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On the identity involving certain Hardy sums and Kloosterman sums

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Abstract

The main purpose of this paper is, using the properties of Gauss sums and the mean value theorem of Dirichlet *L*-functions, to study a hybrid mean value problem involving certain Hardy sums and Kloosterman sums and give two exact computational formulae for them.

MSC: 11L05; 11M20

Keywords: Gauss sums; Kloosterman sums; identity; certain Hardy sums; hybrid mean value; computational formula

1 Introduction

Let c be a natural number and d be an integer prime to c. The classical Dedekind sums

$$S(d,c)=\sum_{j=1}^{c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2} & \text{if } x \text{ is not an integer;}\\ 0 & \text{if } x \text{ is an integer}\end{cases}$$

describes the behavior of the logarithm of the eta-function (see [1] and [2]) under modular transformations. Berndt [3] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z)=\sum_{n=-\infty}^{+\infty}\exp\left(\pi i n^2 z\right),\quad \operatorname{Im}(z)>0.$$

That is, put $Vz=(az+b)(cz+d)$ with $a,b,c,d\in Z$, $c>0$, and $ad-bc=1$. Then we have

$$\log\theta(Vz)=\log\theta(z)+\frac{1}{2}\log(cz+d)-\frac{1}{4}\pi i+\frac{1}{4}\pi i S_1(d,c),\tag{1}$$

where $S_1(d,c)$ are defined as

$$S_1(d,c)=\sum_{j=1}^{c-1}(-1)^{j+1+[\frac{dj}{c}]}.$$

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The sums $S_1(d,c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums. Some authors had studied the properties of $S_1(d,c)$ and related sums and obtained some interesting results, see [4–8] and [9]. For example, Zhang and Yi [8] proved the following conclusion. Let p be an odd prime. Then, for any fixed positive integer m, we have the asymptotic formula

$$\sum_{h=1}^{p-1}\left|S_1(h,p)\right|^{2m} = p^{2m}\cdot\frac{\zeta^2(2m)(1-\frac{1}{4^m})}{\zeta(4m)(1+\frac{1}{4^m})} + O\left(p^{2m-1}\cdot\exp\left(\frac{6\ln p}{\ln\ln p}\right)\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $\exp(y) = e^y$.

On the other hand, we introduce the classical Kloosterman sums $K(n,q)$ which are defined as follows: For any positive integer $q > 1$ and integer n,

$$K(n,q) = {\sum_{b=1}^{q}}' e\left(\frac{nb+\overline{b}}{q}\right),$$

where $\overline{b}$ denotes the solution of the congruence $x\cdot b \equiv 1 \bmod q$, $\sum_{b=1}^{\prime q}$ denotes the summation over all $1 \le b \le q$ with $(b,q) = 1$ and $e(x) = e^{2\pi ix}$. Some elementary properties of $K(n,q)$ can be found in [10] and [11].

The main purpose of this paper is, using the properties of Gauss sums and the mean square value theorem of Dirichlet L-functions, to study a hybrid mean value problem involving certain Hardy sums and Kloosterman sums and give two exact computational formulae. That is, we shall prove the following theorem.

Theorem 1 *Let p be an odd prime. Then we have the identity*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p)\cdot K(n,p)\cdot S_1(2m\cdot\overline{n},p) = \begin{cases} 2p^2 & \text{if } p\equiv 3 \bmod 4; \\ 0 & \text{if } p\equiv 1 \bmod 4. \end{cases}$$

Theorem 2 *Let p be an odd prime, then we have the identity*

$$\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\left|K(m,p)\right|^2\cdot\left|K(n,p)\right|^2\cdot S_1(2m\cdot\overline{n},p) = \begin{cases} 2p^3+4\cdot p^2\cdot h_p^2 & \text{if } p\equiv 7 \bmod 8; \\ 2p^3-36\cdot p^2\cdot h_p^2 & \text{if } p\equiv 3 \bmod 8; \\ 0 & \text{if } p\equiv 1 \bmod 4, \end{cases}$$

where h_p denotes the class number of the quadratic field $\mathbf{Q}(\sqrt{-p})$.

