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Some results for demimartingales and N-demimartingales

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Abstract
In this paper, we obtain some results such as maximal and minimal type inequalities for demisubmartingales and demimartingales. Meanwhile, by giving an example, we point out that the Chow type maximal inequality of N-demimartingales is not true, which affects some maximal type inequalities for N-demimartingales.

MSC: 60E15; 60F15

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1 Introduction

Let $S_1, S_2, \ldots, S_n, \ldots$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $S_0 = 0$.

Definition 1.1 Let $\{S_j, j \geq 1\}$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \ldots,$

$$E\{(S_{j+1} - S_j)f(S_1, \ldots, S_j)\} \geq 0 \tag{1.1}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called a demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_j, j \geq 1\}$ is called a demisubmartingale.

Definition 1.2 Let $\{S_j, j \geq 1\}$ be an L^1 sequence of random variables. Assume that for $j = 1, 2, \ldots,$

$$E\{(S_{j+1} - S_j)f(S_1, \ldots, S_j)\} \leq 0 \tag{1.2}$$

for all coordinatewise nondecreasing functions f such that the expectation is defined. Then $\{S_j, j \geq 1\}$ is called an N-demimartingale. If in addition the function f is assumed to be nonnegative, the sequence $\{S_j, j \geq 1\}$ is called an N-demisupermartingale.

The concepts of demimartingales and demisubmartingales were due to Newman and Wright [1]. It can be checked that a submartingale with the natural choice of σ-algebras is a demisubmartingale, but the converse statement cannot always be true. Newman and Wright [1] proved that the partial sums of mean zero associated random variables form a demimartingale. Similarly, the notion of N-demimartingales and N-demisupermartingales can be found in Christofides [2]. It is trivial to verify that the partial sums of mean zero negatively associated random variables form an N-demimartingale, and a supermartingale with the natural choice of σ-algebras is an N-demisupermartingale, but the converse statement cannot always be true (see Christofides [2]). Various results and examples of demisubmartingales and demimartingales have been obtained. For example, Newman and Wright [1] obtained Doob type maximal inequalities and upcrossing inequality for demisubmartingales; Wood [3] investigated more properties of demimartingales; Christofides [4] generalized the Chow type maximal inequalities for demisubmartingales; Prakasa Rao [5] investigated the Whittle type maximal inequality for demisubmartingales; Christofides [6] constructed some U-statistics based on associated random variables and proved them to be demimartingales; Wang [7] studied some maximal inequalities for associated random variables and demimartingales; Prakasa Rao [8] obtained more maximal and minimal type inequalities for demisubmartingales; Wang and Hu [9] also studied some maximal inequalities for demimartingales and their applications; Wang *et al.* [10] gave a Doob type inequality and a strong law of large numbers for demimartingales; Wang *et al.* [11] also studied the maximal and minimal type inequalities for demimartingales; Christofides and Hadjikyriakou [12] gave some maximal and moment inequalities for demimartingales; Hu *et al.* [13] investigated the Marshall type inequalities for demimartingales; Wang *et al.* [14] got some maximal inequalities for demimartingales based on concave Young functions. Meanwhile, for the results of N-demisupermartingales and N-demimartingales, Christofides [2] gave some maximal type inequalities for N-demimartingales; Prakasa Rao [15] studied the Chow type maximal inequality for N-demimartingales, Christofides and Hadjikyriakou [16] got some exponential inequalities for N-demimartingales; Hu *et al.* [17] gave a note on the inequalities for N-demimartingales; Hadjikyriakou [18] obtained a Marcinkiewicz-Zygmund type inequality for nonnegative N-demimartingales; Wang *et al.* [19] studied some maximal type inequalities for N-demimartingales and provided a strong law of large numbers as an application; Yang and Hu [20] investigated more maximal type inequalities for N-demimartingales, *etc.* For more results and examples of demimartingales and N-demimartingales, one can refer to Prakasa Rao [21] and Hadjikyriakou [22]. On the other hand, the conditional demimartingales and N-demimartingales have received more attention; we refer to Christofides and Hadjikyriakou [23], Wang and Wang [24], Prakasa Rao [21] and Hadjikyriakou [22], *etc.*

Inspired by the papers above, we investigate some maximal and minimal type inequalities for demisubmartingales and demimartingales. Meanwhile, by giving an example, we point out that the Chow type maximal inequality of N-demimartingales is not true, which affects some maximal type inequalities for N-demimartingales.

