i) and due to the claim and the expression of $u_{0x}(x_0)$, we get $\Psi(t) > 0$ and $\Psi'(t) > 0$. Using the above lemma, (3.10) is an equation of type (3.11) with $C_0 = \frac{1}{2}$. We can conclude that under the conditions (i) and (ii), the solution to system (1.2) blows up in finite time. □

Theorem 3.2 *Suppose $X_0 = (u_0,\rho_0)^T \in H^s \times H^{s-1}$, $s \geq \frac{5}{2}$, there exists δ satisfying when $x \in (x_0-\delta, x_0+\delta)$, $\rho_0(x) \equiv 0$, when $x \in (-\infty, x_0-\delta]$, $\rho_0(x) \geq 0$ and when $x \in [x_0+\delta,\infty)$, $\rho_0(x) \leq 0$. Some portion of the positive part of $y_0(x)$ lies to the left of some portion of its negative part with the changing sign point at x_0, then the solution to system* (1.2) *with the initial value X_0 blows up in finite time.*

Before we prove the above theorem, we draw a picture of y_0 in Figure 1.

Proof In order to prove the theorem, we define the following quantities:

$$A_0 = \int_{-\infty}^{x_0} e^{\xi} y_0(\xi)\,d\xi, \qquad B_0 = \int_{x_0}^{\infty} e^{-\xi} y_0(\xi)\,d\xi.$$

Then concerning the sign of A_0 and B_0, we have four cases.

Case 1: $A_0 > 0$, $B_0 < 0$.

Case 2: $A_0 < 0$, $B_0 < 0$.

Case 3: $A_0 > 0$, $B_0 > 0$.

Case 4: $A_0 < 0$, $B_0 > 0$.

The cases for $A_0 = 0$ or $B_0 = 0$ are easy to handle.

First, we can find that Case 3 is equivalent to Case 2.

In fact, if $(u(x,t),\gamma(x,t))$ is a solution, let $\tilde{u}(x,t) = -u(-x,t)$ and $\tilde{\gamma}(x,t) = -\gamma(-x,t)$, then $(\tilde{u}(x,t),\tilde{\gamma}(x,t))$ is also a solution with $\tilde{u_0}(x) = -u_0(-x)$ and $\tilde{\gamma_0}(x) = -\gamma_0(-x)$. Let $\tilde{y_0}(x) = (1-\partial_x^2)\tilde{u_0}(x) = -y_0(-x)$ with positive part on $(-x_2,-x_0)$ and negative part on $(-x_0,-x_1)$, then we have

$$\tilde{A_0} = \int_{-\infty}^{-x_0} e^{\xi}\tilde{y_0}(\xi)\,d\xi = -\int_{-\infty}^{-x_0} e^{\xi} y_0(-\xi)\,d\xi = -\int_{x_0}^{\infty} e^{-\eta} y_0(\eta)\,d\eta = -B_0.$$

By the same reasoning, we have $\tilde{B_0} = -A_0$.

Set

$$A\big(q(x,t),t\big) := \int_{-\infty}^{q(x,t)} e^{\xi} y(\xi,t)\,d\xi \quad \text{and} \quad B\big(q(x,t),t\big) := \int_{q(x,t)}^{\infty} e^{-\xi} y(\xi,t)\,d\xi$$

for any $x \in (x_0 - \delta, x_0 + \delta)$, then we have

$$\begin{aligned}
\frac{dA(q(x,t),t)}{dt} &= \frac{1}{2} e^{q(x,t)} \big[u\big(q(x,t),t\big) - u_x\big(q(x,t),t\big)\big]^2 \\
&\quad + \frac{1}{2}\int_{-\infty}^{q(x,t)} e^{\xi}\big[u(\xi,t) - u_\xi(\xi,t)\big]^2\,d\xi - \frac{1}{2} e^{q(x,t)}\big(\gamma^2 - \gamma_x^2\big)\big(q(x,t),t\big) \\
&\quad + \frac{1}{2}\int_{-\infty}^{q(x,t)} e^{\xi}\big(\gamma^2 - \gamma_x^2\big)(\xi,t)\,d\xi, \qquad (3.12) \\
\frac{dB(q(x,t),t)}{dt} &= -\frac{1}{2} e^{-q(x,t)} \big[u\big(q(x,t),t\big) + u_x\big(q(x,t),t\big)\big]^2 \\
&\quad - \frac{1}{2}\int_{q(x,t)}^{\infty} e^{-\xi}\big[u(\xi,t) + u_\xi(\xi,t)\big]^2\,d\xi + \frac{1}{2} e^{-q(x,t)}\big(\gamma^2 - \gamma_x^2\big)\big(q(x,t),t\big) \\
&\quad - \frac{1}{2}\int_{q(x,t)}^{\infty} e^{-\xi}\big(\gamma^2 - \gamma_x^2\big)(\xi,t)\,d\xi. \qquad (3.13)
\end{aligned}$$

