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Approximation properties of the modification of q-Stancu-Beta operators which preserve x^2

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Abstract

In this paper, we introduce a new kind of modification of q-Stancu-Beta operators which preserve x^2 based on the concept of q-integer. We investigate the moments and central moments of the operators by computation, obtain a local approximation theorem, and get the pointwise convergence rate theorem and also a weighted approximation theorem.

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1 Introduction

In 2012, Aral and Gupta [1] introduced the q analog of Stancu-Beta operators as

$$L_{n,q}^*(f;x) = \frac{K(A;[n]_q x)}{B_q([n]_q x;[n]_q+1)} \int_0^{\infty/A} \frac{u^{[n]_q x-1}}{(1+u)_q^{[n]_q x+[n]_q+1}} f\left(q^{[n]_q x} u\right) d_q u, \tag{1}$$

for every $n \in \mathbb{N}$, $q \in (0,1)$, $x \in [0,\infty)$. They estimated moments, established direct result in terms of modulus of continuity and present an asymptotic formula.

Since the types of operators which preserve x^2 are important in approximation theory, in this paper, we will introduce a modification of q-Stancu-Beta operators which will be defined in (5). The advantage of these new operators is that they reproduce not only constant functions but also x^2.

Firstly, we recall some concepts of q-calculus. All of the results can be found in [2]. For any fixed real number $0 < q \le 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where

$$[k]_q = \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1; \\ k, & q = 1. \end{cases}$$

Also the q-factorial and q-binomial coefficients are defined as follows:

$$[k]_q! = \begin{cases} [k]_q[k-1]_q \cdots [1]_q, & k = 1,2,\ldots; \\ 1, & k = 0 \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n \geq k \geq 0).$$

The q-improper integrals are defined as

$$\int_0^{\infty/A} f(x)\, d_q x = (1-q) \sum_{-\infty}^{\infty} f\left(\frac{q^n}{A}\right) \frac{q^n}{A}, \quad A > 0, \tag{2}$$

provided the sums converge absolutely.

The q-Beta integral is defined as

$$B_q(t;s) = K(A;t) \int_0^{\infty/A} \frac{x^{t-1}}{(1+x)_q^{t+s}}\, d_q x, \tag{3}$$

where $K(x;t) = \frac{1}{x+1} x^t (1+\frac{1}{x})_q^t (1+x)_q^{1-t}$, and $(1+x)_q^\tau = \frac{(1+x)(1+qx)(1+q^2x)\cdots}{(1+q^\tau x)(1+q^{\tau+1}x)(1+q^{\tau+2}x)\cdots}$, $\tau > 0$ ($\tau = t+s$).

In particular for any positive integer n,

$$K(x;n) = q^{\frac{n(n-1)}{2}}, \qquad K(x;0) = 1 \quad \text{and} \quad B_q(t;s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(t+s)}. \tag{4}$$

For $f \in C[0,\infty)$, $q \in (0,1)$, and $n \in \mathbb{N}$, we introduce the new modification of q-Stancu-Beta operators $L_{n,q}(f,x)$ as

$$L_{n,q}(f;x) = \frac{K(A;[n]_q v_n(x))}{B_q([n]_q v_n(x);[n]_q+1)} \int_0^{\infty/A} \frac{u^{[n]_q v_n(x)-1}}{(1+u)_q^{[n]_q v_n(x)+[n]_q+1}} f\left(q^{[n]_q v_n(x)} u\right) d_q u, \tag{5}$$

where

$$v_n(x) = \sqrt{\frac{q[n]_q - q}{[n]_q} x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}. \tag{6}$$

2 Some preliminary results

In this section we give the following lemmas, which we need to prove our theorems.

Lemma 1 (see [1, Lemma 1]) *The following equalities hold*:

$$L^*_{n,q}(1;x) = 1, \qquad L^*_{n,q}(t;x) = x, \qquad L^*_{n,q}\left(t^2;x\right) = \frac{([n]_q x + 1)x}{q([n]_q - 1)}.$$

Lemma 2 *Let $q \in (0,1)$, $x \in [0,\infty)$, we have*

$$L_{n,q}(1;x) = 1, \qquad L_{n,q}(t;x) = \sqrt{\frac{q[n]_q - q}{[n]_q} x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}, \qquad L_{n,q}\left(t^2;x\right) = x^2. \tag{7}$$

