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Explicit bounds derived by some new inequalities and applications in fractional integral equations

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Abstract
In this paper, we present some new Gronwall-type inequalities. Explicit bounds for the unknown functions concerned are derived based on these inequalities and the properties of the modified Riemann-Liouville fractional derivative. The inequalities established are of new forms compared with the existing results so far in the literature. For illustrating the validity of the inequalities established, we apply them to research the boundedness, quantitative property, and continuous dependence on the initial value for the solution to a certain fractional integral equation.
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1 Introduction

Recently, with the development of the theory of differential equations, many authors have researched various inequalities and investigated the boundedness, global existence, uniqueness, stability, and continuous dependence on the initial value and parameters of solutions to differential equations, integral equations as well as difference equations. The Gronwall-Bellman inequality [1, 2] is widely used in the qualitative and quantitative analysis of differential equations, as it can provide explicit bound for an unknown function lying in the inequality. In the last few decades, many authors have researched various generalizations of the Gronwall-Bellman inequality; for example, we refer the reader to [3–28] and the references therein. These Gronwall-type inequalities established can be used as a handy tool in the research of the theory of differential and integral equations as well as difference equations. However, we notice that the existing results in the literature are inadequate for researching the qualitative and quantitative properties of solutions to some fractional integral equations, for example, the following fractional integral equation:

$$u(t) = u(0) + I^{\alpha}\big(f\big(t, u(t)\big)\big) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\big(s, u(s)\big)\,ds,$$

where $0 < \alpha < 1$, $T \geq 0$ is a constant, I^{α} denotes the Riemann-Liouville fractional integral of order α.

So it is necessary to establish some new Gronwall-type inequalities in order to fulfill the desired analysis result.

The modified Riemann-Liouville fractional derivative, presented by Jumarie in [29, 30], is defined by the following expression.

Definition 1 The modified Riemann-Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))\,d\xi, & 0<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n\le\alpha<n+1, n\ge 1. \end{cases}$$

Definition 2 The Riemann-Liouville fractional integral of order α on the interval $[0,t]$ is defined by

$$I^\alpha f(t) = \frac{1}{\Gamma(1+\alpha)}\int_0^t f(s)(ds)^\alpha = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)\,ds.$$

Some important properties for the modified Riemann-Liouville derivative and fractional integral are listed as follows (see [31, 32] and the interval concerned below is always defined by $[0,t]$):

(a) $D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}$.
(b) $D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t)$.
(c) $D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha$.
(d) $I^\alpha(D_t^\alpha f(t)) = f(t) - f(0)$.
(e) $I^\alpha(g(t)D_t^\alpha f(t)) = f(t)g(t) - f(0)g(0) - I^\alpha(f(t)D_t^\alpha g(t))$.
(f) $D_t^\alpha C = 0$, where C is a constant.

The modified Riemann-Liouville derivative has many excellent characters in handling many fractional calculus problems. Many authors have investigated various applications of the modified Riemann-Liouville fractional derivative. For example, in [32, 33], the authors sought exact solutions for some types of fractional differential equations based on the modified Riemann-Liouville fractional derivative, and in [34], the modified Riemann-Liouville fractional derivative was used in fractional calculus of variations, where a fractional basic problem of the calculus of variations with free boundary conditions as well as problems with isoperimetric and holonomic constraints were considered. In [35], Khan *et al.* presented a fractional homotopy perturbation method (FHPM) for solving fractional differential equations of any fractional order based on the modified Riemann-Liouville fractional derivative. In [36–38], the fractional variational iteration method based on the modified Riemann-Liouville fractional derivative was concerned. In [39], a fractional variational homotopy perturbation iteration method was proposed.

Based on the analysis above, in Section 2, we present some new Gronwall-type inequalities, based on which and some basic properties of the modified Riemann-Liouville fractional derivative, we derive explicit bounds for the unknown functions concerned in these inequalities. In Section 3, we apply the results established in Section 2 to research boundedness, quantitative property, and continuous dependence on the initial data for the solution to a certain fractional integral equation.