In order to get the monotonous property of $A(q(x,t),t)$ and $B(q(x,t),t)$, we need the following claim.

Claim 2 *Under the condition of $\rho_0(x)$ from the theorem, for all $t > 0$ we have*

$$\begin{cases} \rho(q(x,t),t) \geq 0, & x \in (-\infty, x_0 - \delta], \\ \rho(q(x,t),t) \equiv 0, & x \in (x_0 - \delta, x_0 + \delta), \\ \rho(q(x,t),t) \leq 0, & x \in [x_0 + \delta, \infty), \end{cases}$$

and $y(q(x_0,t),t) = 0$.

From the first equation of system (1.1) we have the following equivalent form:

$$y_t + 2yu_x + y_x u + \rho\gamma_x = 0.$$

Applying the particle trajectory method and the second equation in (1.1), we obtain

$$\begin{aligned}
\frac{d}{dt}\big(y\big(q(x,t),t\big)q_x^2(x,t)\big) &= (y_t + 2yu_x + y_x u)\big(q(x,t),t\big)q_x^2(x,t) \\
&= -\rho\big(q(x,t),t\big)\gamma_x\big(q(x,t),t\big)q_x^2(x,t)
\end{aligned}$$

and

$$\frac{d}{dt}\rho\big(q(x,t),t\big)q_x(x,t) = 0,$$

which implies

$$\rho\big(q(x,t),t\big)q_x(x,t) = \rho_0(x).$$

Due to the condition of $\rho_0(x)$ from the theorem and $q_x(x,t) > 0$, we get

$$\begin{cases} \rho(q(x,t),t) \geq 0, & x \in (-\infty, x_0 - \delta], \\ \rho(q(x,t),t) \equiv 0, & x \in (x_0 - \delta, x_0 + \delta), \\ \rho(q(x,t),t) \leq 0, & x \in [x_0 + \delta, \infty), \end{cases}$$

for all $t > 0$. Then

$$\frac{d}{dt} y\big(q(x,t),t\big) q_x^2(x,t) = 0 \quad x \in (x_0 - \delta, x_0 + \delta).$$

Thus $y(q(x,t),t)q_x^2(x,t)$ is independent on time t. By taking $t = 0$, we have

$$y\big(q(x,t),t\big) q_x^2(x,t) = y_0(x) \quad x \in (x_0 - \delta, x_0 + \delta).$$

Since $y_0(x_0) = 0$ and from the above equation, we get $y(q(x_0,t),t) = 0$. Therefore the claim holds.

Claim 3 *For any fixed* t, $\gamma_x^2(x,t) - \gamma^2(x,t) \leq (\gamma_x^2 - \gamma^2)(q(\eta,t),t)$, *for any* $\eta \in (x_0 - \delta, x_0 + \delta)$ *and all* $x \in \mathbb{R}$.

As $\gamma = G * \rho$, where G is the Green's function, it can be expressed as $G(x) = -\frac{1}{2}e^{-|x|}$, and then one has the equation for $\gamma(x,t)$ and $\gamma_x(x,t)$:

$$\gamma(x,t) = \frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}\rho(\xi,t)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi,$$
$$\gamma_x(x,t) = -\frac{1}{2}e^{-x}\int_{-\infty}^{x} e^{\xi}\rho(\xi,t)\,d\xi + \frac{1}{2}e^{x}\int_{x}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi.$$

Therefore,

$$(\gamma + \gamma_x)(x,t) = e^{x}\int_{x}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi,$$
$$(\gamma - \gamma_x)(x,t) = e^{-x}\int_{-\infty}^{-x} e^{\xi}\rho(\xi,t)\,d\xi.$$