Throughout this paper, let $I(A)$ denote the indicator function of the set A and $x^+ = I(x \geq 0)$.

Lemma 1.1 (Christofides [4, Lemma 2.1]) *Let* $\{S_n, n \geq 1\}$ *be a demisubmartingale* (*or a demimartingale*) *and* g *be a nondecreasing convex function such that* $g(S_i) \in L^1$, $i \geq 1$. *Then* $\{g(S_n), n \geq 1\}$ *is a demisubmartingale.*

2 Main results

First, we provide a maximal type inequality for a sequence of demisubmartingales.

Theorem 2.1 *Let $\{S_n, n \geq 1\}$ be a demisubmartingale with $S_0 = 0$ and assume that $\{c_n, n \geq 1\}$ is a nondecreasing sequence of positive numbers. Then, for any $\varepsilon > 0$,*

$$\varepsilon P\Big\{\max_{1\leq k\leq n} c_k S_k \geq \varepsilon\Big\} \leq c_n E\Big[S_n^+ I\Big(\max_{1\leq k\leq n} c_k S_k \geq \varepsilon\Big)\Big]. \tag{2.1}$$

Proof Following Christofides [4], we give the proof of Theorem 2.1. For fixed $n \geq 1$, let $A = \{\max_{1\leq k\leq n} c_k S_k \geq \varepsilon\}$. Then A can be written as $A = \bigcup_{j=1}^n A_j$, where $A_1 = \{c_1 S_1 \geq \varepsilon\}$, $A_j = \{c_i S_i < \varepsilon, 1 \leq i < j, c_j S_j \geq \varepsilon\}$, $1 < j \leq n$, and $A_i \cap A_j = \emptyset$ when $i \neq j$. Therefore, one has

$$\begin{aligned}
\varepsilon P(A) &= \varepsilon \sum_{j=1}^n P(A_j) = \sum_{j=1}^n E(\varepsilon I_{A_j}) \leq \sum_{j=1}^n E(c_j S_j I_{A_j}) = \sum_{j=1}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&= E\big(c_1 S_1^+ I_{A_1}\big) + E\big(c_2 S_2^+ I_{A_2}\big) + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&= E\big(c_1 S_1^+ I_{A_1}\big) + E\big[c_2 S_2^+ (I_{A_1\cup A_2} - I_{A_1})\big] + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&= E\big(c_2 S_2^+ I_{A_1\cup A_2}\big) + E\big[\big(c_1 S_1^+ - c_2 S_2^+\big) I_{A_1}\big] + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&\leq E\big(c_2 S_2^+ I_{A_1\cup A_2}\big) + c_2 E\big[\big(S_1^+ - S_2^+\big) I_{A_1}\big] + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&= E\big(c_2 S_2^+ I_{A_1\cup A_2}\big) - c_2 E\big[\big(S_2^+ - S_1^+\big) I_{A_1}\big] + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big),
\end{aligned} \tag{2.2}$$

which is from the facts that $A_1 \cap A_2 = \emptyset$, $I_{A_2} = I_{A_1\cup A_2} - I_{A_1}$ and $\{c_k, k \geq 1\}$ is a nondecreasing sequence of positive numbers.

