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Hermite-Hadamard type inequalities for n-times differentiable and preinvex functions

Shu-Hong Wang[1*] and Feng Qi[1,2,3]

*Correspondence: shuhong7682@163.com
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region 028043, China
Full list of author information is available at the end of the article

Abstract
In the paper, by creating an integral identity involving an n-times differentiable function, the authors establish some new Hermite-Hadamard type inequalities for preinvex functions and generalize some known results.
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1 Introduction

Throughout this paper, let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{N}$ denote the set of all positive integers.

Let us recall some definitions of various convex functions.

Definition 1 A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \le \lambda f(x) + (1-\lambda)f(y) \tag{1}$$

holds for all $x, y \in I$ and $\lambda \in [0,1]$. If the inequality (1) reverses, then f is said to be concave on I.

Definition 2 [1] A set $S \subseteq \mathbb{R}^n$ is said to be invex with respect to the map $\eta : S \times S \to \mathbb{R}^n$, if $y + t\eta(x,y) \in S$ for every $x, y \in S$ and $t \in [0,1]$.

It is obvious that every convex set is invex with respect to the map $\eta(x,y) = x - y$, but there exist invex sets which are not convex. See [1], for example.

Definition 3 [1] Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. For every $x, y \in S$, the η-path P_{xv} joining the points x and $v = x + \eta(y,x)$ is defined by

$$P_{xv} = \bigl\{z | z = x + t\eta(y,x), t \in [0,1]\bigr\}. \tag{2}$$

Definition 4 [1] Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \to \mathbb{R}^n$. A function $f : S \to \mathbb{R}$ is said to be preinvex with respect to η, if $f(y + t\eta(x,y)) \le tf(x) + (1-t)f(y)$ for every $x, y \in S$ and $t \in [0,1]$.

Every convex function is preinvex with respect to the map $\eta(x,y) = x - y$, but not conversely. For properties and applications of preinvex functions, please refer to [1–3] and closely related references therein.

The most important inequality in the theory of convex functions, the well-known Hermite-Hadamard's integral inequality, may be stated as follows. If f is a convex function on $[a,b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{3}$$

If f is concave on $[a,b]$, then the inequality (3) is reversed.

The inequality (3) has been generalized by many mathematicians. Some of them may be recited as follows.

Theorem 1 [4, Theorem 2.2] *Let $f: I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a,b \in I^\circ$ with $a<b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)[|f'(a)|+|f'(b)|]}{8}. \tag{4}$$

Theorem 2 [5, Theorem 1] *If f is differentiable on $[a,b]$ such that $|f'(x)|^q$ is a convex function on $[a,b]$ for $q \ge 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}. \tag{5}$$

Theorem 3 [6, Theorem 2.3] *Let $f: I \to \mathbb{R}$ be differentiable on I°, $a,b \in I^\circ$ with $a<b$, and $p>1$. If $|f'(x)|^{p/(p-1)}$ is convex on $[a,b]$, then*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\quad \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\big[\big|f'(a)\big|^{p/(p-1)} + 3\big|f'(b)\big|^{p/(p-1)}\big]^{1-1/p} \\
&\qquad + \big[3\big|f'(a)\big|^{p/(p-1)} + \big|f'(b)\big|^{p/(p-1)}\big]^{1-1/p}\Big\}.
\end{aligned} \tag{6}$$

Theorem 4 [2, Theorem 2.1] *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta: A \times A \to \mathbb{R}$ and $f: A \to \mathbb{R}$ be a differentiable function. If $|f'(x)|$ is preinvex on A, then for every $a,b \in A$ with $\eta(a,b) \ne 0$*

$$\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \le \frac{|\eta(a,b)|}{8}\big[\big|f'(a)\big| + \big|f'(b)\big|\big]. \tag{7}$$