By direct computation, if $x \leq q(\eta,t)$, for any $\eta \in (x_0 - \delta, x_0 + \delta)$, then from the above two equations we can get

$$\begin{aligned} \gamma_x^2(x,t) - \gamma^2(x,t) &= -\left(\int_{-\infty}^{q(\eta,t)} e^{\xi}\rho(\xi,t)\,d\xi - \int_{x}^{q(\eta,t)} e^{\xi}\rho(\xi,t)\,d\xi\right) \\ &\quad \times \left(\int_{q(\eta,t)}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi + \int_{x}^{q(\eta,t)} e^{-\xi}\rho(\xi,t)\,d\xi\right) \\ &= \big(\gamma_x^2 - \gamma^2\big)\big(q(\eta,t),t\big) - \int_{-\infty}^{q(\eta,t)} e^{\xi}\rho(\xi,t)\,d\xi \int_{x}^{q(\eta,t)} e^{-\xi}\rho(\xi,t)\,d\xi \\ &\quad + \int_{x}^{q(\eta,t)} e^{\xi}\rho(\xi,t)\,d\xi \int_{q(\eta,t)}^{\infty} e^{-\xi}\rho(\xi,t)\,d\xi \\ &\leq \big(\gamma_x^2 - \gamma^2\big)\big(q(\eta,t),t\big), \end{aligned}$$

where we used the above claim as regards $\rho(x,t)$. Similarly, if $x \le q(\eta,t)$, for any $\eta \in (x_0 - \delta, x_0 + \delta)$, we also have

$$\gamma_x^2(x,t) - \gamma^2(x,t) \le (\gamma_x^2 - \gamma^2)(q(\eta,t),t).$$

This completes the proof of the claim.

For any $x \in (x_0 - \delta, x_0 + \delta)$, by applying our claim to (3.12) and (3.13), we obtain

$$\begin{aligned} \frac{dA(q(x,t),t)}{dt} &\ge \frac{1}{2}e^{q(x,t)}\big[u(q(x,t),t) - u_x(q(x,t),t)\big]^2 \\ &\quad + \frac{1}{2}\int_{-\infty}^{q(x,t)} e^{\xi}\big[u(\xi,t) - u_\xi(\xi,t)\big]^2\,d\xi > 0, \qquad (3.14) \\ \frac{dB(q(x,t),t)}{dt} &\le -\frac{1}{2}e^{-q(x,t)}\big[u(q(x,t),t) + u_x(q(x,t),t)\big]^2 \\ &\quad - \frac{1}{2}\int_{q(x,t)}^{\infty} e^{-\xi}\big[u(\xi,t) + u_\xi(\xi,t)\big]^2\,d\xi < 0, \qquad (3.15) \end{aligned}$$

which implies $A(q(x,t),t)$ is a strictly increasing function, while $B(q(x,t),t)$ is a strictly decreasing one for a nontrivial solution.

Now we prove Case 1.

From (3.1) and Claim 3, we have

$$\begin{aligned} \frac{d}{dt}u_x(q(x_0,t),t) &= (u_{xt} + uu_{xx})(q(x_0,t),t) \\ &\le \frac{1}{2}u^2(q(x_0,t),t) - \frac{1}{2}u_x^2(q(x_0,t),t) \\ &\quad + \frac{1}{2}\gamma^2(q(x_0,t),t) - \frac{1}{2}\gamma_x^2(q(x_0,t),t) \\ &\quad - \frac{1}{2}G * (\gamma^2 - \gamma_x^2) \\ &\le \frac{1}{2}u^2(q(x_0,t),t) - \frac{1}{2}u_x^2(q(x_0,t),t) \\ &= \frac{1}{2}A(q(x_0,t),t)B(q(x_0,t),t). \end{aligned}$$

Due to the increasing property of $A(q(x_0,t),t)$ and the decreasing property of $B(q(x_0,t),t)$ ((3.14) and (3.15)), if we let

$$m(t) := u_x(q(x_0,t),t),$$

then

$$\begin{aligned} \frac{d}{dt}m(t) &\le \frac{1}{2}u^2(q(x_0,t),t) - \frac{1}{2}u_x^2(q(x,t),t) \\ &= \frac{1}{2}A(q(x_0,t),t)B(q(x_0,t),t) \\ &\le \frac{1}{2}A_0B_0 < 0. \end{aligned} \qquad (3.16)$$

