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On multivariate higher order Lyapunov-type inequalities

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Abstract

In this paper, by using the best Sobolev constant method, we obtain some new Lyapunov-type inequalities for a class of even-order partial differential equations; the results of this paper are new which generalize and improve some early results in the literature.

Keywords: Lyapunov-type inequality; even-order partial differential equations; Sobolev constant

1 Introduction

It is well known that the Lyapunov inequality for the second-order linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1}$$

states that if $q \in C[a,b]$, $x(t)$ is a nonzero solution of (1) such that $x(a) = x(b) = 0$, then the following inequality holds:

$$\int_a^b |q(t)|\,dt > \frac{4}{b-a} \tag{2}$$

and the constant 4 is sharp.

There have been many proofs and generalizations as well as improvements on this inequality. For example, the authors in [1–3] generalized the Lyapunov-type inequality to the partial differential equations or systems.

First let us recall some background and notations which are introduced in [1, 2].

Let A be a spherical shell $\subseteq \mathbb{R}^N$ for $N > 1$, *i.e.* $A = B(0,R_2) - \overline{B(0,R_1)}$ for $0 < R_1 < R_2$, where $B(0,R) = \{x \in \mathbb{R}^N : \|x\| < R\}$ for $R > 0$ and $\|\cdot\|$ is the Euclidean norm. Denote $S^{N-1} = \{x \in \mathbb{R}^N : \|x\| = 1\}$, the unit sphere in $\mathbb{R}^N$ with surface area

$$\omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \quad \textit{i.e.} \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. Then every $x \in \mathbb{R}^N - \{0\}$ has a unique representation of the form $x = r\omega$, where $r = \|x\| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$. Therefore, for any $f \in C(\overline{A})$, we have

$$\int_A f(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} f(r\omega) r^{N-1}\,dr \right) d\omega.$$

In [1], Aktaş obtained the following results.

Theorem A *If $f \in C^{2n}(\overline{A})$ is a nonzero solution of the following even-order partial differential equation*:

$$\frac{\partial^{2n} f(x)}{\partial r^{2n}} + q(x) f(x) = 0, \quad x \in A, \tag{4}$$

where $n \in \mathbb{N}$ and $q \in C(\overline{A})$, with the boundary conditions

$$\frac{\partial^{2i} f}{\partial r^{2i}} \big(\partial B(0,R_1)\big) = \frac{\partial^{2i} f}{\partial r^{2i}} \big(\partial B(0,R_2)\big) = 0, \quad i = 0,1,2,\ldots,n-1, \tag{5}$$

then the following inequality holds:

$$\int_A \big|q(x)\big|\,dx > \frac{2^{3n-1}}{(R_2 - R_1)^{2n-1}} \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} R_1^{N-1}. \tag{6}$$

Theorem B *If $f \in C^{2n}(\overline{A})$ is a nonzero solution of* (4) *with the boundary conditions*

$$\frac{\partial^{i} f}{\partial r^{i}} \big(\partial B(0,R_1)\big) = \frac{\partial^{i} f}{\partial r^{i}} \big(\partial B(0,R_2)\big) = 0, \quad i = 0,1,2,\ldots,n-1, \tag{7}$$

then the following inequality holds:

$$\int_A \big|q(x)\big|\,dx > \frac{4^{2n-1}(2n-1)[(n-1)!]^2}{(R_2 - R_1)^{2n-1}} \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} R_1^{N-1}. \tag{8}$$

In this paper, we generalize Theorem A and Theorem B to a more general class of even order partial differential equations. Moreover, as we shall see by the end of this paper, Theorem 1 improves Theorem A significantly.

2 Main results

Let us consider the following even-order partial differential equation:

$$\frac{\partial^{2n} y(x)}{\partial r^{2n}} + \sum_{k=0}^{n} p_k(x) \frac{\partial^k y(x)}{\partial r^k} = 0, \tag{9}$$

where $y(x) \in C^{2n}(\overline{A})$, $p_k(x) \in C(\overline{A})$, $k = 0,1,2,\ldots,n$, and $x \in \mathbb{R}^N$.

The main results of this paper are the following theorems.