Let $h(y) = \lim_{x\to y^-}(x^+ - y^+)/(x - y)$ and $f(x) = x^+ = \max\{0, x\}$. Then f and h are nonnegative nondecreasing functions. By the convexity of the function $f(x) = x^+$, we have

$$S_2^+ - S_1^+ \geq (S_2 - S_1) h(S_1),$$

and then we can get

$$E\big[\big(S_2^+ - S_1^+\big) I_{A_1}\big] \geq E\big[(S_2 - S_1) h(S_1) I_{A_1}\big].$$

Since $h(S_1) I_{A_1}$ is a nonnegative nondecreasing function of S_1 and $\{S_n, n \geq 1\}$ is a demisubmartingale, we have

$$E\big[\big(S_2^+ - S_1^+\big) I_{A_1}\big] \geq E\big[(S_2 - S_1) h(S_1) I_{A_1}\big] \geq 0.$$

So we can get

$$\begin{aligned}
\varepsilon P(A) &\leq E\big(c_2 S_2^+ I_{A_1\cup A_2}\big) + \sum_{j=3}^n E\big(c_j S_j^+ I_{A_j}\big) \\
&= E\big(c_2 S_2^+ I_{A_1\cup A_2}\big) + E\big(c_3 S_3^+ I_{A_3}\big) + \sum_{j=4}^n E\big(c_j S_j^+ I_{A_j}\big).
\end{aligned} \tag{2.3}$$

Since $A_1 \cap A_2 \cap A_3 = \emptyset$, one has $I_{A_3} = I_{A_1 \cup A_2 \cup A_3} - I_{A_1 \cup A_2}$. Thus we have

$$\begin{aligned} \varepsilon P(A) &\le E\bigl(c_2 S_2^+ I_{A_1\cup A_2}\bigr) + E\bigl[c_3 S_3^+ (I_{A_1\cup A_2\cup A_3} - I_{A_1\cup A_2})\bigr] + \sum_{j=4}^{n} E\bigl(c_j S_j^+ I_{A_j}\bigr) \\ &= E\bigl(c_3 S_3^+ I_{A_1\cup A_2\cup A_3}\bigr) + E\bigl[\bigl(c_2 S_2^+ - c_3 S_3^+\bigr) I_{A_1\cup A_2}\bigr] + \sum_{j=4}^{n} E\bigl(c_j S_j^+ I_{A_j}\bigr) \\ &\le E\bigl(c_3 S_3^+ I_{A_1\cup A_2\cup A_3}\bigr) + c_3 E\bigl[\bigl(S_2^+ - S_3^+\bigr) I_{A_1\cup A_2}\bigr] + \sum_{j=4}^{n} E\bigl(c_j S_j^+ I_{A_j}\bigr) \\ &= E\bigl(c_3 S_3^+ I_{A_1\cup A_2\cup A_3}\bigr) - c_3 E\bigl[\bigl(S_3^+ - S_2^+\bigr) I_{A_1\cup A_2}\bigr] + \sum_{j=4}^{n} E\bigl(c_j S_j^+ I_{A_j}\bigr). \end{aligned} \tag{2.4}$$

By the convexity of the function $f(x) = x^+$ again,

$$S_3^+ - S_2^+ \ge (S_3 - S_2) h(S_2), \tag{2.5}$$

then

$$E\bigl[\bigl(S_3^+ - S_2^+\bigr) I_{A_1\cup A_2}\bigr] \ge E\bigl[(S_3 - S_2) h(S_2) I_{A_1\cup A_2}\bigr]. \tag{2.6}$$

Obviously, $A_1 \cup A_2 = \{\max(c_1 S_1, c_2 S_2) \ge \varepsilon\}$ and $I_{A_1\cup A_2}$ is a nonnegative and componentwise nondecreasing function of $\{S_1, S_2\}$, then $h(S_2) I_{A_1\cup A_2}$ is a nonnegative and componentwise nondecreasing function of $\{S_1, S_2\}$. By the demisubmartingale property, the right-hand side of (2.6) is nonnegative. Thus

$$E\bigl[\bigl(S_3^+ - S_2^+\bigr) I_{A_1\cup A_2}\bigr] \ge 0$$

and the right-hand side of (2.4) is bounded by

$$E\bigl(c_3 S_3^+ I_{A_1\cup A_2\cup A_3}\bigr) + \sum_{j=4}^{n} E\bigl(c_j S_j^+ I_{A_j}\bigr).$$