For general odd number $q \ge 3$, whether there exits a computational formula for the hybrid mean value

$${\sum_{m=1}^{q}}'{\sum_{n=1}^{q}}'\left|K(m,q)\right|^2\cdot\left|K(n,q)\right|^2\cdot S_1(2m\cdot\overline{n},q)$$

is an open problem.

2 Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorems. Hereinafter, we shall use many properties of Gauss sums, all of which can be found in [12], so they will not be repeated here. First we have the following lemma.

Lemma 1 *Let p be an odd prime, then we have the identity*

$$\sum_{n=1}^{p-1} \chi(n) \cdot \left|K(n,p)\right|^2 = \overline{\chi}(-1) \cdot \frac{\tau^3(\chi) \cdot \tau(\overline{\chi}^2)}{\tau(\overline{\chi})}.$$

Proof For any non-principal character χ mod p, from the properties of Gauss sums we have

$$\begin{aligned}
\sum_{n=1}^{p-1} \chi(n)\left|K(n,p)\right|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{n=1}^{p-1} \chi(n) e\left(\frac{n(a-b)+(\overline{a}-\overline{b})}{p}\right) \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{n=1}^{p-1} \chi(n) e\left(\frac{nb(a-1)+\overline{b}(\overline{a}-1)}{p}\right) \\
&= \tau(\chi) \cdot \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}\big(b(a-1)\big) e\left(\frac{\overline{b}(\overline{a}-1)}{p}\right) \\
&= \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \overline{\chi}(a-1)\overline{\chi}(\overline{a}-1) \\
&= \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \chi(a)\overline{\chi}\big(-(a-1)^2\big) = \overline{\chi}(-1) \cdot \tau^2(\chi) \cdot \sum_{a=1}^{p-2} \chi(a+1)\overline{\chi}\big(a^2\big) \\
&= \overline{\chi}(-1) \cdot \tau^2(\chi) \cdot \sum_{a=1}^{p-2} \chi\big(\overline{a}+\overline{a}^2\big) = \overline{\chi}(-1) \cdot \tau^2(\chi) \cdot \sum_{a=1}^{p-1} \chi\big(a^2+a\big) \\
&= \overline{\chi}(-1) \cdot \tau^2(\chi) \cdot \frac{1}{\tau(\overline{\chi})} \sum_{a=1}^{p-1}\sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b(a^2+a)}{p}\right) \\
&= \overline{\chi}(-1) \cdot \tau^2(\chi) \cdot \frac{1}{\tau(\overline{\chi})} \cdot \sum_{b=1}^{p-1} \overline{\chi}(b) e\left(\frac{b}{p}\right) \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ba}{p}\right) \\
&= \overline{\chi}(-1) \cdot \tau^3(\chi) \cdot \frac{1}{\tau(\overline{\chi})} \cdot \sum_{b=1}^{p-1} \overline{\chi}^2(b) e\left(\frac{b}{p}\right) \\
&= \overline{\chi}(-1) \cdot \frac{\tau^3(\chi) \cdot \tau(\overline{\chi}^2)}{\tau(\overline{\chi})}.
\end{aligned}$$

This proves Lemma 1. □

Lemma 2 *Let $q > 2$ be an integer, then, for any integer a with $(a,q) = 1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d \mid q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a)\left|L(1,\chi)\right|^2,$$

where $L(1,\chi)$ denotes the Dirichlet L-function corresponding to character χ mod d.