Theorem 5 [2, Theorem 4.1] *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta: A \times A \to \mathbb{R}$ and $\eta(a,b) \ne 0$ for all $a \ne b$. Suppose that $f: A \to \mathbb{R}$ is a twice differentiable function on A. If $|f''(x)|$ is preinvex on A and f'' is integrable on the η-path P_{bc} for $c = b+\eta(a,b)$, then*

$$\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \le \frac{[\eta(a,b)]^2}{24}\big[\big|f''(a)\big| + \big|f''(b)\big|\big]. \tag{8}$$

Theorem 6 [2, Theorem 4.3] *Let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta: A \times A \to \mathbb{R}$ and $\eta(a,b) \ne 0$ for all $a \ne b$. Suppose that $f: A \to \mathbb{R}$ is a twice differentiable function on A*

and $|f''(x)|$ is preinvex on A. If $q > 1$ and f'' is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$, then

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x\right| \\
&\quad\le \frac{[\eta(a,b)]^2}{12}\left(\frac{1}{2}\right)^{1/q}\left[\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{1/q}.
\end{aligned} \tag{9}$$

Recently, some related inequalities for preinvex functions were also obtained in [7, 8]. Some integral inequalities of Hermite-Hadamard type for other kinds of convex functions were also established in [9–16] and references cited therein.

In this paper, by creating an integral identity involving an n-times differentiable function, the authors will establish some new Hermite-Hadamard type inequalities for preinvex functions and generalize some of the above mentioned results.

2 A lemma

In order to obtain our main results, we need the following lemma.

Lemma 1 *For $n \in \mathbb{N}$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and let $a, b \in A$ with $\eta(a,b) \neq 0$ for all $a \neq b$. If $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$, then*

$$\begin{aligned}
&\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\qquad+\sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\left[f^{(k)}(b)+(-1)^k f^{(k)}\big(b+\eta(a,b)\big)\right] \\
&\quad=\frac{[\eta(a,b)]^n}{4(n!)}\int_0^1\left[(1-t)^{n-1}(2t+n-2)+(-t)^{n-1}(2t-n)\right]f^{(n)}\big(b+t\eta(a,b)\big)\,\mathrm{d}t,
\end{aligned} \tag{10}$$

where the above summation is zero for $n = 1$.

Proof Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. When $n = 1$, integrating by parts in the right-hand side of (1) gives

$$\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x=\frac{\eta(a,b)}{2}\int_0^1(2t-1)f'\big(b+t\eta(a,b)\big)\,\mathrm{d}t.$$

Hence, the identity (1) holds for $n = 1$.

When $n = m - 1$ and $m \ge 2$, suppose that the identity (1) is valid.

When $n = m$, by the hypothesis, we have

$$\begin{aligned}
&\frac{[\eta(a,b)]^m}{4(m!)}\int_0^1\left[(1-t)^{m-1}(2t+m-2)+(-t)^{m-1}(2t-m)\right]f^{(m)}\big(b+t\eta(a,b)\big)\,\mathrm{d}t \\
&\quad=\frac{[\eta(a,b)]^{m-1}}{4(m!)}\Bigg\{(-1)^{m-1}(2-m)f^{(m-1)}\big(b+\eta(a,b)\big)-(m-2)f^{(m-1)}(b) \\
&\qquad-m\int_0^1\left[(1-t)^{m-2}(3-2t-m)+(-t)^{m-2}(m-1-2t)\right]f^{(m-1)}\big(b+t\eta(a,b)\big)\,\mathrm{d}t\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[\eta(a,b)]^{m-1}(2-m)}{4(m!)}\bigl[f^{(m-1)}(b)+(-1)^{m-1}f^{(m-1)}\bigl(b+\eta(a,b)\bigr)\bigr] \\
&\quad + \frac{[\eta(a,b)]^{m-1}}{4[(m-1)!]}\int_0^1 \bigl[(1-t)^{m-2}(2t+m-3) \\
&\quad + (-t)^{m-2}(2t-m+1)\bigr]f^{(m-1)}\bigl(b+t\eta(a,b)\bigr)\,\mathrm{d}t \\
&= \frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{m-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)]!}\bigl[f^{(k)}(b)+(-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr].
\end{aligned}
$$

Therefore, when $n = m$, the identity (1) holds. By induction, the proof of Lemma 1 is complete. □

Remark 1 When $n = 1$ and $n = 2$ in (1), respectively, we obtain the identities

$$
\begin{aligned}
&\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad = \frac{\eta(a,b)}{2}\int_0^1 (2t-1)f'\bigl(b+t\eta(a,b)\bigr)\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad = \frac{[\eta(a,b)]^2}{2}\int_0^1 t(1-t)f''\bigl(b+t\eta(a,b)\bigr)\,\mathrm{d}t,
\end{aligned}
$$

which may be found in [2].

