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Some fixed point results for generalized contraction mappings with cyclic (α, β)-admissible mapping in multiplicative metric spaces

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Abstract

The purpose of this work is to introduce new types of contraction mappings in the sense of a multiplicative metric space. Fixed point results for these contraction mappings in multiplicative metric spaces are obtained. Our presented results generalize, extend, and improve results on the topic in the literature. Moreover, our results cannot be directly obtained as a consequence from the corresponding results in metric spaces. We also state some illustrative examples to claim that our results properly generalize some results in the literature. We apply our main results for proving a fixed point theorem involving a cyclic mapping.

MSC: 47H09; 47H10

Keywords: multiplicative (α, β)-Banach-contraction mappings; multiplicative (α, β)-Kannan-contraction mappings; multiplicative (α, β)-Chatterjea-contraction mappings; multiplicative metric spaces

1 Introduction and preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}^+$, and $\mathbb{R}$ the sets of positive integers, positive real numbers, and real numbers, respectively.

The Banach-contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory. Also its significance lies in its vast applicability in a number of branches of mathematics. This principle is given by the next theorem.

Theorem 1.1 ([1]) *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Banach-contraction mapping, i.e.,*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, where $k \in [0, 1)$. Then T has a unique fixed point.

Theorem 1.1 was used to establish the existence of a solution for an integral equation. Since then, because of its simplicity and usefulness, it has become a very famous and popular tool in solving existence problems in many branches of mathematical analysis.

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In 1968, Kannan [2, 3] introduced a new type of contraction mapping, which is called the Kannan-contraction type. He also established a fixed point result for such a type.

Theorem 1.2 ([2, 3]) *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Kannan-contraction mapping, i.e.,*

$$d(Tx, Ty) \leq k\big[d(x, Tx) + d(y, Ty)\big] \tag{1.2}$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

The Kannan fixed point theorem is very important because Subrahmanyam [4] proved that the Kannan theorem characterizes the metric completeness of underlying spaces, that is, a metric space X is complete if and only if every Kannan-contraction mapping on X has a fixed point. There is a large literature dealing with Kannan-contraction mappings and their generalizations, some of which are found in [5, 6], and [7].

A similar contractive condition and fixed point result for this contraction has been introduced by Chatterjea [8].

Theorem 1.3 ([8]) *Let (X, d) be a complete metric spaces and $T : X \to X$ be a Chatterjea-contraction mapping, i.e.,*

$$d(Tx, Ty) \leq k\big[d(x, Ty) + d(y, Tx)\big] \tag{1.3}$$

for all $x, y \in X$, where $k \in [0, \frac{1}{2})$. Then T has a unique fixed point.

We see that the conditions (1.1), (1.2), and (1.3) are interesting to study because all conditions are independent.

In 2008, Bashirov *et al.* [9] defined a new distance, the so-called multiplicative distance, by using the concept of a multiplicative absolute value. After that, by using the idea of a multiplicative distance, Özavşar and Çevikel [10] studied multiplicative metric spaces and some topological properties.

Definition 1.4 ([9]) Let X be a nonempty set. A mapping $d : X \times X \to \mathbb{R}$ is said to be a multiplicative metric if it satisfies the following conditions:

1. $d(x, y) \geq 1$ for all $x, y \in X$ and $d(x, y) = 1$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$ for all $x, y \in X$,
3. $d(x, z) \leq d(x, y) \cdot d(y, z)$ for all $x, y, z \in X$.

Also, (X, d) is called a multiplicative metric space.

Example 1.5 ([9]) Let $d^* : (\mathbb{R}^+)^n \times (\mathbb{R}^+)^n \to \mathbb{R}$ be defined as follows:

$$d^*(x, y) := \left|\frac{x_1}{y_1}\right|^* \cdot \left|\frac{x_2}{y_2}\right|^* \cdots \left|\frac{x_n}{y_n}\right|^*,$$

where $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in (\mathbb{R}^+)^n$ and $|\cdot|^* : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as follows:

$$|a|^* = \begin{cases} a, & \text{if } a \geq 1, \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}$$

It is easy to see that all conditions of the multiplicative metric are satisfied.