Proof From Lemma 1, we get $L_{n,q}(1;x) = 1$ and $L_{n,q}(t;x) = \sqrt{\frac{q[n]_q-q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}$ easily. Finally, we have

$$\begin{aligned} L_{n,q}\left(t^2;x\right) &= \frac{([n]_q v_n(x)+1)v_n(x)}{q([n]_q-1)} \\ &= \frac{[n]_q}{q[n]_q-q}\left(\frac{1}{4[n]_q^2} + \frac{q[n]_q-q}{[n]_q}x^2 + \frac{1}{4[n]_q^2} - \frac{1}{[n]_q}\sqrt{\frac{q[n]_q-q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}}\right) \\ &\quad + \frac{1}{q[n]_q-q}\left(\sqrt{\frac{q[n]_q-q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}\right) = x^2. \end{aligned}$$

Lemma 2 is proved. □

Remark 1 Let $n \in \mathbb{N}$ and $x \in [0,\infty)$, then for every $q \in (0,1)$, by Lemma 2, we have

$$L_{n,q}\left(1+t^2;x\right) = 1 + x^2. \tag{8}$$

Lemma 3 *For every $q \in (0,1)$ and $x \in [0,\infty)$, we have*

$$L_{n,q}\left((t-x)^2;x\right) = 2x^2 - 2x\sqrt{\frac{q[n]_q-q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} + \frac{x}{[n]_q}. \tag{9}$$

Proof Since $L_{n,q}((t-x)^2;x) = L_{n,q}(t^2;x) - 2xL_{n,q}(t;x) + x^2$ and from Lemma 2, we get Lemma 3 easily. □

Remark 2 Let the sequence $q = \{q_n\}$ satisfy that $q_n \in (0,1)$ and $q_n \to 1$ as $n \to \infty$, then for any fixed $x \in [0,\infty)$, by Lemma 3, we have

$$\lim_{n\to\infty} L_{n,q_n}\left((t-x)^2;x\right) = 0. \tag{10}$$

3 Local approximation

In this section we establish direct local approximation theorem in connection with the operators $L_{n,q}(f;x)$.

We denote the space of all real valued continuous bounded functions f defined on the interval $[0,\infty)$ by $C_B[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$.

Further let us consider Peetre's K-functional:

$$K_2(f;\delta) = \inf_{g\in W^2}\left\{\|f-g\| + \delta\left\|g''\right\|\right\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

For $f \in C_B[0,\infty)$, the modulus of continuity of second order is defined by

$$\omega_2(f;\delta) = \sup_{0<h\leq\delta}\ \sup_{x\in[0,\infty)}\left|f(x+2h) - 2f(x+h) + f(x)\right|,$$

by [3, p.177] there exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}), \quad \delta > 0. \tag{11}$$

For $f \in C_B[0,\infty)$, the modulus of continuity is defined by

$$\omega(f;\delta) = \sup_{0<h\le\delta}\ \sup_{x\in[0,\infty)} \left|f(x+h) - f(x)\right|.$$

Our first result is a direct local approximation theorem for the operators $L_{n,q}(f;x)$.

Theorem 1 *For $q \in (0,1)$, $x \in [0,\infty)$, $n \in \mathbb{N}$, and $f \in C_B[0,\infty)$, we have*

$$\begin{aligned} &\left|L_{n,q}(f,x) - f(x)\right| \\ &\quad\le C\omega_2\left(f;\sqrt{2x^2 - 2x\sqrt{\frac{q[n]_q - q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} + \frac{x}{[n]_q} + \left(x + \frac{1}{2[n]_q} - \sqrt{\frac{q[n]_q - q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}}\right)^2}\right) \\ &\qquad + \omega\left(f; x + \frac{1}{2[n]_q} - \sqrt{\frac{q[n]_q - q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}}\right). \end{aligned} \tag{12}$$

Proof For $x \in (0,\infty]$, we define the auxiliary operators $\overline{L_{n,q}}(f;x)$

$$\overline{L_{n,q}}(f;x) = L_{n,q}(f;x) - f\left(\sqrt{\frac{q[n]_q - q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}\right) + f(x). \tag{13}$$

Obviously, we have

$$\overline{L_{n,q}}(t-x;x) = 0. \tag{14}$$

Let $g \in W^2$, by Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)\,du, \quad x,t \in [0,\infty).$$