Working in this manner we prove that

$$\begin{aligned} \varepsilon P(A) &\le E\bigl(c_{n-1} S_{n-1}^+ I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\bigr) + E\bigl(c_n S_n^+ I_{A_n}\bigr) \\ &= E\bigl(c_{n-1} S_{n-1}^+ I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\bigr) + E\bigl[c_n S_n^+ (I_{A_1\cup A_2\cup\cdots\cup A_n} - I_{A_1\cup A_2\cup\cdots\cup A_{n-1}})\bigr] \\ &\le c_n E\bigl(S_n^+ I_A\bigr) - c_n E\bigl[\bigl(S_n^+ - S_{n-1}^+\bigr) I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\bigr]. \end{aligned} \tag{2.7}$$

By the convexity of the function $f(x) = x^+$, we have

$$S_n^+ - S_{n-1}^+ \ge (S_n - S_{n-1}) h(S_{n-1}).$$

Hence

$$E\bigl[\bigl(S_n^+ - S_{n-1}^+\bigr) I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\bigr] \ge E\bigl[(S_n - S_{n-1}) h(S_{n-1}) I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\bigr]. \tag{2.8}$$

Since $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{\max(c_1S_1, c_2S_2, \ldots, c_{n-1}S_{n-1}) \geq \varepsilon\}$, $I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}$ is a nonnegative and componentwise nondecreasing function of $\{S_1, S_2, \ldots, S_{n-1}\}$. Then $h(S_{n-1}) \times I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}$ is a nonnegative and componentwise nondecreasing function of $\{S_1, S_2, \ldots, S_{n-1}\}$. As $\{S_n, n \geq 1\}$ forms a demisubmartingale and $\{c_n, n \geq 1\}$ is a sequence of positive numbers, we have

$$c_nE\big[(S_n - S_{n-1})h(S_{n-1})I_{A_1\cup A_2\cup\cdots\cup A_{n-1}}\big] \geq 0. \tag{2.9}$$

Consequently, it follows from (2.7), (2.8) and (2.9) that

$$\varepsilon P(A) \leq E\big(c_nS_n^+I_A\big).$$

So (2.1) is proved. □

Corollary 2.1 *Assume that $\{S_n, n \geq 1\}$ is a demisubmartingale or a demimartingale with $S_0 = 0$. Let g be a nondecreasing convex function such that $g(S_n) \in L^1$, $n \geq 1$ and $\{c_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then, for any $\varepsilon > 0$,*

$$\varepsilon P\Big\{\max_{1\leq k\leq n} c_kg(S_k) \geq \varepsilon\Big\} \leq E\Big[c_ng^+(S_n)I\Big(\max_{1\leq k\leq n} c_kg(S_k) \geq \varepsilon\Big)\Big]. \tag{2.10}$$

Proof By Lemma 1.1, $\{g(S_n), n \geq 1\}$ is a demisubmartingale. By Theorem 2.1, we obtain the result of (2.10). □

Remark 2.1 Chow [25] proved a maximal inequality for submartingales, which contains the Hajek-Renyi inequality and other inequalities as special cases (see Theorem 1 of Chow [25]). Christofides [4] generalized Theorem 1 of Chow [25] and obtained a Chow type maximal inequality for demimartingales (see Theorem 2.1 of Christofides [4]). Wang [7] generalized Theorem 2.1 of Christofides [4] to the nonnegative convex functions (see Theorem 2.1 of Wang [7]). Based on Christofides [4] and Wang [7], Wang and Hu [9] obtained some similar maximal inequalities for demisubmartingales and demimartingales (see Theorem 2.1 and Theorem 2.2 of Wang and Hu [9]). Inspired by these papers, we also get some similar Chow type maximal inequality for demisubmartingales and demimartingales (see Theorem 2.1 and Corollary 2.1).

Second, we provide a minimal type inequalities for a sequence of nonnegative demimartingales.