Proof See Lemma 2 of [9]. □

Lemma 3 *Let $q > 0$ and $(h,q) = 1$. Then we have the identity*

$$S_1(h,q) = -8S(h+q,2q) + 4S(h,q).$$

Proof This formula is an immediate consequence of (5.9) and (5.10) in [7]. □

Lemma 4 *Let p be an odd prime and $0 < h < p$. Then we have the identity*

$$S_1(2h,p) = -20 \cdot S(2h,p) + 8 \cdot S(4h,p) + 8 \cdot S(h,p).$$

Proof Note that the divisors of $2p$ are 1, 2, p and $2p$. So from Lemma 2 and Lemma 3 we have

$$\begin{aligned} S_1(2h,p) &= -8 \cdot S(2h+p,2p) + 4 \cdot S(2h,p) \\ &= -\frac{4}{\pi^2 p}\sum_{d|2p}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(2h+p)\left|L(1,\chi)\right|^2 \\ &\quad + \frac{4}{\pi^2 p}\sum_{d|p}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}}\chi(2h)\left|L(1,\chi)\right|^2 \\ &= -\frac{4}{\pi^2 p}\cdot\frac{(2p)^2}{\phi(2p)}\sum_{\substack{\chi \bmod 2p \\ \chi(-1)=-1}}\chi(2h+p)\left|L(1,\chi)\right|^2 \\ &= -\frac{16p}{\pi^2(p-1)}\cdot\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\chi(2h+p)\lambda(2h+p)\left|L(1,\chi\lambda)\right|^2 \\ &= -\frac{16p}{\pi^2(p-1)}\cdot\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\chi(2h)\left|L(1,\chi\lambda)\right|^2, \end{aligned} \tag{2}$$

where λ denotes the principal character mod 2.

From the Euler infinite product formula we have

$$\begin{aligned} \left|L(1,\chi\lambda)\right|^2 &= \prod_{p_1}\left|1-\frac{\chi(p_1)\lambda(p_1)}{p_1}\right|^{-2} = \prod_{p_1>2}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2} \\ &= \left|1-\frac{\chi(2)}{2}\right|^2\cdot\prod_{p_1}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2} = \left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\overline{\chi}(2)}{2}\right)\cdot\left|L(1,\chi)\right|^2, \end{aligned} \tag{3}$$

where $\prod_p$ denotes the product over all primes p.

From Lemma 2 we also have the identity

$$S(n,p) = \frac{1}{\pi^2}\cdot\frac{p}{p-1}\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}}\chi(n)\left|L(1,\chi)\right|^2. \tag{4}$$

Now, combining (2), (3) and (4), we have the identity

$$\begin{aligned} S_1(2h,p) &= -\frac{16p}{\pi^2(p-1)} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2h)\left|L(1,\chi\lambda)\right|^2 \\ &= -16 \cdot \frac{p}{\pi^2(p-1)} \cdot \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2h)\left(\frac{5}{4} - \frac{\chi(2)}{2} - \frac{\overline{\chi}(2)}{2}\right) \cdot \left|L(1,\chi)\right|^2 \\ &= -20 \cdot S(2h,p) + 8 \cdot S(4h,p) + 8 \cdot S(h,p). \end{aligned}$$

This proves Lemma 4. □

Lemma 5 *Let p be an odd prime. Then we have the identities*

(A) $$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2 \cdot (p-2)}{p^2};$$

(B) $$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{24} \cdot \frac{(p-1)^2 \cdot (p-5)}{p^2};$$

(C) $$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \cdot \left|L(1,\chi)\right|^2 = \begin{cases} \frac{\pi^2}{48} \cdot \frac{(p-1)^2 \cdot (p-17)}{p^2} & \text{if } p \equiv 1 \bmod 4; \\ \frac{\pi^2}{48} \cdot \frac{(p-1) \cdot (p^2-6p+17)}{p^2} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

Proof From the definition of Dedekind sums we have

$$S(1,c) = \sum_{a=1}^{c-1}\left(\frac{a}{c} - \frac{1}{2}\right)^2 = \frac{(c-1)(c-2)}{12c}. \tag{5}$$