2 Main results

Lemma 1 *Suppose* $0<\alpha<1$, f *is a continuous function, then* $D^\alpha(I_t^\alpha f(t)) = f(t)$.

Proof Since f is continuous, then there exists a constant M such that $|f(t)| \leq M$ for $t \in [0,\varepsilon]$, where $\varepsilon > 0$. So, for $t \in [0,\varepsilon]$, we have $|I_t^\alpha f(t)| = |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\,ds| \leq \frac{M}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds = \frac{M}{\alpha\Gamma(\alpha)} t^\alpha$. Then one can see $I_t^\alpha f(0) = 0$. Therefore,

$$\begin{aligned} D^\alpha\left(I_t^\alpha f(t)\right) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t (t-\xi)^{-\alpha}\left(I_t^\alpha f(\xi) - I_t^\alpha f(0)\right) d\xi\right\} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t (t-\xi)^{-\alpha}\int_0^\xi (\xi-s)^{\alpha-1} f(s)\,ds\,d\xi\right\} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t \int_0^\xi (t-\xi)^{-\alpha}(\xi-s)^{\alpha-1} f(s)\,ds\,d\xi\right\} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t f(s)\int_s^t (t-\xi)^{-\alpha}(\xi-s)^{\alpha-1}\,d\xi\,ds\right\}. \end{aligned}$$

Letting $\xi = s + (t-s)x$, we obtain that

$$\begin{aligned} D^\alpha\left(I_t^\alpha f(t)\right) &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t f(s)\int_0^1 (1-x)^{-\alpha}x^{\alpha-1}\,dx\,ds\right\} \\ &= \frac{B(\alpha,1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\left\{\int_0^t f(s)\,ds\right\} = f(t), \end{aligned}$$

where $B(\cdot,\cdot)$ denotes the beta function. The proof is complete. □

Theorem 2 *Suppose* $0 < \alpha < 1$, *the functions* u, g *are nonnegative continuous functions defined on* $t \geq 0$, $T \geq 0$ *is a constant. If the following inequality is satisfied*

$$\begin{aligned} &u(t) \leq C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)u(s)\,ds, \\ &\quad t \in [0,T], \end{aligned} \tag{1}$$

then we have the following explicit estimate for $u(t)$:

$$u(t) \leq \frac{C\exp[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})\,ds]}{2 - \exp[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} g((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})\,ds]}, \quad t \in [0,T], \tag{2}$$

provided that $\exp[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} g((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})\,ds] < 2$.

Proof Denote the right-hand side of (1) by $v(t)$. Then we have

$$u(t) \leq v(t), \quad t \in [0,T], \tag{3}$$

and, by use of Lemma 1 and the property (f), we obtain

$$D_t^\alpha v(t) = g(t)u(t) \leq g(t)v(t).$$

Furthermore, by the properties (a), (b), (c), we have:

$$\begin{aligned}
&D_t^\alpha\left\{v(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right]\right\} \\
&\quad= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] D_t^\alpha v(t) \\
&\qquad+ v(t) D_t^\alpha\left\{\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right]\right\} \\
&\quad= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] D_t^\alpha v(t) \\
&\qquad- g(t)v(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] D_t^\alpha\left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \\
&\quad= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right]\big[D_t^\alpha v(t) - g(t)v(t)\big] \le 0.
\end{aligned} \tag{4}$$

Substituting t with τ, fulfilling a fractional integral of order α for (3) with respect to τ from 0 to t, we deduce that

$$v(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] \le v(0),$$

which implies

$$v(t) \le \exp\left[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] v(0), \quad t \in [0, T]. \tag{5}$$

On the other hand, we have

$$2v(0) - C = v(T) \le \exp\left[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} g\big(\big(s\Gamma(1+\alpha)\big)^{\frac{1}{\alpha}}\big)\,ds\right] v(0),$$

which is followed by

$$v(0) \le \frac{C}{2 - \exp[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} g((s\Gamma(1+\alpha))^{\frac{1}{\alpha}})\,ds]}. \tag{6}$$