Theorem 2.2 *Let $\{S_n, n \geq 1\}$ be a nonnegative demimartingale with $S_0 = 0$ and $\{c_n, n \geq 1\}$ be a nonincreasing sequence of positive numbers. Then, for any $\varepsilon > 0$,*

$$\varepsilon P\Big\{\min_{1\leq k\leq n} c_kS_k \leq \varepsilon\Big\} \geq c_nE\Big[S_nI\Big(\min_{1\leq k\leq n} c_kS_k \leq \varepsilon\Big)\Big]. \tag{2.11}$$

Proof Following Christofides [4], we let $A = \{\min_{1\leq k\leq n} c_kS_k \leq \varepsilon\}$, $n \geq 1$. Then A can be written as $A = \bigcup_{j=1}^n A_j$, where $A_1 = \{c_1S_1 \leq \varepsilon\}$, $A_j = \{c_iS_i > \varepsilon, 1 \leq i < j, c_jS_j \leq \varepsilon\}$, $1 < j \leq n$, and

$A_i \cap A_j = \emptyset$ when $i \neq j$. Thus, similar to the proof of (2.2),

$$\begin{aligned}
\varepsilon P(A) &= \varepsilon \sum_{j=1}^{n} P(A_j) = \sum_{j=1}^{n} E(\varepsilon I_{A_j}) \geq \sum_{j=1}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_1 S_1 I_{A_1}) + E(c_2 S_2 I_{A_2}) + \sum_{j=3}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_1 S_1 I_{A_1}) + E\bigl[c_2 S_2 (I_{A_1 \cup A_2} - I_{A_1})\bigr] + \sum_{j=3}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_2 S_2 I_{A_1 \cup A_2}) + E\bigl[(c_1 S_1 - c_2 S_2) I_{A_1}\bigr] + \sum_{j=3}^{n} E(c_j S_j I_{A_j}) \\
&\geq E(c_2 S_2 I_{A_1 \cup A_2}) + c_2 E\bigl[(S_1 - S_2) I_{A_1}\bigr] + \sum_{j=3}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_2 S_2 I_{A_1 \cup A_2}) + c_2 E\bigl[(S_2 - S_1)(-I_{A_1})\bigr] + \sum_{j=3}^{n} E(c_j S_j I_{A_j}),
\end{aligned}$$

which is from the fact that $A_1 \cap A_2 = \emptyset$ and $I_{A_2} = I_{A_1 \cup A_2} - I_{A_1}$. Since I_{A_1} is a nonincreasing function of S_1, $-I_{A_1}$ is a nondecreasing function of S_1. By the definition of a demimartingale, one has

$$E\bigl[(S_2 - S_1)(-I_{A_1})\bigr] \geq 0. \tag{2.12}$$

So we can get

$$\begin{aligned}
\varepsilon P(A) &\geq E(c_2 S_2 I_{A_1 \cup A_2}) + \sum_{j=3}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_2 S_2 I_{A_1 \cup A_2}) + E(c_3 S_3 I_{A_3}) + \sum_{j=4}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_2 S_2 I_{A_1 \cup A_2}) + E\bigl[c_3 S_3 (I_{A_1 \cup A_2 \cup A_3} - I_{A_1 \cup A_2})\bigr] + \sum_{j=4}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_3 S_3 I_{A_1 \cup A_2 \cup A_3}) + E\bigl[(c_2 S_2 - c_3 S_3) I_{A_1 \cup A_2}\bigr] + \sum_{j=4}^{n} E(c_j S_j I_{A_j}) \\
&\geq E(c_3 S_3 I_{A_1 \cup A_2 \cup A_3}) + c_3 E\bigl[(S_2 - S_3) I_{A_1 \cup A_2}\bigr] + \sum_{j=4}^{n} E(c_j S_j I_{A_j}) \\
&= E(c_3 S_3 I_{A_1 \cup A_2 \cup A_3}) + c_3 E\bigl[(S_3 - S_2)(-I_{A_1 \cup A_2})\bigr] + \sum_{j=4}^{n} E(c_j S_j I_{A_j}).
\end{aligned} \tag{2.13}$$

Since $A_1 \cup A_2 = \{\min(c_1 S_1, c_2 S_2) \leq \varepsilon\}$, $I_{A_1 \cup A_2}$ is a componentwise nonincreasing function of $\{S_1, S_2\}$ and $-I_{A_1 \cup A_2}$ is a componentwise nondecreasing function of $\{S_1, S_2\}$. By the definition of a demimartingale,

$$E\big[(S_3 - S_2)(-I_{A_1 \cup A_2})\big] \geq 0. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\varepsilon P(A) \geq E(c_3 S_3 I_{A_1 \cup A_2 \cup A_3}) + \sum_{j=4}^{n} E(c_j S_j I_{A_j}).$$