Suppose the corresponding solution exists globally in time. Since $m(t)$ is strictly decreasing with initial assumption $m(0) < 0$, there exists a t_1 such that for all $t > t_1$, we have

$$m(t) < -\sqrt{E(0)} < 0,$$

where

$$E(0) = \int_{\mathbb{R}} u_0^2 + u_{0x}^2 + \gamma_0^2 + \gamma_{0x}^2 \, dx.$$

Thanks to (3.16) and the following fact:

$$\|u(x,t)\|_{L^\infty(\mathbb{R})}^2 \le \frac{1}{2}\|u(x,t)\|_{H^1(\mathbb{R})}^2 \le \frac{1}{2}E(t) = \frac{1}{2}E(0),$$

we have the following inequality for $t > t_0$:

$$\begin{aligned}\frac{d}{dt}m(t) &\le -\frac{1}{2}m^2(t) + \frac{1}{2}u^2\bigl(q(x_0,t),t\bigr)\\ &\le -\frac{1}{2}m^2(t) + \frac{1}{4}E(0)\\ &\le -\frac{1}{4}m^2(t).\end{aligned}$$

Then we need the following lemma to finish our proof for Case 1.

Lemma 3.2 [31] *Assume that a differentiable function $y(t)$ satisfies*

$$y'(t) \le -Cy^2(t) + K$$

with constant $C, K > 0$. If we have the initial datum $y(0) = y_0 < -\sqrt{\frac{K}{C}}$, then the solution goes to $-\infty$ before t tends to $\frac{1}{-Cy_0 + \frac{K}{y_0}}$.

Through this lemma, we can see that $m(t)$ goes to $-\infty$ within finite time and Case 1 has been proved.

Now we prove Case 2.

We will prove that after some time Case 2 will change to Case 1. So it is sufficient to show that there exists a time $T_0 \in (0,\infty)$, such that $\int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi > 0$ as $t > T_0$.

Suppose not, *i.e.*, $\int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi \le 0$ for any $t > 0$. As in Case 2, we have

$$\int_{-\infty}^{x_0} e^{\xi} y_0(\xi)\,d\xi < 0, \qquad \int_{x_0}^{\infty} e^{-\xi} y_0(\xi)\,d\xi = -\epsilon_0 < 0.$$

Then

$$\lim_{x\to x_0}\int_x^{\infty} e^{-\xi} y_0(\xi)\,d\xi = -\epsilon_0,$$

which shows that, for $\frac{\epsilon_0}{2} > 0$, there exists a $\delta' > 0$, and for any $x \in U(x_0,\delta') \doteq \{x \mid |x - x_0| < \delta'\}$, we have

$$\left|\int_x^{\infty} e^{-\xi} y_0(\xi)\,d\xi + \epsilon_0\right| < \frac{\epsilon_0}{2}.$$

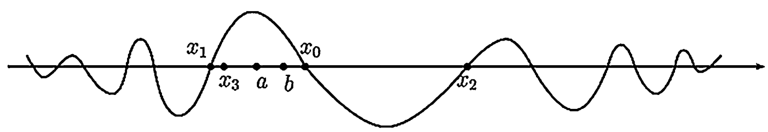

Figure 2 y_0 **for Case 2.**

That is to say

$$\int_x^\infty e^{-\xi} y_0(\xi)\,d\xi < -\frac{\epsilon_0}{2} < 0, \quad \text{for any } x \in U(x_0,\rho).$$

If we set $x_1^* = \max\{x_1, x_0 - \delta\}$ and $x_3 = \max\{x_1^*, x_0 - \delta'\}$ (see Figure 2), then

$$\int_{x_3}^\infty e^{-\xi} y_0(\xi)\,d\xi < -\frac{\epsilon_0}{2} < 0.$$

Therefore, for any $x \in [x_3, x_0]$,

$$\int_{-\infty}^x e^{\xi} y_0(\xi)\,d\xi < 0 \quad \text{and} \quad \int_x^\infty e^{-\xi} y_0(\xi)\,d\xi < 0.$$