Using (14), we get

$$\overline{L_{n,q}}(g;x) = g(x) + \overline{L_{n,q}}\left(\int_x^t (t-u)g''(u)\,du;x\right),$$

hence, by Lemma 3, we have

$$\begin{aligned} &\left|\overline{L_{n,q}}(g;x) - g(x)\right| \\ &\quad = \left|L_{n,q}\left(\int_x^t (t-u)g''(u)\,du;x\right)\right| \\ &\qquad + \left|\int_{\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}-\frac{1}{2[n]_q}}^{x} \left[u - \left(\sqrt{\frac{q[n]_q - q}{[n]_q}x^2 + \frac{1}{4[n]_q^2}} - \frac{1}{2[n]_q}\right)\right]g''(u)\,du\right| \end{aligned}$$

$$\begin{aligned}
&\le L_{n,q}\left(\left|\int_x^t (t-u)\left|g''(u)\right|du\right|;x\right) \\
&\quad + \int_{\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}-\frac{1}{2[n]_q}}^{x} \left|u-\left(\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}-\frac{1}{2[n]_q}\right)\right|\left|g''(u)\right|du \\
&\le \left[2x^2-2x\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}+\frac{x}{[n]_q}\right. \\
&\quad \left. + \left(x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right)^2\right]\|g''\|.
\end{aligned}$$

On the other hand, using (13) and Lemma 2, we have

$$\begin{aligned}
\left|\overline{L_{n,q}}(f;x)\right| &\le \left|L_{n,q}(f;x)\right| + 2\|f\| \\
&\le \|f\|L_{n,q}(1;x) + 2\|f\| \\
&\le 3\|f\|.
\end{aligned} \tag{15}$$

Thus,

$$\begin{aligned}
&\left|L_{n,q}(f;x)-f(x)\right| \\
&\quad \le \left|\overline{L_{n,q}}(f-g;x)-(f-g)(x)\right| + \left|\overline{L_{n,q}}(g;x)-g(x)\right| \\
&\qquad + \left|f\left(\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}-\frac{1}{2[n]_q}\right)-f(x)\right| \\
&\quad \le 4\|f-g\| + \left|f\left(\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}-\frac{1}{2[n]_q}\right)-f(x)\right| \\
&\qquad + \left[2x^2-2x\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}+\frac{x}{[n]_q}\right. \\
&\qquad \left. + \left(x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right)^2\right]\|g''\|.
\end{aligned}$$

Hence taking the infimum on the right-hand side over all $g \in W^2$, we get

$$\begin{aligned}
&\left|L_{n,q}(f;x)-f(x)\right| \\
&\quad \le 4K_2\left(f;2x^2-2x\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}+\frac{x}{[n]_q}\right. \\
&\qquad \left. + \left(x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right)^2\right) \\
&\qquad + \omega\left(f;x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right).
\end{aligned}$$

By (11), for every $q \in (0,1)$, we have

$$\begin{aligned}
&\left|L_{n,q}(f,x)-f(x)\right| \\
&\quad \le C\omega_2\left(f;\sqrt{2x^2-2x\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}+\frac{x}{[n]_q}+\left(x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right)^2}\right) \\
&\qquad +\omega\left(f;x+\frac{1}{2[n]_q}-\sqrt{\frac{q[n]_q-q}{[n]_q}x^2+\frac{1}{4[n]_q^2}}\right).
\end{aligned}$$

This completes the proof of Theorem 1. □

4 Rate of convergence

Let $B_{x^2}[0,\infty)$ be the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \le M_f(1+x^2)$, where M_f is a constant depending only on f. We denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$ by $C_{x^2}[0,\infty)$. Also, let $C^*_{x^2}[0,\infty)$ be the subspace of all functions $f \in C_{x^2}[0,\infty)$ for which $\lim_{x\to\infty}\frac{f(x)}{1+x^2}$ is finite. The norm on $C^*_{x^2}[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)}\frac{|f(x)|}{1+x^2}$. We denote the usual modulus of continuity of f on the closed interval $[0,a]$ $(a>0)$ by

$$\omega_a(f,\delta) = \sup_{|t-x|\le\delta}\ \sup_{x,t\in[0,a]}\left|f(t)-f(x)\right|.$$

Obviously, for a function $f \in C_{x^2}[0,\infty)$, the modulus of continuity $\omega_a(f,\delta)$ tends to zero as $\delta \to 0$.