Theorem 1 *If $y(x)$ is a nonzero solution of* (9) *satisfying boundary conditions* (5), *then the following inequality holds*:

$$\begin{aligned}1 &< \sqrt{\frac{(2^{2n}-1)\zeta(2n)(R_2-R_1)^{2n-1}\Gamma(\frac{N}{2})}{2^{2n}\pi^{2n+\frac{N}{2}}R_1^{N-1}}}\left(\int_A p_n^2(x)\,dx\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1}\frac{(R_2-R_1)^{2n-k-1}\Gamma(\frac{N}{2})}{(2\pi)^{2n-k}R_1^{N-1}\pi^{\frac{N}{2}}}\sqrt{\left(2^{2n}-1\right)\left(2^{2(n-k)}-1\right)\zeta(2n)\zeta\left(2(n-k)\right)} \\ &\quad \times \int_A \left|p_k(x)\right|dx, \end{aligned} \tag{10}$$

where $\zeta(s) = \sum_{n=1}^{+\infty}\frac{1}{n^s}$ *is the Riemann zeta function.*

Theorem 2 *If* $y(x)$ *is a nonzero solution of* (9) *satisfying boundary conditions* (7), *then the following inequality holds*:

$$\begin{aligned}1 &< \frac{1}{(n-1)!2^{2n-1}}\sqrt{\frac{(R_2-R_1)^{2n-1}\Gamma(\frac{N}{2})}{(2n-1)R_1^{N-1}2\pi^{\frac{N}{2}}}}\left(\int_A p_n^2(x)\,dx\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1}\frac{(R_2-R_1)^{2n-k-1}\Gamma(\frac{N}{2})}{\sqrt{(2n-1)(2n-2k-1)}(n-1)!(n-k-1)!4^{2n-k-1}R_1^{N-1}2\pi^{\frac{N}{2}}}\int_A\left|p_k(x)\right|dx. \end{aligned} \tag{11}$$

3 Proofs of theorems

For the proofs of Theorem 1 and Theorem 2, let us consider first the following ordinary even-order linear ordinary differential equation:

$$x^{(2n)}(t) + \sum_{k=0}^{n} p_k(t)x^{(k)}(t) = 0, \tag{12}$$

where $p_k(t) \in C[a,b]$, $k = 0,1,2,\dots,n$.

Proposition 3 *If* (12) *has a nonzero solution* $x(t)$ *satisfying the following boundary value conditions*:

$$x^{(2i)}(a) = x^{(2i)}(b) = 0, \quad i = 0,1,2,\dots,n-1, \tag{13}$$

then the following inequality holds:

$$\begin{aligned}1 &< \sqrt{\frac{(2^{2n}-1)\zeta(2n)(b-a)^{2n-1}}{2^{2n-1}\pi^{2n}}}\left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1}\frac{(b-a)^{2n-k-1}\sqrt{(2^{2n}-1)(2^{2n-2k}-1)\zeta(2n)\zeta(2(n-k))}}{2^{2n-k-1}\pi^{2n-k}}\int_a^b\left|p_k(t)\right|dt, \end{aligned} \tag{14}$$

where $\zeta(s)$ *is the Riemann zeta function*: $\zeta(s) = \sum_{k=1}^{+\infty}\frac{1}{k^s}$, $s > 1$.

Proposition 4 *If* (12) *has a nonzero solution* $x(t)$ *satisfying the following boundary value conditions*:

$$x^{(i)}(a) = x^{(i)}(b) = 0, \quad i = 0,1,2,\dots,n-1, \tag{15}$$

then we have the following inequality:

$$1 < \frac{1}{(n-1)!2^{2n-1}}\sqrt{\frac{(b-a)^{2n-1}}{(2n-1)}}\left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} + \sum_{k=0}^{n-1}\frac{(b-a)^{2n-k-1}}{(n-1)!(n-k-1)!4^{2n-k-1}\sqrt{(2n-1)(2n-2k-1)}}\int_a^b \left|p_k(t)\right|dt. \tag{16}$$

In order to prove the above propositions, we need the following lemmas.

Lemma 5 ([4, Proposition 2.1]) *Let $M \in \mathbb{N}$ and*

$$H_C = \left\{u|u^{(M)} \in L^2(a,b), u^{(2k)}(a) = u^{(2k)}(b) = 0, 0 \le k \le \left[(M-1)/2\right]\right\}.$$

Then there exists a positive constant C such that, for any $u \in H_C$, the Sobolev inequality

$$\left(\sup_{a\le t\le b}\left|u(t)\right|\right)^2 \le C\int_a^b \left|u^{(M)}(t)\right|^2 dt$$

holds. Moreover, the best constant $C = C(M)$ is as follows:

$$C(M) = \frac{(2^{2M}-1)\zeta(2M)(b-a)^{2M-1}}{2^{2M-1}\pi^{2M}}.$$

Lemma 6 ([5, Theorem 1.2 and Corollary 1.3]) *Let $M \in \mathbb{N}$ and*

$$H_D = \left\{u|u^{(M)} \in L^2(a,b), u^{(k)}(a) = u^{(k)}(b) = 0, 0 \le k \le M-1\right\}.$$

Then there exists a positive constant D such that for any $u \in H_D$, the Sobolev inequality

$$\left(\sup_{a\le t\le b}\left|u(t)\right|\right)^2 \le D\int_a^b \left|u^{(M)}(t)\right|^2 dt$$

holds. Moreover, the best constant $D = D(M)$ is as follows:

$$D(M) = \frac{(b-a)^{2M-1}}{(2M-1)[(M-1)!]^2 4^{2M-1}}. \tag{17}$$

We give the first seven values of $\zeta(2n)$, $C(n)$, and $D(n)$ in Table 1.