When $x \in [x_3, x_0] \subseteq (x_0 - \delta, x_0 + \delta)$, we see that $A(q(x,t),t)$ is increasing and $B(q(x,t),t)$ is decreasing, then from the hypothesis we know for all $t > 0$

$$\int_{-\infty}^{q(x,t)} e^{\xi} y(\xi,t)\,d\xi < \int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi \le 0$$

and

$$\int_{q(x,t)}^\infty e^{-\xi} y(\xi,t)\,d\xi < 0.$$

Then we obtain

$$\begin{aligned}\left(u^2 - u_x^2\right)\left(q(x,t),t\right) &= \int_{-\infty}^{q(x,t)} e^{\xi} y(\xi,t)\,d\xi \int_{q(x,t)}^\infty e^{-\xi} y(\xi,t)\,d\xi \\ &> 0, \quad \text{for } x \in [x_3, x_0].\end{aligned}$$

That is to say $|u_x| \le |u| \le E(0)$.

Case 3.1 For any x, y satisfying $x < y \in [\frac{x_3+5x_0}{6}, x_0]$, there exists a constant $M > 0$, such that $0 < q(y,t) - q(x,t) \le M$.

For any $a < b \in [\frac{x_3+5x_0}{6}, x_0]$, assume that $b - a = \delta_0$. In view of $u(q(x,t),t) < 0$ for $x \in [x_3, x_0]$, we have

$$\begin{aligned}\left(\int_a^b \sqrt{y_0(x)}\,dx\right)^2 &\le \int_{q(a,t)}^{q(b,t)} y(\xi,t)\,d\xi \left(q(b,t) - q(a,t)\right) \\ &\le -\int_{q(a,t)}^{q(b,t)} u_{\xi\xi}\,d\xi \left(q(b,t) - q(a,t)\right)\end{aligned}$$

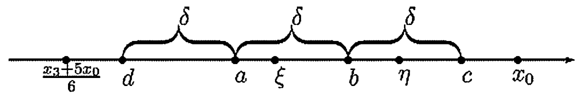

Figure 3 Case 3.1.

$$\begin{aligned} &= \big[u_x\big(q(a,t),t\big) - u_x\big(q(b,t),t\big)\big]\big(q(b,t) - q(a,t)\big) \\ &\le M\big[u_x\big(q(a,t),t\big) - u_x\big(q(b,t),t\big)\big]. \end{aligned} \tag{3.17}$$

On the other hand, from the boundedness of u and u_x, it follows that

$$\left(\int_a^b \sqrt{y_0(x)}\,dx\right)^2 \le 2E(0)\big(q(b,t) - q(a,t)\big),$$

which implies that

$$q(b,t) - q(a,t) \ge \frac{(\int_a^b \sqrt{y_0(x)}\,dx)^2}{2E(0)}.$$

From the expression of $q(x,t)$, we know that

$$\frac{d}{dt}\frac{q_x(a,t)}{q_x(b,t)} = \frac{q_x(a,t)}{q_x(b,t)}\big[u_x\big(q(a,t),t\big) - u_x\big(q(b,t),t\big)\big].$$

Hence

$$\int_0^t \big[u_x\big(q(a,s),s\big) - u_x\big(q(b,s),s\big)\big]\,ds = \ln\frac{q_x(a,t)}{q_x(b,t)}.$$

Fix some points $d < a < b < c \in [\frac{x_3+5x_0}{6}, x_0]$ (see Figure 3), such that $b - a = a - d = c - b = \delta_0 > 0$, and δ_0 is small enough. Because of the convexity of $q(x,t)$, we can deduce that

$$\begin{aligned} M &> q(a,t) - q(d,t) > q(b,t) - q(a,t) > q(c,t) - q(b,t) \\ &\ge \frac{(\int_b^c \sqrt{y_0(x)}\,dx)^2}{2E(0)}. \end{aligned}$$

Therefore, there exist $\eta \in (b,c)$ and $\xi \in (a,b)$, such that

$$\begin{aligned} \frac{(\int_b^c \sqrt{y_0(x)}\,dx)^2}{2E(0)\delta_0} &\le \frac{q(c,t) - q(b,t)}{c - b} = q_x(\eta,t) < q_x(b,t) < q_x(\xi,t) \\ &= \frac{q(b,t) - q(a,t)}{b - a} \le \frac{M}{\delta_0}. \end{aligned} \tag{3.18}$$