3 Hermite-Hadamard type inequalities for preinvex functions

Now we start out to establish some new Hermite-Hadamard type inequalities for n-times differentiable and preinvex functions.

Theorem 7 *For $n \in \mathbb{N}$ and $n \ge 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) \ne 0$ for all $a \ne b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$. If $|f^{(n)}|^q$ is preinvex on A for $q \ge 1$, then*

$$
\begin{aligned}
&\biggl|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\qquad + \sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\bigl[f^{(k)}(b)+(-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr]\biggr| \\
&\quad \le \frac{|\eta(a,b)|^n(n-1)^{1-1/q}}{4[(n+1)!](n+2)^{1/q}}\Bigl\{\bigl[n\bigl|f^{(n)}(a)\bigr|^q + \bigl(n^2-2\bigr)\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q} \\
&\qquad + \bigl[\bigl(n^2-2\bigr)\bigl|f^{(n)}(a)\bigr|^q + n\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q}\Bigr\}. \qquad (11)
\end{aligned}
$$

Proof Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. Using Lemma 1 and Hölder's inequality yields

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \right.\\
&\quad\left. + \sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\left[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\right]\right|\\
&\quad\le \frac{|\eta(a,b)|^n}{4(n!)}\left[\int_0^1 (1-t)^{n-1}(2t+n-2)\left|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\right|\mathrm{d}t\right.\\
&\quad\left. + \int_0^1 t^{n-1}(n-2t)\left|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\right|\mathrm{d}t\right]\\
&\quad\le \frac{|\eta(a,b)|^n}{4(n!)}\left\{\left[\int_0^1 (1-t)^{n-1}(2t+n-2)\,\mathrm{d}t\right]^{1-1/q}\right.\\
&\quad\times\left[\int_0^1 (1-t)^{n-1}(2t+n-2)\bigl(t\left|f^{(n)}(a)\right|^q + (1-t)\left|f^{(n)}(b)\right|^q\bigr)\,\mathrm{d}t\right]^{1/q}\\
&\quad+\left[\int_0^1 t^{n-1}(n-2t)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 t^{n-1}(n-2t)\bigl(t\left|f^{(n)}(a)\right|^q\right.\\
&\quad\left.\left. + (1-t)\left|f^{(n)}(b)\right|^q\bigr)\,\mathrm{d}t\right]^{1/q}\right\}\\
&\quad= \frac{|\eta(a,b)|^n(n-1)^{1-1/q}}{4[(n+1)!](n+2)^{1/q}}\left\{\left[n\left|f^{(n)}(a)\right|^q + \bigl(n^2-2\bigr)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.\\
&\quad\left. + \left[\bigl(n^2-2\bigr)\left|f^{(n)}(a)\right|^q + n\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

Theorem 7 is thus proved. □

Corollary 1 *Under the assumptions of Theorem* 7,

1. *if* $q = 1$, *then*

$$\begin{aligned}
&\left|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \right.\\
&\quad\left. + \sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\left[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\right]\right|\\
&\quad\le \frac{(n-1)|\eta(a,b)|^n}{4[(n+1)!]}\left[\left|f^{(n)}(a)\right| + \left|f^{(n)}(b)\right|\right];
\end{aligned}$$