By iterations,

$$\begin{aligned}\varepsilon P(A) &\geq E(c_{n-1} S_{n-1} I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}}) + E(c_n S_n I_{A_n}). \\ &= E(c_{n-1} S_{n-1} I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}}) + E\big[c_n S_n (I_{A_1 \cup A_2 \cup \cdots \cup A_n} - I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}})\big] \\ &\geq c_n E(S_n I_A) + c_n E\big[(S_n - S_{n-1})(-I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}})\big].\end{aligned} \tag{2.15}$$

Since $A_1 \cup A_2 \cup \cdots \cup A_{n-1} = \{\min(c_1 S_1, c_2 S_2, \ldots, c_{n-1} S_{n-1}) \leq \varepsilon\}$, $I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}}$ is a componentwise nonincreasing function of $\{S_1, S_2, \ldots, S_{n-1}\}$ and $-I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}}$ is a componentwise nondecreasing function of $\{S_1, S_2, \ldots, S_{n-1}\}$. By the fact that $\{S_n, n \geq 1\}$ is a nonnegative demimartingale and $\{c_k, k \geq 1\}$ is a nonincreasing sequence of positive numbers, it is checked that

$$c_n E\big[(S_n - S_{n-1})(-I_{A_1 \cup A_2 \cup \cdots \cup A_{n-1}})\big] \geq 0. \tag{2.16}$$

Finally, by (2.15) and (2.16), we get

$$\varepsilon P(A) \geq c_n E(S_n I_A).$$

So (2.11) holds. □

Corollary 2.2 *Let $\{S_n, n \geq 1\}$ be a demimartingale. Then, for any $\varepsilon > 0$,*

$$\varepsilon P\Big\{\min_{1 \leq k \leq n} S_k \leq \varepsilon\Big\} \geq \int_{\{\min_{1 \leq k \leq n} S_k \leq \varepsilon\}} S_n \, dP.$$

Proof By the proof of Theorem 2.2 with $c_k \equiv 1$, we can get the minimal inequality for demimartingales without the assumption of nonnegativeness. □

Remark 2.2 Newman and Wright [1] obtained some inequalities for demisubmartingales and demimartingales, including maximal and minimal inequalities (see Theorem 3 of Newman and Wright [1]). Prakasa Rao [8] generalized some results of Newman and Wright [1] and got minimal type inequalities for demisubmartingales (see Theorems 2.8-2.10 of Prakasa Rao [8]). Wang *et al.* [11] also obtained some minimal inequalities for nonnegative demimartingales (see Theorem 2.1, Corollary 2.1 and Corollary 2.2 of Wang *et al.* [11]). Similar to Theorem 2.8 of Prakasa Rao [8] and Theorem 2.1 of Wang *et al.* [11], we get some minimal type inequalities for nonnegative demimartingales in Theorem 2.2 and Corollary 2.3. It is pointed out that Corollary 2.2 is not a new result (see Theorem 2.9 of Prakasa Rao [8], Corollary 2.1 of Hu *et al.* [17], Corollary 2.1 of Wang *et al.* [11]).

Third, we consider the Chow type maximal inequality for N-demimartingales. Similar to Chow [25] and Christofides [4], Prakasa Rao [15] obtained a Chow type maximal inequality for N-demimartingales.