Theorem 2 *Let $f \in C_{x^2}[0,\infty)$, $q \in (0,1)$ and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0,a+1] \subset [0,\infty)$, where $a>0$. Then we have*

$$\begin{aligned}
\left\|L_{n,q}(f)-f\right\|_{C[0,a]} &\le 4M_f\left(1+a^2\right)\left(2a^2-2a\sqrt{\frac{q[n]_q-q}{[n]_q}a^2+\frac{1}{4[n]_q^2}}+\frac{a}{[n]_q}\right) \\
&\quad + 2\omega_{a+1}\left(f;\sqrt{2a^2-2a\sqrt{\frac{q[n]_q-q}{[n]_q}a^2+\frac{1}{4[n]_q^2}}+\frac{a}{[n]_q}}\right). \qquad (16)
\end{aligned}$$

Proof For $x \in [0,a]$ and $t > a+1$, we have $t-x \ge t-a > 1$. Hence $(t-x)^2 > 1$. Thus $2+3x^2+2(t-x)^2 \le (2+3x^2)(t-x)^2+2(t-x)^2 = (4+3x^2)(t-x)^2 \le (4+3a^2)(t-x)^2 \le 4(1+a^2)(t-x)^2$. Hence, we obtain

$$\left|f(t)-f(x)\right| \le 4M_f\left(1+a^2\right)(t-x)^2. \qquad (17)$$

For $x \in [0,a]$ and $t \le a+1$, we have

$$\left|f(t)-f(x)\right| \le \omega_{a+1}\left(f;|t-x|\right) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta), \quad \delta>0. \qquad (18)$$

From (17) and (18), we get

$$\left|f(t)-f(x)\right| \le 4M_f\left(1+a^2\right)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f;\delta). \qquad (19)$$

For $x \in [0,a]$ and $t \geq 0$, by Schwarz's inequality, Lemma 2, and Lemma 3, we have

$$\begin{aligned}
&\left|L_{n,q}(f;x) - f(x)\right| \\
&\quad \leq L_{n,q}\left(\left|f(t) - f(x)\right|;x\right) \\
&\quad \leq 4M_f\left(1+a^2\right)L_{n,q}\left((t-x)^2;x\right) + \omega_{a+1}(f;\delta)\left(1 + \frac{1}{\delta}\sqrt{L_{n,q}\left((t-x)^2;x\right)}\right) \\
&\quad \leq 4M_f\left(1+a^2\right)\left(2a^2 - 2a\sqrt{\frac{q[n]_q - q}{[n]_q}a^2 + \frac{1}{4[n]_q}} + \frac{a}{[n]_q}\right) \\
&\qquad + \omega_{a+1}(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{2a^2 - 2a\sqrt{\frac{q[n]_q - q}{[n]_q}a^2 + \frac{1}{4[n]_q}} + \frac{a}{[n]_q}}\right).
\end{aligned}$$

By taking $\delta = \sqrt{2a^2 - 2a\sqrt{\frac{q[n]_q-q}{[n]_q}a^2 + \frac{1}{4[n]_q}} + \frac{a}{[n]_q}}$, we get the assertion of Theorem 2. □

5 Weighted approximation

Now we will discuss the weighted approximation theorems.

Theorem 3 *Let the sequence* $\{q_n\}$ *satisfy* $0 < q_n < 1$ *and* $q_n \to 1$ *as* $n \to \infty$, *for* $f \in C^*_{x^2}[0,\infty)$, *we have*

$$\lim_{n\to\infty}\left\|L_{n,q_n}(f) - f\right\|_{x^2} = 0. \tag{20}$$

Proof By using the Korovkin theorem in [4], we see that it is sufficient to verify the following three conditions:

$$\lim_{n\to\infty}\left\|L_{n,q_n}\left(t^\nu;x\right) - x^\nu\right\|_{x^2}, \quad \nu = 0,1,2. \tag{21}$$

Since $L_{n,q_n}(1;x) = 1$ and $L_{n,q_n}(t^2;x) = x^2$ (see Lemma 2), (21) holds true for $\nu = 0$ and $\nu = 2$.

Finally, for $\nu = 1$, we have

$$\begin{aligned}
\left\|L_{n,q_n}(t;x) - x\right\|_{x^2} &= \sup_{x\in[0,\infty)}\frac{|L_{n,q_n}(t;x) - x|}{1+x^2} \\
&\leq \left(1 - \sqrt{\frac{q[n]_q - q}{[n]_q}}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2} + \frac{1}{2[n]_q}\sup_{x\in[0,\infty)}\frac{1}{1+x^2} \\
&\leq 1 - \sqrt{\frac{q[n]_q - q}{[n]_q}} + \frac{1}{2[n]_q},
\end{aligned}$$

since $\lim_{n\to\infty} q_n = 1$, we get $\lim_{n\to\infty}(1 - \sqrt{\frac{q[n]_q-q}{[n]_q}}) = 0$ and $\lim_{n\to\infty}\frac{1}{2[n]_q} = 0$, so the second condition of (21) holds for $\nu = 1$ as $n \to \infty$, then the proof of Theorem 3 is completed. □

Competing interests

The author declares that they have no competing interests.

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