2. *if* $q = 1$ *and* $n = 2$, *then the inequality* (8) *is valid.*

Theorem 8 *For* $n \in \mathbb{N}$ *and* $n \ge 2$, *let* $A \subseteq \mathbb{R}$ *be an open invex set with respect to* $\eta : A \times A \to \mathbb{R}$ *and* $a, b \in A$ *with* $\eta(a,b) \ne 0$ *for all* $a \ne b$. *Suppose that* $f : A \to \mathbb{R}$ *is an n-times differentiable function on* A *and* $f^{(n)}$ *is integrable on the* η*-path* P_{bc} *for* $c = b + \eta(a,b)$. *If*

$|f^{(n)}|^q$ *is preinvex on A for $q > 1$, then*

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x \\
&\quad+\frac{1}{4}\sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{(k+1)!}\big[f^{(k)}(b)+(-1)^kf^{(k)}\big(b+\eta(a,b)\big)\big]\Bigg| \\
&\quad\le\frac{|\eta(a,b)|^n}{16(n!)[(q+1)(q+2)]^{1/q}}\left[\frac{4(q-1)}{nq-1}\right]^{1-1/q} \\
&\quad\times\Big\{\Big[\big((n-2)^{q+2}-(n-2q-4)n^{q+1}\big)\big|f^{(n)}(a)\big|^q \\
&\quad+\big(n^{q+2}-(n+2q+2)(n-2)^{q+1}\big)\big|f^{(n)}(b)\big|^q\Big]^{1/q} \\
&\quad+\Big[\big(n^{q+2}-(n+2q+2)(n-2)^{q+1}\big)\big|f^{(n)}(a)\big|^q \\
&\quad+\big((n-2)^{q+2}-(n-2q-4)n^{q+1}\big)\big|f^{(n)}(b)\big|^q\Big]^{1/q}\Big\}. \qquad (12)
\end{aligned}$$

Proof For every $t\in[0,1]$, we have $b+t\eta(a,b)\in A$. By Lemma 1 and Hölder's inequality, it follows that

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2}-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\,\mathrm{d}x \\
&\quad+\sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\big[f^{(k)}(b)+(-1)^kf^{(k)}\big(b+\eta(a,b)\big)\big]\Bigg| \\
&\quad\le\frac{|\eta(a,b)|^n}{4(n!)}\bigg[\int_0^1(1-t)^{n-1}(2t+n-2)\big|f^{(n)}\big(b+t\eta(a,b)\big)\big|\,\mathrm{d}t \\
&\quad+\int_0^1t^{n-1}(n-2t)\big|f^{(n)}\big(b+t\eta(a,b)\big)\big|\,\mathrm{d}t\bigg] \\
&\quad\le\frac{|\eta(a,b)|^n}{4(n!)}\Bigg\{\bigg[\int_0^1(1-t)^{q(n-1)/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q} \\
&\quad\times\bigg[\int_0^1(2t+n-2)^q\big(t\big|f^{(n)}(a)\big|^q+(1-t)\big|f^{(n)}(b)\big|^q\big)\,\mathrm{d}t\bigg]^{1/q} \\
&\quad+\bigg[\int_0^1t^{q(n-1)/(q-1)}\,\mathrm{d}t\bigg]^{1-1/q}\bigg[\int_0^1(n-2t)^q\big(t\big|f^{(n)}(a)\big|^q+(1-t)\big|f^{(n)}(b)\big|^q\big)\,\mathrm{d}t\bigg]^{1/q}\Bigg\} \\
&\quad=\frac{|\eta(a,b)|^n}{16(n!)[(q+1)(q+2)]^{1/q}}\left[\frac{4(q-1)}{nq-1}\right]^{1-1/q} \\
&\quad\times\Big\{\Big[\big((n-2)^{q+2}-(n-2q-4)n^{q+1}\big)\big|f^{(n)}(a)\big|^q \\
&\quad+\big(n^{q+2}-(n+2q+2)(n-2)^{q+1}\big)\big|f^{(n)}(b)\big|^q\Big]^{1/q} \\
&\quad+\Big[\big(n^{q+2}-(n+2q+2)(n-2)^{q+1}\big)\big|f^{(n)}(a)\big|^q \\
&\quad+\big((n-2)^{q+2}-(n-2q-4)n^{q+1}\big)\big|f^{(n)}(b)\big|^q\Big]^{1/q}\Big\}.
\end{aligned}$$