Example 1.6 ([9]) Let $a > 1$ be a fixed real number. Then $d_a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$d_a(x,y) := a^{\sum_{i=1}^n |x_i - y_i|},$$

where $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, holds for the multiplicative metric conditions.

Remark 1.7 One can extend the multiplicative metric in Example 1.6 to $\mathbb{C}^n$ by the following definition:

$$d_a(\omega,z) := a^{\sum_{i=1}^n |\omega_i - z_i|},$$

where $\omega = (\omega_1, \dots, \omega_n), z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

Definition 1.8 ([10]) Let (X,d) be a multiplicative metric space, $x \in X$ and $\varepsilon > 1$. We now define a set

$$B_\varepsilon(x) := \{y \in X | d(x,y) < \varepsilon\},$$

which is called a multiplicative open ball of radius ε with center x. Similarly, one can describe a multiplicative closed ball as

$$\overline{B}_\varepsilon(x) := \{y \in X | d(x,y) \le \varepsilon\}.$$

Definition 1.9 ([10]) Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X, and $x \in X$. If, for every multiplicative open ball $B_\varepsilon(x)$, there exists a natural number N such that $n \ge N \Rightarrow x_n \in B_\varepsilon(x)$, then the sequence $\{x_n\}$ is said to be multiplicative convergent to x, denoted by $x_n \to_* x$ as $n \to \infty$.

Lemma 1.10 ([10]) *Let (X,d) be a multiplicative metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then*

$$x_n \to_* x \quad \text{as } n \to \infty \quad \text{if and only if} \quad d(x_n,x) \to_* 1 \quad \text{as } n \to \infty.$$

Lemma 1.11 ([10]) *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. If the sequence $\{x_n\}$ is multiplicative convergent, then the multiplicative limit point is unique.*

Definition 1.12 ([10]) Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a multiplicative Cauchy sequence if, for all $\varepsilon > 1$, there exists $N \in \mathbb{N}$ such that $d(x_m,x_n) < \varepsilon$ for $m,n \ge N$.

Lemma 1.13 ([10]) *Let (X,d) be a multiplicative metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a multiplicative Cauchy sequence if and only if $d(x_n,x_m) \to_* 1$ as $m,n \to \infty$.*

Definition 1.14 ([10]) Let (X,d) be a multiplicative metric space. The multiplicative metric spaces X is said to be complete if and only if every Cauchy sequence $\{x_n\}$ in X for all $n \in \mathbb{N}$ converges in X.

Definition 1.15 ([10]) Let (X,d) be a multiplicative metric space. A point $x \in X$ is said to be a multiplicative limit point of $S \subseteq X$ if and only if $(B_\varepsilon(x) - \{x\}) \cap S \neq \emptyset$ for every $\varepsilon > 1$. The set of all multiplicative limit points of the set S is denoted by S'.

Definition 1.16 ([10]) Let (X,d) be a multiplicative metric space. We call a set $S \subseteq X$ multiplicative closed in (X,d) if S contains all of its multiplicative limit points.

Özavşar and Çevikel also introduced the concepts of Banach-contraction, Kannan-contraction, and Chatterjea-contraction mappings in the sense of multiplicative metric spaces.

Definition 1.17 ([10]) Let (X,d) be a multiplicative metric space. A self-mapping f is said to be multiplicative Banach-contraction if

$$d(fx,fy) \leq d(x,y)^\lambda, \tag{1.4}$$

for all $x,y \in X$, where $\lambda \in [0,1)$.

Definition 1.18 ([10]) Let (X,d) be a multiplicative metric space. A self-mapping f is said to be multiplicative Kannan-contraction if

$$d(fx,fy) \leq \bigl(d(fx,x) \cdot d(fy,y)\bigr)^\lambda, \tag{1.5}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{2})$.

Definition 1.19 ([10]) Let (X,d) be a multiplicative metric space. A self-mapping f is said to be a multiplicative Chatterjea-contraction if

$$d(fx,fy) \leq \bigl(d(fx,y) \cdot d(fy,x)\bigr)^\lambda, \tag{1.6}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{2})$.