Similarly, we also get

$$\frac{(\int_a^b \sqrt{y_0(x)}\,dx)^2}{2E(0)\delta_0} < q_x(a,t) < \frac{M}{\delta_0}. \tag{3.19}$$

Combining (3.18) and (3.19), it follows that

$$\begin{aligned}\int_0^\infty \bigl[u_x\bigl(q(a,s),s\bigr) - u_x\bigl(q(b,s),s\bigr)\bigr]\,ds &= \ln \frac{q_x(a,t)}{q_x(b,t)}\bigg|_{t=\infty}\\ &< \ln \frac{2E(0)M}{(\int_b^c \sqrt{y_0(x)}\,dx)^2} < \infty.\end{aligned}$$

Then a contradiction is obtained from (3.17): $u_x(q(a,t),t) - u_x(q(b,t),t)$ is summable with respect to t, but $(\int_a^b \sqrt{y_0(x)}\,dx)^2$ is not.

Case 3.2. There exist some points, say $a' < b' \in [\frac{x_3+5x_0}{6}, x_0]$, such that $q(b',t) - q(a',t) > 0$ is unbounded.

Different from (3.17), we can deal with the same term as

$$\begin{aligned}\left(\int_a^b \sqrt{y_0(x)}\,dx\right)^2 &= \left(\int_{q(a,t)}^{q(b,t)} \sqrt{y(\xi)}\,d\xi\right)^2\\ &= \left(\int_{q(a,t)}^{q(b,t)} \sqrt{y(\xi)}\cdot\bigl(\xi - q(c,t)\bigr)^{\frac{1}{2}}\cdot\frac{1}{(\xi - q(c,t))^{\frac{1}{2}}}\,d\xi\right)^2\\ &\le \int_{q(a,t)}^{q(b,t)} y(\xi)\bigl(\xi - q(c,t)\bigr)\,d\xi\cdot\int_{q(a,t)}^{q(b,t)}\frac{1}{\xi - q(c,t)}\,d\xi\\ &:= J_1\cdot J_2. \end{aligned}\tag{3.20}$$

According to the convexity of $q(x,t)$, we have

$$\begin{aligned}J_2 &= \ln\frac{q(b,t) - q(c,t)}{q(a,t) - q(c,t)} = \ln\left(\frac{q(b,t) - q(a,t)}{q(a,t) - q(c,t)} + 1\right)\\ &< \ln\left(\frac{b-a}{a-c} + 1\right) = \ln\frac{b-c}{a-c}.\end{aligned}\tag{3.21}$$

Next we will consider the first term J_1 in (3.20),

$$\begin{aligned}J_1 &\le -\int_{q(c,t)}^{q(b,t)} u_{\xi\xi}\bigl(\xi - q(c,t)\bigr)\,d\xi\\ &= -u_\xi\bigl(\xi - q(c,t)\bigr)\big|_{q(c,t)}^{q(b,t)} + \int_{q(c,t)}^{q(b,t)} u_\xi\,d\xi\\ &= -u_x\bigl(q(b,t),t\bigr)\bigl(q(b,t) - q(c,t)\bigr) + u\bigl(q(b,t),t\bigr) - u\bigl(q(c,t),t\bigr)\\ &< -u_x\bigl(q(b,t),t\bigr)\bigl(q(b,t) - q(c,t)\bigr) - u\bigl(q(c,t),t\bigr).\end{aligned}$$

From the hypothesis, we know that $q(b',t) - q(a',t)$ may reach ∞, which means that there exist some times t_1 and t_2, such that

$$\frac{d}{dt}\bigl(q(b',t) - q(a',t)\bigr) > 0, \quad \text{for } t_1 \le t \le t_2.$$

Let $d < c < a < b < a' < b'$ with $c - d = a - c = b - a = a' - b = b' - a' = \delta_0$ (see Figure 4). Thanks to $a' < b' \in [\frac{x_3+5x_0}{6}, x_0]$, we know that the above points all belong to the interval $[x_3, x_0]$. First, we prove the following claim.

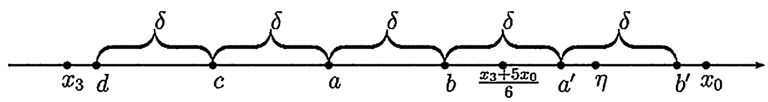

Figure 4 Case 3.2.