Combining (3), (5), (6), we can get the desired result. □

Now we study the inequality of the following form:

$$\begin{aligned}
u^p(t) \le{}& C + \int_0^t h(s)u^p(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)u^q(s)\,ds \\
&+ \int_0^T h(s)u^p(s)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} g(s)u^q(s)\,ds, \quad t \in [0, T],
\end{aligned} \tag{7}$$

where $0 < \alpha < 1$, the functions u, g, h are nonnegative continuous functions defined on $t \ge 0$, and $T \ge 0$ is a constant, p, q are constants with $p \ge q > 0$.

The following lemma is useful in deriving explicit bound for the function $u(t)$ in (7).

Lemma 3 [24] *Assume that* $a \geq 0$, $p \geq q \geq 0$, *and* $p \neq 0$, *then for any* $K > 0$,

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}.$$

Theorem 4 *The inequality admits the following explicit estimate for* $u(t)$:

$$\begin{aligned} u(t) \leq{} & \Biggl\{ \biggl\{ \frac{p-q}{p} K^{\frac{q}{p}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\biggl[\frac{q}{p} \int_0^s h(\xi)\,d\xi \biggr] ds \\ & + \frac{C + [\frac{p-q}{p} K^{\frac{q}{p}} \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) \exp[\frac{q}{p} \int_0^s h(\xi)\,d\xi]\,ds] \exp[\int_0^T h(s)\,ds]}{2 - \exp[\int_0^T h(s)\,ds]} \\ & + \exp\biggl[\int_0^T h(s)\,ds \biggr] \\ & \times \biggl(\biggl\{ \frac{1}{\Gamma(\alpha)} \exp\biggl[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\bigl(\bigl(s\Gamma(1+\alpha) \bigr)^{\frac{1}{\alpha}} \bigr)\,ds \biggr] \int_0^T (T-\tau)^{\alpha-1} a(\tau) \widetilde{g}(\tau) \\ & \times \exp\biggl\{ - \int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\bigl(\bigl(s\Gamma(1+\alpha) \bigr)^{\frac{1}{\alpha}} \bigr)\,ds \biggr\}\,d\tau \biggr\} \biggr) \Big/ \biggl(2 - \exp\biggl[\int_0^T h(s)\,ds \biggr] \biggr) \\ & + \frac{1}{\Gamma(\alpha)} \exp\biggl[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\bigl(\bigl(s\Gamma(1+\alpha) \bigr)^{\frac{1}{\alpha}} \bigr)\,ds \biggr] \int_0^t (t-\tau)^{\alpha-1} a(\tau) \widetilde{g}(\tau) \\ & \times \exp\biggl\{ - \int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\bigl(\bigl(s\Gamma(1+\alpha) \bigr)^{\frac{1}{\alpha}} \bigr)\,ds \biggr\}\,d\tau \biggr\} \\ & \times \exp\biggl[\int_0^t h(s)\,ds \biggr] \Biggr\}^{\frac{1}{p}}, \quad t \in [0,T], \end{aligned} \tag{8}$$

provided that $\exp[\int_0^T h(s)\,ds] < 2$, *where* $K > 0$, *and*

$$\widetilde{g}(t) = \frac{q}{p} K^{\frac{q-p}{p}} g(t) \exp\biggl[\frac{q}{p} \int_0^t h(\xi)\,d\xi \biggr].$$

Proof Denote the right-hand side of (7) by $v(t)$. Then we have

$$u(t) \leq v^{\frac{1}{p}}(t), \quad t \in [0,T], \tag{9}$$

and considering $v(0) = C + \int_0^T h(s) u^p(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} g(s) u^q(s)\,ds$, it follows that

$$v(t) \leq v(0) + \int_0^t h(s) v(s)\,ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) v^{\frac{q}{p}}(s)\,ds, \quad t \in [0,T]. \tag{10}$$