Since the proof of Proposition 4 is similar to that of Proposition 3, we give only the proof of Proposition 3 below.

Table 1 The first seven values of $\zeta(2n)$, $C(n)$ and $D(n)$

n	1	2	3	4	5	6	7
$\zeta(2n)$	$\frac{\pi^2}{6}$	$\frac{\pi^4}{90}$	$\frac{\pi^6}{945}$	$\frac{\pi^8}{9{,}450}$	$\frac{\pi^{10}}{93{,}555}$	$\frac{691\pi^{12}}{638{,}512{,}875}$	$\frac{2\pi^{14}}{18{,}243{,}225}$
$C(n)$	$\frac{b-a}{4}$	$\frac{(b-a)^3}{48}$	$\frac{(b-a)^5}{480}$	$\frac{17(b-a)^7}{80{,}640}$	$\frac{31(b-a)^9}{1{,}451{,}520}$	$\frac{691(b-a)^{11}}{9{,}123{,}840}$	$\frac{(2^{14}-1)(b-a)^{13}}{2^{13}\times 18{,}243{,}225}$
$D(n)$	$\frac{b-a}{4}$	$\frac{(b-a)^3}{192}$	$\frac{(b-a)^5}{20{,}480}$	$\frac{(b-a)^7}{4{,}128{,}768}$	$\frac{(b-a)^9}{1{,}358{,}954{,}496}$	$\frac{(b-a)^{11}}{664{,}377{,}753{,}600}$	$\frac{(b-a)^{13}}{13(6!)^2 4^{13}}$

Proof of Proposition 3 Multiplying both sides of (12) by $x(t)$ and integrating from a to b by parts and using the boundary value condition (13), we can obtain

$$\int_a^b x^{(2n)}(t)x(t)\,dt = (-1)^n \int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt = -\sum_{k=0}^{n} \int_a^b p_k(t)x^{(k)}(t)x(t)\,dt.$$

This yields

$$\begin{aligned}\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt &\le \sum_{k=0}^{n} \int_a^b \bigl|p_k(t)\bigr|\bigl|x^{(k)}(t)x(t)\bigr|\,dt \\ &= \int_a^b \bigl|p_n(t)\bigr|\bigl|x^{(n)}(t)x(t)\bigr|\,dt + \sum_{k=0}^{n-1} \int_a^b \bigl|p_k(t)\bigr|\bigl|x^{(k)}(t)x(t)\bigr|\,dt. \end{aligned} \tag{18}$$

Now, by using Lemma 5, we get for any $t \in [a,b]$, $k = 1,2,\ldots,n-1$,

$$\bigl|x(t)\bigr| \le \sqrt{C(n)}\left(\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt\right)^{\frac{1}{2}} \tag{19}$$

and

$$\bigl|x^{(k)}(t)\bigr| \le \sqrt{C(n-k)}\left(\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt\right)^{\frac{1}{2}}. \tag{20}$$

Substituting (19) and (20) into (18), we obtain

$$\begin{aligned}\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt &\le \sqrt{C(n)} \int_a^b \bigl|p_n(t)\bigr|\bigl|x^{(n)}(t)\bigr|\,dt \left(\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1} \sqrt{C(n)C(n-k)} \int_a^b \bigl|p_k(t)\bigr|\,dt \int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt. \end{aligned} \tag{21}$$

Now by applying Hölder's inequality, we get

$$\int_a^b \bigl|p_n(t)x^{(n)}(t)\bigr|\,dt \le \left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} \left(\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt\right)^{\frac{1}{2}}. \tag{22}$$

Substituting (22) into (21) and by using the fact that $x(t)$ is not a constant function, we obtain the following strict inequality:

$$\begin{aligned}\int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt &< \sqrt{C(n)}\left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} \int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt \\ &\quad + \sum_{k=0}^{n-1} \sqrt{C(n)C(n-k)} \int_a^b \bigl|p_k(t)\bigr|\,dt \int_a^b \bigl(x^{(n)}(t)\bigr)^2\,dt. \end{aligned} \tag{23}$$

Dividing both sides of (23) by $\int_a^b (x^{(n)}(t))^2\,dt$, which can be proved to be positive by using the boundary value condition (13) and the assumption that $x(t) \not\equiv 0$, we obtain

$$1 < \sqrt{C(n)}\left(\int_a^b p_n^2(t)\,dt\right)^{\frac{1}{2}} + \sum_{k=0}^{n-1}\sqrt{C(n)C(n-k)}\int_a^b \left|p_k(t)\right|dt.$$