Claim 4 *For any two adjacent points belong to* $[x_3, a']$, *say* e *and* f *with* $e < f$, *it satisfies* $\frac{d}{dt}(q(f,t) - q(e,t)) > 0$ *as* $t_1 \le t \le t_2$.

In fact, there exist $\xi \in (q(e,t), q(f,t))$, and $\eta \in (q(a',t), q(b',t))$, such that

$$\begin{aligned}\frac{d}{dt}\big(q(b',t) - q(a',t)\big) &= u\big(q(b',t),t\big) - u\big(q(a',t),t\big) \\ &= u_x(\eta,t)\big(q(b',t) - q(a',t)\big), \\ \frac{d}{dt}\big(q(f,t) - q(e,t)\big) &= u\big(q(f,t),t\big) - u\big(q(e,t),t\big) \\ &= u_x(\xi,t)\big(q(f,t) - q(e,t)\big).\end{aligned}$$

According to $0 < u_x(\eta,t) < u_x(\xi,t)$ and the convexity of $q(x,t)$, we have

$$\frac{d}{dt}\big(q(f,t) - q(e,t)\big) > 0 \quad \text{for } t \in [t_1, t_2].$$

So the claim is true.

Therefore,

$$0 < \frac{d}{dt}\big(q(a',t) - q(b,t)\big) = \int_{q(b,t)}^{q(a',t)} u_\xi \, d\xi \le u_x\big(q(b,t),t\big)\big(q(a',t) - q(b,t)\big),$$

for all $t \in [t_1, t_2]$, which implies

$$u_x\big(q(b,t),t\big) \ge \frac{d(q(a',t) - q(b,t))}{dt} \cdot \frac{1}{q(a',t) - q(b,t)}.$$

Since $y = u - u_{\xi\xi} > 0$,

$$-u\big(q(c,t),t\big)\big(q(c,t) - q(d,t)\big) \le -\int_{q(d,t)}^{q(c,t)} u(\xi)\, d\xi \le -\int_{q(d,t)}^{q(c,t)} u_{\xi\xi}\, d\xi \le 2E(0).$$

Then

$$-u\big(q(c,t),t\big) \le \frac{2E(0)}{q(c,t) - q(d,t)}.$$

Summarizing these estimates and using the convexity of $q(x,t)$, we can get

$$\begin{aligned}J_1 &< -u_x\big(q(b,t),t\big)\big(q(b,t) - q(c,t)\big) - u\big(q(c,t),t\big) \\ &\le -\frac{d(q(a',t) - q(b,t))}{dt} \cdot \frac{1}{q(a',t) - q(b,t)}\big(q(b,t) - q(c,t)\big)\end{aligned}$$

$$\begin{aligned}
&\quad + \frac{2E(0)}{q(c,t)-q(d,t)} \\
&\le -\frac{d(q(a',t)-q(b,t))}{dt} \cdot \frac{1}{q(a',t)-q(b,t)} \bigl(q(a',t)-q(b,t)\bigr) \\
&\quad + \frac{2E(0)}{q(a',t)-q(b,t)} \\
&\le -\frac{d(q(a',t)-q(b,t))}{dt} + \frac{2E(0)}{q(a',t)-q(b,t)} \\
&< 0,
\end{aligned} \tag{3.22}$$

for $q(a',t)-q(b,t)$ large enough and in the time interval it is increasing.

Putting (3.21) and (3.22) into (3.20), we know that $(\int_a^b \sqrt{y_0(x)}\,dx)^2$ becomes negative. This is a contradiction.

Therefore, we finish the proof for Case 2.

Finally we finish the proof of our theorem with proving Case 4.

We want to prove that Case 4 can be reduced to the first or the third case, so it is sufficient to prove that there exists a time $T_0 \in (0,\infty)$, such that $\int_{-\infty}^{q(x_0,t)} e^{\xi} y(\xi,t)\,d\xi > 0$ as $t > T_0$.