Let $z(t) = v(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) v^{\frac{q}{p}}(s)\,ds$. Then

$$v(t) \leq z(t) + \int_0^t h(s) v(s)\,ds, \quad t \in [0,T],$$

which implies that

$$v(t) \le z(t) \exp\left[\int_0^t h(s)\,ds\right], \quad t \in [0,T]. \tag{11}$$

So

$$z(t) \le v(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right] z^{\frac{q}{p}}(s)\,ds, \quad t \in [0,T].$$

Using Lemma 3, we get that

$$\begin{aligned} z(t) &\le v(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right]\left[\frac{q}{p}K^{\frac{q-p}{p}} z(s) + \frac{p-q}{p}K^{\frac{q}{p}}\right] ds \\ &= v(0) + \frac{p-q}{p}K^{\frac{q}{p}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right] ds \\ &\quad + \frac{q}{p}K^{\frac{q-p}{p}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right] z(s)\,ds \\ &= a(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{g}(s) z(s)\,ds, \quad t \in [0,T], \end{aligned}$$

where $\widetilde{g}(t)$ is defined as above, and

$$a(t) = v(0) + \frac{p-q}{p}K^{\frac{q}{p}} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s) \exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right] ds.$$

Let $w(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \widetilde{g}(s) z(s)\,ds$. Then

$$z(t) \le a(t) + w(t), \quad t \in [0,T], \tag{12}$$

and

$$D_t^\alpha w(t) = \widetilde{g}(t) z(t) \le a(t)\widetilde{g}(t) + \widetilde{g}(t) w(t).$$

By the properties (a), (b), and (c), we get that

$$\begin{aligned} &D_t^\alpha \left\{ w(t) \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]\right\} \\ &\quad = \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right] D_t^\alpha w(t) \\ &\qquad + w(t) D_t^\alpha \left\{ \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]\right\} \\ &\quad = \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right] D_t^\alpha w(t) \\ &\qquad - \widetilde{g}(t) w(t) \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right] D_t^\alpha \left(\frac{t^\alpha}{\Gamma(1+\alpha)}\right) \end{aligned}$$

$$= \exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]\left[D_t^\alpha w(t) - \widetilde{g}(t)w(t)\right]$$
$$\leq a(t)\widetilde{g}(t)\exp\left[-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right], \quad t \in [0,T]. \tag{13}$$

Substituting t with τ, fulfilling a fractional integral of order α for (13) with respect to τ from 0 to t, and using $w(0) = 0$, we deduce that

$$w(t)\exp\left\{-\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right\}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}a(\tau)\widetilde{g}(\tau)\exp\left[-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right] d\tau,$$

which implies

$$w(t) \leq \frac{1}{\Gamma(\alpha)}\exp\left[\int_0^{\frac{t^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]$$
$$\times \int_0^t (t-\tau)^{\alpha-1}a(\tau)\widetilde{g}(\tau)\exp\left\{-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right\} d\tau. \tag{14}$$

Combining (11), (12), and (14), we get that

$$2v(0) - C = v(T) \leq z(T)\exp\left[\int_0^T h(s)\,ds\right] \leq \left[a(T) + w(T)\right]\exp\left[\int_0^T h(s)\,ds\right]$$
$$\leq \left\{v(0) + \frac{p-q}{p}K^{\frac{q}{p}}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s)\exp\left[\frac{q}{p}\int_0^s h(\xi)\,d\xi\right] ds\right.$$
$$+ \frac{1}{\Gamma(\alpha)}\exp\left[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]$$
$$\left.\times \int_0^T (T-\tau)^{\alpha-1}a(\tau)\widetilde{g}(\tau)\exp\left\{-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right\} d\tau\right\}$$
$$\times \exp\left[\int_0^T h(s)\,ds\right],$$