Theorem 2.3 (see Theorem 3.1 of Prakasa Rao [15] or Theorem 3.5.1 of Prakasa Rao [21]) *Assume that $\{S_n, n \geq 1\}$ is an N-demimartingale with $S_0 = 0$ and $m(\cdot)$ is a nonnegative nondecreasing function on $\mathbb{R}$ with $m(0) = 0$. Let $g(\cdot)$ be a function on $\mathbb{R}$ with $g(0) = 0$ and suppose that*

$$g(x) - g(y) \geq (y - x)h(y) \tag{2.17}$$

for all x, y, where $h(\cdot)$ is a nonnegative and nondecreasing function. Further assume that $\{c_k, 1 \leq k \leq n\}$ is a sequence of positive numbers such that $(c_k - c_{k+1})g(S_k) \geq 0$ for $1 \leq k \leq n-1$. Define $Y_k = \max_{1\leq j\leq k} c_j g(S_j)$, $k \geq 1$, $Y_0 = 0$. Then

$$E\biggl(\int_0^{Y_n} u\,dm(u)\biggr) \leq \sum_{i=1}^n c_i E\bigl[\bigl(g(S_i) - g(S_{i-1})\bigr)m(Y_n)\bigr]. \tag{2.18}$$

Let $\varepsilon > 0$ and define $m(t) = 1$ if $t \geq \varepsilon$ and $m(t) = 0$ if $t < \varepsilon$. By Theorem 2.3,

$$\varepsilon P(Y_n \geq \varepsilon) \leq \sum_{i=1}^n c_i E\bigl[\bigl(g(S_i) - g(S_{i-1})\bigr)I(Y_n \geq \varepsilon)\bigr] \tag{2.19}$$

(see (3.5.10) of Prakasa Rao [21]) was obtained. It can be seen that $g(x) = -\alpha x$, $\alpha \geq 0$, and $g(x) = -\alpha x^+$, $\alpha \geq 0$, satisfy the condition of (2.17) (see Prakasa Rao [15, 21]).

It is a fact that if $\{S_n\}_{n\geq 1}$ is an N-demimartingale, then $\{-S_n\}_{n\geq 1}$ is also an N-demimartingale (see Christofides [2] or Prakasa Rao [21]). By using Theorem 2.3, Hadjikyriakou [22] got the following maximal inequality for N-demimartingales.

Corollary 2.3 (Hadjikyriakou [22, Theorem 3.2.1]) *Assume that $\{S_n, n \geq 1\}$ is an N-demimartingale. Then, for every $\varepsilon > 0$,*

$$\varepsilon P\Bigl(\max_{1\leq k\leq n} S_k \geq \varepsilon\Bigr) \leq E\Bigl(S_n I\Bigl(\max_{1\leq k\leq n} S_k \geq \varepsilon\Bigr)\Bigr). \tag{2.20}$$

But we find that the Chow type maximal inequality for N-demimartingales, *i.e.*, Theorem 2.3, is not true. We give an example as follows.

An example for N-demimartingales Let $g(x) = -x$, $m(x) = x^+$, $c_1 = c_2 = 1$, $S_0 = 0$. Assume that S_1 and S_2 are independent random variables with probability distributions

$$S_1 \sim \begin{pmatrix} -2 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \qquad S_2 \sim \begin{pmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

In addition, let $Y_1 = c_1 g(S_1) = -S_1$ and

$$Y_2 = \max\bigl\{c_1 g(S_1), c_2 g(S_2)\bigr\} = \max\{-S_1, -S_2\}.$$

It is easy to check that for any nondecreasing function f,

$$\begin{aligned} &E\big[(S_2 - S_1)f(S_1)\big] \\ &\quad = \big[-1-(-2)\big]f(-2) \times \frac{1}{4} + (-1-2)f(2) \times \frac{1}{4} + \big[1-(-2)\big]f(-2) \\ &\qquad \times \frac{1}{4} + (1-2)f(2) \times \frac{1}{4} \\ &\quad = f(-2) - f(2) \le 0. \end{aligned}$$

Hence $\{S_1, S_2\}$ is an N-demimartingale. It follows from the distribution of S_1 that

$$Y_1 \sim \begin{pmatrix} -2 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Meanwhile,

$$\begin{aligned} &P(Y_2 = 1) = P(S_1 = 2, S_2 = -1) = \frac{1}{4}, \\ &P(Y_2 = -1) = P(S_1 = 2, S_2 = 1) = \frac{1}{4}, \\ &P(Y_2 = 2) = P(S_1 = -2, S_2 = 1) + P(S_1 = -2, S_2 = -1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}, \end{aligned}$$

so

$$Y_2 \sim \begin{pmatrix} -1 & 1 & 2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