We suppose that for all $t \in (0,\infty)$, we have $\int_{-\infty}^{q(x_0,t)} e^{x} y(x,t)\,dx < 0$, then get a contradiction. For any $a \in [x_1^*, x_0]$,

$$\begin{aligned}
(u_x - u)\bigl(q(a,t),t\bigr) &= -e^{-q(a,t)} \int_{-\infty}^{q(a,t)} e^{\xi} y(\xi,t)\,d\xi > 0, \\
(u_x + u)\bigl(q(a,t),t\bigr) &= e^{q(a,t)} \int_{q(a,t)}^{\infty} e^{-\xi} y(\xi,t)\,d\xi > 0.
\end{aligned}$$

Summarizing the above two inequalities, we obtain

$$u_x\bigl(q(x,t),t\bigr) \ge \bigl|u\bigl(q(x,t),t\bigr)\bigr|, \quad \text{for } x_1^* < x < x_0, t \ge 0.$$

After the above preparation, we have

$$\begin{aligned}
\left(\int_a^{x_0} \sqrt{y_0(x)}\,dx\right)^2 &= \left(\int_{q(a,t)}^{q(x_0,t)} \sqrt{y(q)}\,dq\right)^2 \\
&\le \left(\int_{q(a,t)}^{q(x_0,t)} y(q)e^{q}\,dq\right)\left(\int_{q(a,t)}^{q(x_0,t)} e^{-q}\,dq\right) \\
&= \left(\int_{q(a,t)}^{q(x_0,t)} (u-u_{xx})(q)e^{q}\,dq\right)\bigl(e^{-q(a,t)} - e^{-q(x_0,t)}\bigr) \\
&\le (u_x - u)\bigl(q(a,t),t\bigr)\bigl(1 - e^{q(a,t)-q(x_0,t)}\bigr) \\
&\le (u_x - u)\bigl(q(a,t),t\bigr),
\end{aligned}$$

which implies

$$\left(\int_a^{x_0} \sqrt{y_0(x)}\,dx\right)^4 \le (u_x - u)^2\bigl(q(a,t),t\bigr), \quad \text{for } x_1^* < x < x_0, t \ge 0. \tag{3.23}$$

On the other hand, by (3.14), we have

$$\begin{aligned}\infty &> -\int_0^\infty e^q (u_x - u)^2(q)\,dt\Big|_{x=a}^{x=x_0} \\ &= \int_0^\infty dt \int_a^{x_0} e^q\big[(u_x-u)^2 + 2(u_x-u)y\big](q)\,dq \\ &\geq \int_0^\infty dt \int_a^{x_0} e^q q_x (u_x-u)^2(q)\,dx.\end{aligned}$$

Since

$$q + \ln(q_x) = \int_0^t (u_x + u)(q)\,dt + x \geq x,$$

which yields

$$\int_0^\infty dt \int_a^{x_0} e^x (u_x - u)^2(q)\,dx \leq \int_0^\infty dt \int_a^{x_0} e^q q_x (u_x-u)^2(q)\,dx < \infty,$$

we get

$$\int_0^\infty (u_x - u)^2\big(q(x,t),t\big)\,dt < \infty, \quad \text{for almost every } x \in \big[x_1^*, x_0\big].$$

Then a contradiction is obtained from (3.23): $(u_x - u)^2(q)$ taken at $x = a$ is summable with respect to t, but $(\int_a^{x_0} \sqrt{y_0(x)}\,dx)^4$ is not.

So there exists a time T, such that when $t > T$, $\int_{-\infty}^{q(x_0,t)} e^\xi y(\xi,t)\,d\xi > 0$. This completes the proof for Case 4.

Remark 3.2 Scrutinizing the proof, we find that the condition of $\rho_0(x)$ guarantees that Claim 3 holds. Therefore it can be replaced by

$$\begin{cases} \rho_0(x) \leq 0, & x \in (-\infty, x_0 - \delta], \\ \rho_0(x) \equiv 0, & x \in (x_0 - \delta, x_0 + \delta), \\ \rho_0(x) \geq 0, & x \in [x_0 + \delta, \infty), \end{cases}$$

for all $t > 0$. Then the theorem still holds.

Remark 3.3 This blow-up theorem has nothing to do with the initial energy $E(0)$. It is the sign of the initial density $\rho_0(x)$ and the sign of $y_0(x)$ that play an important role in wave breaking, it is not the size of them that affects it. It is very similar to the necessary and sufficient blow-up condition for the Camassa-Holm equation given by McKean in [20]. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
LJ proposed the problems and finished the whole manuscript. YJ proved Theorem 3.1. CM proved Theorem 3.2. All authors read and approved the final manuscript.

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