which implies

$$v(0) \leq \frac{C + [\frac{p-q}{p}K^{\frac{q}{p}}\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}g(s)\exp[\frac{q}{p}\int_0^s h(\xi)\,d\xi]\,ds]\exp[\int_0^T h(s)\,ds]}{2 - \exp[\int_0^T h(s)\,ds]}$$
$$+ \exp\left[\int_0^T h(s)\,ds\right]$$
$$\times \left(\left\{\frac{1}{\Gamma(\alpha)}\exp\left[\int_0^{\frac{T^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right]\int_0^T (T-\tau)^{\alpha-1}a(\tau)\widetilde{g}(\tau)\right.\right.$$
$$\left.\left.\times \exp\left\{-\int_0^{\frac{\tau^\alpha}{\Gamma(1+\alpha)}} \widetilde{g}\left(\left(s\Gamma(1+\alpha)\right)^{\frac{1}{\alpha}}\right) ds\right\} d\tau\right\}\right) \Big/ \left(2 - \exp\left[\int_0^T h(s)\,ds\right]\right), \tag{15}$$

under the condition $\exp[\int_0^T h(s)\,ds] < 2$.

The desired result can be obtained by the combination of (11), (12), (14), and (15). □

Theorem 5 *Suppose* $0 < \alpha < 1$, *the function* u *is a nonnegative continuous function defined on* $t \geq 0$, p, T *are constants with* $p \geq 1$, $T \geq 0$, $L \in C(R_+^2, R_+)$ *satisfying* $0 \leq L(t,u) - L(t,v) \leq M(u-v)$ *for* $\forall u \geq v$, $t \geq 0$, *where* $M > 0$ *is a constant. If the following inequality is satisfied*

$$u^p(t) \leq C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\bigl(s, u(s)\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} L\bigl(s, u(s)\bigr)\,ds,$$
$$t \in [0,T], \tag{16}$$

then we have the following explicit estimate for $u(t)$:

$$u(t) \leq \frac{\exp[\frac{Mt^\alpha}{p\Gamma(1+\alpha)} K^{\frac{1-p}{p}}]}{2 - \exp[\frac{MT^\alpha}{p\Gamma(1+\alpha)} K^{\frac{1-p}{p}}]}\left[C + \frac{2T^\alpha}{\alpha\Gamma(\alpha)} L\left(s, \frac{p-1}{p} K^{\frac{1}{p}}\right)\right], \quad t \in [0,T], \tag{17}$$

provided that $\exp[\frac{MT^\alpha}{p\Gamma(1+\alpha)} K^{\frac{1-p}{p}}] < 2$.

Proof Denote the right-hand side of (16) by $v(t)$. Then we have

$$u(t) \leq v^{\frac{1}{p}}(t), \quad t \in [0,T], \tag{18}$$

and

$$\begin{aligned}
v(t) &\leq C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\bigl(s, v^{\frac{1}{p}}(s)\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} L\bigl(s, v^{\frac{1}{p}}(s)\bigr)\,ds \\
&\leq C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\left(s, \frac{1}{p}K^{\frac{1-p}{p}} v(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right) ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} L\left(s, \frac{1}{p}K^{\frac{1-p}{p}} v(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right) ds \\
&= C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left[L\left(s, \frac{1}{p}K^{\frac{1-p}{p}} v(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\right. \\
&\quad \left. - L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) + L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right)\right] ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left[L\left(s, \frac{1}{p}K^{\frac{1-p}{p}} v(s) + \frac{p-1}{p}K^{\frac{1}{p}}\right)\right. \\
&\quad \left. - L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) + L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right)\right] ds \\
&\leq C + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) ds + \frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^t (t-s)^{\alpha-1} v(s)\,ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) ds + \frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^T (T-s)^{\alpha-1} v(s)\,ds \\
&= C + \frac{t^\alpha}{\alpha\Gamma(\alpha)} L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) + \frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^t (t-s)^{\alpha-1} v(s)\,ds \\
&\quad + \frac{T^\alpha}{\alpha\Gamma(\alpha)} L\left(s, \frac{p-1}{p}K^{\frac{1}{p}}\right) + \frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^T (T-s)^{\alpha-1} v(s)\,ds
\end{aligned}$$