If $p \equiv 1 \bmod c$, then from (5), and noting the reciprocity theorem of Dedekind sums (see [5]), we have the computational formula

$$\begin{aligned} S(c,p) &= \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - S(p,c) = \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - S(1,c) \\ &= \frac{p^2+c^2+1}{12pc} - \frac{1}{4} - \frac{(c-1)(c-2)}{12c} = \frac{(p-1)(p-1-c^2)}{12pc}. \end{aligned} \tag{6}$$

If $p \equiv 3 \bmod 4$, then we also have

$$\begin{aligned} S(4,p) &= \frac{p^2+16+1}{48p} - \frac{1}{4} - S(p,4) = \frac{p^2+17}{48p} - \frac{1}{4} - S(3,4) \\ &= \frac{p^2+17}{48p} - \frac{1}{4} + \frac{1}{8} = \frac{p^2-6p+17}{48p}. \end{aligned} \tag{7}$$

Now taking $c = 1$ in (6), from (4) we may immediately deduce the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \frac{\pi^2}{12} \cdot \frac{(p-1)^2 \cdot (p-2)}{p^2}. \tag{8}$$

Taking $c = 2$ in (6), from (4) we can also deduce the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{24} \cdot \frac{(p-1)^2 \cdot (p-5)}{p^2}. \tag{9}$$

If $p \equiv 1 \bmod 4$, then taking $c = 4$ in (6), from (4) we can deduce the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{48} \cdot \frac{(p-1)^2 \cdot (p-17)}{p^2}. \tag{10}$$

If $p \equiv 3 \bmod 4$, then from (4) and (7) we have the identity

$$\sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{48} \cdot \frac{(p-1) \cdot (p^2-6p+17)}{p^2}. \tag{11}$$

Now Lemma 5 follows from (8)-(11). □

3 Proof of the theorems

In this section, we shall complete the proof of our theorems. Note that if χ is a non-principal character mod p, then $|\tau(\chi)| = \sqrt{p}$ and

$$\left|\sum_{m=1}^{p-1} \chi(m) K(m,p)\right| = \left|\sum_{a=1}^{p-1}\sum_{m=1}^{p-1} \chi(m) e\left(\frac{ma+\overline{a}}{p}\right)\right| = \left|\tau^2(\chi)\right| = p. \tag{12}$$

If $p \equiv 3 \bmod 4$, then from (12), Lemma 4 and Lemma 5 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p) \cdot K(n,p) \cdot S_1(2m \cdot \overline{n}, p) \\
&\quad = -\frac{20 \cdot p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \left|\sum_{n=1}^{p-1} \chi(n) \cdot K(n,p)\right|^2 \cdot \left|L(1,\chi)\right|^2 \\
&\qquad + \frac{8 \cdot p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \left|\sum_{n=1}^{p-1} \chi(n) \cdot K(n,p)\right|^2 \cdot \left|L(1,\chi)\right|^2 \\
&\qquad + \frac{8 \cdot p}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|\sum_{n=1}^{p-1} \chi(n) \cdot K(n,p)\right|^2 \cdot \left|L(1,\chi)\right|^2 \\
&\quad = -\frac{20 \cdot p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 + \frac{8 \cdot p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 \\
&\qquad + \frac{8 \cdot p^3}{\pi^2(p-1)} \sum_{\substack{\chi \bmod p \\ \chi(-1)=-1}} \chi(4) \cdot \left|L(1,\chi)\right|^2 \\
&\quad = -\frac{5}{6}p(p-1)(p-5) + \frac{2}{3}p(p-1)(p-2) + \frac{1}{6}p\left(p^2-6p+17\right) = 2p^2.
\end{aligned} \tag{13}$$

If $p \equiv 1 \bmod 4$, then from (12), Lemma 4 and Lemma 5 we also have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} K(m,p)\cdot K(n,p)\cdot S_1(2m\cdot\overline{n},p) \\
&\quad= -\frac{20\cdot p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\cdot\left|L(1,\chi)\right|^2 + \frac{8\cdot p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2 \\
&\qquad+ \frac{8\cdot p^3}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\cdot\left|L(1,\chi)\right|^2 \\
&\quad= -\frac{5}{6}p(p-1)(p-5) + \frac{2}{3}p(p-1)(p-2) + \frac{1}{6}p(p-1)(p-17) = 0. \qquad (14)
\end{aligned}$$

It is clear that Theorem 1 follows from (13) and (14).