Theorem 8 is thus proved. □

Theorem 9 *For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) \neq 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$. If $|f^{(n)}|^q$ is preinvex on A for $q > 1$, then*

$$\begin{aligned}
&\left| \frac{f(b) + f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)} \int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \right. \\
&\qquad \left. + \sum_{k=1}^{n-1} \frac{[\eta(a,b)]^k (1-k)}{4[(k+1)!]} \bigl[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr] \right| \\
&\quad \leq \frac{|\eta(a,b)|^n}{4(n!)[(nq-q+1)(nq-q+2)]^{1/q}} \\
&\qquad \times \left\{ \frac{(q-1)[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}]}{2(2q-1)} \right\}^{1-1/q} \\
&\qquad \times \bigl\{ \bigl[\bigl|f^{(n)}(a)\bigr|^q + (nq-q+1) \bigl|f^{(n)}(b)\bigr|^q \bigr]^{1/q} \\
&\qquad + \bigl[(nq-q+1) \bigl|f^{(n)}(a)\bigr|^q + \bigl|f^{(n)}(b)\bigr|^q \bigr]^{1/q} \bigr\}.
\end{aligned} \tag{13}$$

Proof Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. Utilizing Lemma 1 and Hölder's inequality results in

$$\begin{aligned}
&\left| \frac{f(b) + f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)} \int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \right. \\
&\qquad \left. + \sum_{k=1}^{n-1} \frac{[\eta(a,b)]^k (1-k)}{4[(k+1)!]} \bigl[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr] \right| \\
&\quad \leq \frac{|\eta(a,b)|^n}{4(n!)} \left[\int_0^1 (1-t)^{n-1} (2t+n-2) \bigl|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\bigr|\,\mathrm{d}t \right. \\
&\qquad \left. + \int_0^1 t^{n-1}(n-2t) \bigl|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\bigr|\,\mathrm{d}t \right] \\
&\quad \leq \frac{|\eta(a,b)|^n}{4(n!)} \left\{ \left[\int_0^1 (2t+n-2)^{q/(q-1)}\,\mathrm{d}t \right]^{1-1/q} \right. \\
&\qquad \times \left[\int_0^1 (1-t)^{q(n-1)} \bigl(t\bigl|f^{(n)}(a)\bigr|^q + (1-t)\bigl|f^{(n)}(b)\bigr|^q \bigr)\,\mathrm{d}t \right]^{1/q} \\
&\qquad \left. + \left[\int_0^1 (n-2t)^{q/(q-1)}\,\mathrm{d}t \right]^{1-1/q} \left[\int_0^1 t^{q(n-1)} \bigl(t\bigl|f^{(n)}(a)\bigr|^q + (1-t)\bigl|f^{(n)}(b)\bigr|^q \bigr)\,\mathrm{d}t \right]^{1/q} \right\} \\
&\quad = \frac{|\eta(a,b)|^n}{4(n!)[(nq-q+1)(nq-q+2)]^{1/q}} \\
&\qquad \times \left\{ \frac{(q-1)[n^{(2q-1)/(q-1)} - (n-2)^{(2q-1)/(q-1)}]}{2(2q-1)} \right\}^{1-1/q} \\
&\qquad \times \bigl\{ \bigl[\bigl|f^{(n)}(a)\bigr|^q + (nq-q+1) \bigl|f^{(n)}(b)\bigr|^q \bigr]^{1/q} \\
&\qquad + \bigl[(nq-q+1) \bigl|f^{(n)}(a)\bigr|^q + \bigl|f^{(n)}(b)\bigr|^q \bigr]^{1/q} \bigr\}.
\end{aligned}$$