This is equivalent to (14). Thus we finished the proof of Proposition 3. □

Lemma 7 *For any $f \in C(A)$, we have*

$$\int_A \left|f(x)\right|dx \geq \frac{R_1^{N-1}2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{R_1}^{R_2}\left|f(r\omega)\right|dr. \tag{24}$$

Proof Similar to the proofs given in [1] and [2], we have

$$\int_{R_1}^{R_2}\left|f(r\omega)\right|dr = \int_{R_1}^{R_2} r^{1-N}r^{N-1}\left|f(r\omega)\right|dr \leq \left(\int_{R_1}^{R_2} r^{N-1}\left|f(r\omega)\right|dr\right)R_1^{1-N},$$

which implies that

$$\begin{aligned}\int_A \left|f(x)\right|dx &= \int_{S^{N-1}}\left(\int_{R_1}^{R_2} r^{N-1}\left|f(r\omega)\right|dr\right)d\omega \\ &\geq \int_{S^{N-1}}\left(R_1^{N-1}\int_{R_1}^{R_2}\left|f(r\omega)\right|dr\right)d\omega \\ &= \left(\int_{R_1}^{R_2}\left|f(r\omega)\right|dr\right)\frac{R_1^{N-1}2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}.\end{aligned}$$

□

Proof of Theorem 1 It follows from (14) and Lemma 7 that for any fixed $\omega \in S^{N-1}$, we have

$$\begin{aligned}1 &< \sqrt{\frac{(2^{2n}-1)\zeta(2n)(R_2-R_1)^{2n-1}}{2^{2n-1}\pi^{2n}}}\left(\int_{R_1}^{R_2} p_n^2(r\omega)\,dr\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1}\frac{(R_2-R_1)^{2n-k-1}\sqrt{(2^{2n}-1)(2^{2n-2k}-1)\zeta(2n)\zeta(2(n-k))}}{2^{2n-k-1}\pi^{2n-k}}\int_a^b \left|p_k(r\omega)\right|dr \\ &\leq \sqrt{\frac{(2^{2n}-1)\zeta(2n)(R_2-R_1)^{2n-1}\Gamma(\frac{N}{2})}{2^{2n}\pi^{2n+\frac{N}{2}}R_1^{N-1}}}\left(\int_A p_n^2(x)\,dx\right)^{\frac{1}{2}} \\ &\quad + \sum_{k=0}^{n-1}\frac{(R_2-R_1)^{2n-k-1}\Gamma(\frac{N}{2})}{(2\pi)^{2n-k}R_1^{N-1}\pi^{\frac{N}{2}}}\sqrt{\left(2^{2n}-1\right)\left(2^{2(n-k)}-1\right)\zeta(2n)\zeta\left(2(n-k)\right)}\int_A \left|p_k(x)\right|dx,\end{aligned}$$

which is (10). This finishes the proof of Theorem 1. □

The proof of Theorem 2 is similar to that of Theorem 1, so we omit it for simplicity.

Let us compare Theorem 1 and Theorem 2 with Theorem A and Theorem B. It is evident that Theorem 2 is a natural generalization of Theorem B. If we let $p_n(x) = p_{n-1}(x) = \cdots =$

Table 2 The first eight values of δ_n

n	1	2	3	4	5	6	7	8
δ_n	1	1.50	1.42	2.32	2.86	3.53	4.35	5.37

$p_1(x) \equiv 0$, $p_0(x) = q(x)$, $\forall x \in A$, then (10) reduces to the following inequality:

$$\int_A |q(x)|\,dx > \frac{2^{2n-1}\pi^{2n}}{(2^{2n}-1)\zeta(2n)(R_2-R_1)^{2n-1}}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}R_1^{N-1}. \tag{25}$$

Let us compare the right sides of inequalities (6) and (25): if we denote $\delta_n = \frac{2^{2n-1}\pi^{2n}}{(2^{2n}-1)\zeta(2n)2^{3n-1}}$, then we have

$$\delta_n = \frac{\pi^{2n}}{2^n(2^{2n}-1)\zeta(2n)} > \frac{\pi^{2n}}{2^{3n}\zeta(2n)} = \left(\frac{\pi^2}{8}\right)^n \frac{1}{\zeta(2n)} \to \infty, \quad \text{as } n \to \infty,$$

since $\zeta(2n) \to 1$ as $n \to \infty$. Table 2 gives the first eight values of δ_n.

From Table 2 we see that δ_n increases very quickly, so Theorem 1 improves Theorem A significantly even in the special case of (4).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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