$$\begin{aligned}&\leq C+\frac{2T^{\alpha}}{\alpha\Gamma(\alpha)}L\left(s,\frac{p-1}{p}K^{\frac{1}{p}}\right)+\frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^t(t-s)^{\alpha-1}v(s)\,ds\\&\quad+\frac{M}{p\Gamma(\alpha)}K^{\frac{1-p}{p}}\int_0^T(T-s)^{\alpha-1}v(s)\,ds,\quad t\in[0,T].\end{aligned}\tag{19}$$

Then a suitable application of Theorem 2 to (19) yields the desired result. □

3 Applications

In this section, we present one example for the results established above, in which the boundedness, quantitative property, and continuous dependence on the initial value for the solutions to one certain fractional integral equation are researched.

Example Consider the following fractional integral equation:

$$u(t)=u(0)+I^{\alpha}\big(f\big(t,u(t)\big)\big)+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f\big(s,u(s)\big)\,ds,\quad t\in[0,T],\tag{20}$$

where $0<\alpha<1$, $f\in C(R\times R,R)$, $T\geq 0$ is a constant, I^{α} denotes the Riemann-Liouville fractional integral of order α on the interval $[0,t]$ as defined in Definition 2.

Theorem 6 *For Eq.* (20), *if* $|f(t,u)|\leq M|u|$, *where* $g\in C(R,R_+)$, *then under the condition* $\exp[\frac{MT^{\alpha}}{\Gamma(1+\alpha)}]<2$, *we have the following estimate*:

$$\big|u(t)\big|\leq\big|u(0)\big|\frac{\exp[\frac{Mt^{\alpha}}{\Gamma(1+\alpha)}]}{2-\exp[\frac{MT^{\alpha}}{\Gamma(1+\alpha)}]},\quad t\in[0,T].\tag{21}$$

Proof By Eq. (20) we in fact have

$$\begin{aligned}&u(t)=u(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f\big(s,u(s)\big)\,ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}f\big(s,u(s)\big)\,ds,\\&\quad t\in[0,T].\end{aligned}$$

So,

$$\begin{aligned}\big|u(t)\big|&\leq\big|u(0)\big|+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\big|f\big(s,u(s)\big)\big|\,ds+\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\big|f\big(s,u(s)\big)\big|\,ds\\&\leq\big|u(0)\big|+\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\big|u(s)\big|\,ds+\frac{M}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}\big|u(s)\big|\,ds,\\&\quad t\in[0,T].\end{aligned}\tag{22}$$

Then a suitable application of Theorem 2 to (22) yields the desired result. □

Remark 1 The result of Theorem 6 shows that the trivial solution to Eq. (20) is uniformly stable on the initial value.

Theorem 7 *If the function* f *satisfies the Lipschitz condition with* $|f(t,u)-f(t,v)|\leq A|u-v|$, *where* A *is the Lipschitz constant, then under the condition of the same initial value, Eq.* (20) *has at most one solution*.

Proof Suppose that Eq. (20) has two solutions $u_1(t)$, $u_2(t)$ with the same initial value $u(0)$. Then we have

$$u_1(t) = u(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s, u_1(s)\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\bigl(s, u_1(s)\bigr)\,ds, \tag{23}$$

$$u_2(t) = u(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s, u_2(s)\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\bigl(s, u_2(s)\bigr)\,ds. \tag{24}$$