Now we prove Theorem 2. If $p \equiv 1 \bmod 4$, then from Lemma 1 and the method of proving Theorem 1 we have

$$\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1} \left|K(m,p)\right|^2\cdot\left|K(n,p)\right|^2\cdot S_1(2m\cdot\overline{n},p) \\
&\quad= -\frac{20\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2 \\
&\qquad+ \frac{8\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2 \\
&\qquad+ \frac{8\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2 \\
&\quad= -\frac{20\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\cdot\left|L(1,\chi)\right|^2 + \frac{8\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2 \\
&\qquad+ \frac{8\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\cdot\left|L(1,\chi)\right|^2 \\
&\quad= -\frac{5}{6}p^2(p-1)(p-5) + \frac{2}{3}p^2(p-1)(p-2) + \frac{1}{6}p^2(p-1)(p-17) = 0. \qquad (15)
\end{aligned}$$

If $p \equiv 3 \bmod 4$, then note that the Legendre symbol $(\frac{-1}{p}) = \chi_2(-1) = -1$, $L(1,\chi_2) = \pi\cdot h_p/\sqrt{p}$, and

$$\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)^2 e\left(\frac{a}{p}\right) = -1,$$

so from Lemma 1 and the method of proving Theorem 1 we have

$$
\begin{aligned}
&\sum_{m=1}^{p-1}\sum_{n=1}^{p-1}\left|K(m,p)\right|^2\cdot\left|K(n,p)\right|^2\cdot S_1(2m\cdot\overline{n},p)\\
&= -\frac{20\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2\\
&\quad+\frac{8\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2\\
&\quad+\frac{8\cdot p}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|\sum_{n=1}^{p-1}\chi(n)\cdot\left|K(n,p)\right|^2\right|^2\cdot\left|L(1,\chi)\right|^2\\
&= -\frac{20\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(2)\cdot\left|L(1,\chi)\right|^2+\frac{8\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left|L(1,\chi)\right|^2\\
&\quad+\frac{8\cdot p^4}{\pi^2(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\chi(4)\cdot\left|L(1,\chi)\right|^2+\frac{20\cdot p^3}{\pi^2}\cdot\left(\frac{2}{p}\right)\cdot\left|L(1,\chi_2)\right|^2\\
&\quad-\frac{8\cdot p^3}{\pi^2}\cdot\left(\frac{4}{p}\right)\cdot\left|L(1,\chi_2)\right|^2-\frac{8\cdot p^3}{\pi^2}\cdot\left|L(1,\chi_2)\right|^2\\
&= -\frac{5}{6}p^2(p-1)(p-5)+\frac{2}{3}p^2(p-1)(p-2)+\frac{1}{6}p^2\left(p^2-6p+17\right)\\
&\quad+20\cdot p^2\cdot h_p^2\cdot\left(\frac{2}{p}\right)-16\cdot p^2\cdot h_p^2\\
&= 2p^3+20\cdot\left(\frac{2}{p}\right)\cdot p^2\cdot h_p^2-16\cdot p^2\cdot h_p^2. \qquad (16)
\end{aligned}
$$

Note that $(\frac{2}{p})=(-1)^{\frac{p^2-1}{8}}=-1$ if $p\equiv 3 \bmod 8$, and $(\frac{2}{p})=1$ if $p\equiv 7 \bmod 8$, then from (15) and (16) we may immediately deduce Theorem 2. This completes the proof of our theorems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HZ obtained the theorems and completed the proof. WZ corrected and improved the final version. Both authors read and approved the final manuscript.

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