The proof of Theorem 9 is complete. □

Theorem 10 *For $n \in \mathbb{N}$ and $n \geq 2$, let $A \subseteq \mathbb{R}$ be an open invex set with respect to $\eta : A \times A \to \mathbb{R}$ and $a, b \in A$ with $\eta(a,b) \neq 0$ for all $a \neq b$. Suppose that $f : A \to \mathbb{R}$ is an n-times differentiable function on A and $f^{(n)}$ is integrable on the η-path P_{bc} for $c = b + \eta(a,b)$. If $|f^{(n)}|^q$ is preinvex on A for $q > 1$, then*

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad + \frac{1}{4}\sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{(k+1)!}\bigl[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr]\Bigg| \\
&\quad \leq \frac{|\eta(a,b)|^n}{24(n!)}\left[\frac{6(q-1)(nq-2)(n-1)}{(nq-1)(nq+q-2)}\right]^{1-1/q} \\
&\quad \times \Bigl\{\bigl[(3n-2)\bigl|f^{(n)}(a)\bigr|^q + (3n-4)\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q} \\
&\quad + \bigl[(3n-4)\bigl|f^{(n)}(a)\bigr|^q + (3n-2)\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q}\Bigr\}.
\end{aligned} \tag{14}$$

Proof Since $a, b \in A$ and A is an invex set with respect to η, for every $t \in [0,1]$, we have $b + t\eta(a,b) \in A$. Employing Lemma 1 and Hölder's inequality leads to

$$\begin{aligned}
&\Bigg|\frac{f(b)+f(b+\eta(a,b))}{2} - \frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)} f(x)\,\mathrm{d}x \\
&\quad + \sum_{k=1}^{n-1}\frac{[\eta(a,b)]^k(1-k)}{4[(k+1)!]}\bigl[f^{(k)}(b) + (-1)^k f^{(k)}\bigl(b+\eta(a,b)\bigr)\bigr]\Bigg| \\
&\quad \leq \frac{|\eta(a,b)|^n}{4(n!)}\left[\int_0^1 (1-t)^{n-1}(2t+n-2)\bigl|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\bigr|\,\mathrm{d}t\right. \\
&\quad \left. + \int_0^1 t^{n-1}(n-2t)\bigl|f^{(n)}\bigl(b+t\eta(a,b)\bigr)\bigr|\,\mathrm{d}t\right] \\
&\quad \leq \frac{|\eta(a,b)|^n}{4(n!)}\left\{\left[\int_0^1 (1-t)^{q(n-1)/(q-1)}(2t+n-2)\,\mathrm{d}t\right]^{1-1/q}\right. \\
&\quad \times \left[\int_0^1 (2t+n-2)\bigl(t\bigl|f^{(n)}(a)\bigr|^q + (1-t)\bigl|f^{(n)}(b)\bigr|^q\bigr)\,\mathrm{d}t\right]^{1/q} \\
&\quad + \left[\int_0^1 t^{q(n-1)/(q-1)}(n-2t)\,\mathrm{d}t\right]^{1-1/q}\left[\int_0^1 (n-2t)\bigl(t\bigl|f^{(n)}(a)\bigr|^q\right. \\
&\quad \left.\left. + (1-t)\bigl|f^{(n)}(b)\bigr|^q\bigr)\,\mathrm{d}t\right]^{1/q}\right\} \\
&\quad = \frac{|\eta(a,b)|^n}{24(n!)}\left[\frac{6(q-1)(nq-2)(n-1)}{(nq-1)(nq+q-2)}\right]^{1-1/q}\Bigl\{\bigl[(3n-2)\bigl|f^{(n)}(a)\bigr|^q \\
&\quad + (3n-4)\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q} + \bigl[(3n-4)\bigl|f^{(n)}(a)\bigr|^q + (3n-2)\bigl|f^{(n)}(b)\bigr|^q\bigr]^{1/q}\Bigr\}.
\end{aligned}$$

The proof of Theorem 10 is complete. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

Author details
[1]College of Mathematics, Inner Mongolia University for Nationalities, Tongliao City, Inner Mongolia Autonomous Region 028043, China. [2]Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin City, 300387, China. [3]Institute of Mathematics, Henan Polytechnic University, Jiaozuo City, 454010, China.

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