Furthermore,

$$\begin{aligned} u_1(t) - u_2(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl[f\bigl(s, u_1(s)\bigr) - f\bigl(s, u_2(s)\bigr)\bigr]\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl[f\bigl(s, u_1(s)\bigr) - f\bigl(s, u_2(s)\bigr)\bigr]\,ds, \end{aligned} \tag{25}$$

which implies

$$\begin{aligned} \bigl|u_1(t) - u_2(t)\bigr| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|f\bigl(s, u_1(s)\bigr) - f\bigl(s, u_2(s)\bigr)\bigr|\,ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl|f\bigl(s, u_1(s)\bigr) - f\bigl(s, u_2(s)\bigr)\bigr|\,ds \\ &\le \frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\bigl|u_1(s) - u_2(s)\bigr|\,ds \\ &\quad + \frac{A}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\bigl|u_1(s) - u_2(s)\bigr|\,ds. \end{aligned} \tag{26}$$

After a suitable application of Theorem 2 to (26) (with $|u_1(t) - u_2(t)|$ being treated as one independent function), we obtain that $|u_1(t) - u_2(t)| \le 0$, which implies $u_1(t) \equiv u_2(t)$. So the proof is complete. □

Theorem 8 *Let $u(t)$ be the solution of Eq.* (20), *and let $\widetilde{u}(t)$ be the solution of the following fractional integral equation*:

$$\widetilde{u}(t) = \widetilde{u}(0) + I^\alpha\bigl(f\bigl(t, \widetilde{u}(t)\bigr)\bigr) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\bigl(s, \widetilde{u}(s)\bigr)\,ds, \quad t \in [0, T]. \tag{27}$$

If f satisfies the Lipschitz condition with A being the Lipschitz constant, then we have the following estimate:

$$\bigl|u(t) - \widetilde{u}(t)\bigr| \le \bigl|u(0) - \widetilde{u}(0)\bigr| \frac{\exp[\frac{Mt^\alpha}{\Gamma(1+\alpha)}]}{2 - \exp[\frac{MT^\alpha}{\Gamma(1+\alpha)}]}, \quad t \in [0, T]. \tag{28}$$

Proof By Eq. (27) we have

$$\widetilde{u}(t) = \widetilde{u}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\bigl(s, \widetilde{u}(s)\bigr)\,ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} f\bigl(s, \widetilde{u}(s)\bigr)\,ds. \tag{29}$$

So, we have

$$\begin{aligned} u(t)-\widetilde{u}(t) &= u(0)-\widetilde{u}(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big[f\big(s,u(s)\big)-f\big(s,\widetilde{u}(s)\big)\big]\,ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\big[f\big(s,u(s)\big)-f\big(s,\widetilde{u}(s)\big)\big]\,ds. \end{aligned} \tag{30}$$

Furthermore,

$$\begin{aligned} \big|u(t)-\widetilde{u}(t)\big| &\le \big|u(0)-\widetilde{u}(0)\big|+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|f\big(s,u(s)\big)-f\big(s,\widetilde{u}(s)\big)\big|\,ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\big|f\big(s,u(s)\big)-f\big(s,\widetilde{u}(s)\big)\big|\,ds \\ &\le \big|u(0)-\widetilde{u}(0)\big|+\frac{A}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\big|u(s)-\widetilde{u}(s)\big|\,ds \\ &\quad+\frac{A}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\big|u(s)-\widetilde{u}(s)\big|\,ds. \end{aligned} \tag{31}$$

Applying Theorem 2 to (31), after some basic computation, we can get the desired result. □

Remark 2 The result of Theorem 8 shows that the solution to Eq. (20) depends continuously on the initial value.

4 Conclusions

In this paper, we have derived new explicit bounds for the unknown functions concerned in some new Gronwall-type inequalities. In the proof for the main results, we have used the properties of the modified Riemann-Liouville fractional derivative. As for applications, we have presented one example, in which the boundedness, uniqueness, and continuous dependence on the initial value for the solution to a certain fractional integral equation are investigated. Finally, we note that these inequalities can be generalized to more general forms, as well as be generalized to 2D cases.

Competing interests
The author declares that they have no competing interests.

Author's contributions
BZ carried out the main part of this article. The author read and approved the final manuscript.

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