It can be seen that if $Y_2(\omega) \ge 0$, then

$$\int_0^{Y_2} u\,dm(u) = \int_0^{Y_2} u\,du = \frac{1}{2}Y_2^2.$$

Otherwise, for the case $Y_2(\omega) < 0$, one has

$$\int_0^{Y_2} u\,dm(u) = 0.$$

Consequently,

$$\int_0^{Y_2} u\,dm(u) = \frac{1}{2}Y_2^2 I(Y_2 \ge 0) \sim \begin{pmatrix} 0 & \frac{1}{2} & 2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}.$$

On the other hand, we can calculate that

$$E\bigg[\int_0^{Y_2} u\,dm(u)\bigg] = \frac{9}{8}$$

and

$$\begin{aligned}\sum_{k=1}^{2} c_k E\bigl\{\bigl[g(S_k)-g(S_{k-1})\bigr]m(Y_2)\bigr\} &= E\bigl[g(S_2)m(Y_2)\bigr] = -E\bigl(S_2 Y_2^{+}\bigr)\\ &= -\biggl[-1\times 1\times\frac{1}{4}+1\times 2\times\frac{1}{4}+(-1)\times 2\times\frac{1}{4}\biggr]\\ &= \frac{1}{4}.\end{aligned}$$

But we have

$$\frac{9}{8} = E\biggl[\int_0^{Y_2} u\,dm(u)\biggr] > \sum_{k=1}^{2} c_k E\bigl\{\bigl[g(S_k)-g(S_{k-1})\bigr]m(Y_2)\bigr\} = \frac{1}{4},$$

which is contrary to (2.18). Therefore, Theorem 2.3 is not true. In fact, in the proof of Theorem 3.1 of Prakasa Rao [15] or the proof of Theorem 3.5.1 of Prakasa Rao [8], it was given that $h(S_i)m(Y_i)$ is a nondecreasing function of $S_1, S_2, \dots, S_i$. But by checking the proof carefully, we find that one cannot find that $h(S_i)m(Y_i)$ is a nondecreasing function of $S_1, S_2, \dots, S_i$ under the conditions of Theorem 2.3.

Similarly, it can be checked that

$$P(Y_2 \ge 1) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

and

$$\begin{aligned}\sum_{k=1}^{2} E\bigl[\bigl(g(S_k)-g(S_{k-1})\bigr)I(Y_2\ge 1)\bigr] &= -E\bigl[S_2 I(Y_2 \ge 1)\bigr]\\ &= -\biggl[-1\times 1\times\frac{1}{4}+1\times 1\times\frac{1}{4}+(-1)\times 1\times\frac{1}{4}\biggr]\\ &= \frac{1}{4}.\end{aligned}$$

Then

$$P(Y_2 \ge 1) > -E\bigl[S_2 I(Y_2 \ge 1)\bigr],$$

which is contrary to (2.19). So (2.19) is not true.

Meanwhile, one has

$$P\Bigl(\max_{1\le k\le 2} S_k \ge 1\Bigr) = P(S_1 = 2, S_2 = -1) + P(S_1 = 2, S_2 = 1) + P(S_1 = -2, S_2 = 1) = \frac{3}{4}$$

and

$$E\bigl[S_2 I\bigl(\max(S_1,S_2)\bigr) \ge 1\bigr] = -1\times\frac{1}{4} + 1\times\frac{1}{4} + 1\times\frac{1}{4} = \frac{1}{4}.$$

So

$$\frac{3}{4} = P\Bigl(\max_{1\le k\le 2} S_k \ge 1\Bigr) > E\bigl[S_2 I\bigl(\max(S_1,S_2)\bigr) \ge 1\bigr] = \frac{1}{4},$$

which is contrary to (2.20). Thus, (2.20) is not true. There are some problems of maximal type inequalities for N-demimartingales in the literature such as Wang *et al.* [19], Hu *et al.* [13], Wang *et al.* [14] and Yang and Hu [20]. It is interesting to investigate the maximal type inequalities of N-demimartingales for researchers in the future.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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