By using these ideas, they proved some fixed point theorems on complete multiplicative metric spaces. In fact, fixed point results in the framework of multiplicative metric spaces of Özavşar and Çevikel [10] can be directly obtained as a consequence from the corresponding results in metric spaces.

In this paper, we introduce new types of nonlinear mappings, the so-called (α,β)-Banach-contraction, (α,β)-Kannan-contraction and (α,β)-Chatterjea-contraction mappings, in the sense of multiplicative metric spaces and establish the existence of fixed point theorems for such mappings in multiplicative metric spaces by using the concept of cyclic (α,β)-admissible mappings.

Definition 1.20 ([11]) Let X be a nonempty set, f be a self-mapping on X, and $\alpha,\beta : X \to [0,\infty)$ be two mappings. We say that f is a cyclic (α,β)-admissible mapping if

$$x \in X, \quad \alpha(x) \geq 1 \quad \Rightarrow \quad \beta(fx) \geq 1$$

and

$$x \in X, \quad \beta(x) \geq 1 \quad \Rightarrow \quad \alpha(fx) \geq 1.$$

We also give the example of a nonlinear mapping which is not in the range of application of the results of Özavşar and Çevikel [10], but it can be applied to our results. Our fixed point results generalize, extend, and improve results on the topic in the literature. Moreover, our results cannot be directly obtained as a consequence of the corresponding results in metric spaces. Finally, we apply our main results for proving a fixed point theorem involving a cyclic mapping.

2 Fixed point results in multiplicative metric spaces

First of all, we will introduce the concept of a (α,β)-Banach-contraction mapping in the sense of a multiplicative metric distance.

Definition 2.1 Let (X,d) be a multiplicative metric space and let $\alpha,\beta : X \to [0,\infty)$ be two mappings. The mapping $f : X \to X$ is said to be a multiplicative (α,β)-Banach-contraction if

$$\alpha(x)\beta(y) \cdot d(fx,fy) \leq d(x,y)^{\lambda} \tag{2.1}$$

for all $x,y \in X$, where $\lambda \in [0,1)$.

Example 2.2 Let $X = [0.1,100]$ and $d^* : X \times X \to \mathbb{R}$ be defined as follows:

$$d^*(x,y) = \left|\frac{x}{y}\right|^*$$

for all $x,y \in X$, where $|\cdot|^* : X \to X$ is defined by

$$|a|^* = \begin{cases} a, & a \geq 1; \\ \frac{1}{a}, & a < 1. \end{cases} \tag{2.2}$$

Then (X,d^*) is a multiplicative metric space.

Define the mappings $\alpha,\beta : X \to [0,\infty)$ and $f : X \to X$ as follows:

$$\alpha(x) = \begin{cases} 1, & x \in [0.1,0.8]; \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

$$\beta(x) = \begin{cases} 1, & x \in [0.4,0.8]; \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

and

$$fx = \begin{cases} e^{x-1-\frac{x^3}{10}}, & x \in [0.1,0.8]; \\ \frac{3x-1}{4}, & x \in (0.8,100]. \end{cases} \tag{2.5}$$

Now we will show that f is a multiplicative (α,β)-Banach-contraction with $\lambda = 0.952$.

For $x, y \in [0.4, 0.8]$ and $x \geq y$, we get

$$\begin{aligned}
\alpha(x)\beta(y) \cdot d^*(fx, fy) &= \left|\frac{fx}{fy}\right|^* \\
&= \left|\frac{e^{x-1-\frac{x^3}{10}}}{e^{y-1-\frac{y^3}{10}}}\right|^* \\
&= \left|e^{x-y-\frac{x^3-y^3}{10}}\right|^* \\
&= \left|e^{(x-y)-(x-y)(\frac{x^2+xy+y^2}{10})}\right|^* \\
&= \left|e^{(x-y)(1-\frac{x^2+xy+y^2}{10})}\right|^* \\
&= e^{(x-y)(1-\frac{x^2+xy+y^2}{10})} \\
&\leq e^{(x-y)(1-\frac{(0.4)^2+(0.4)^2+(0.4)^2}{10})} \\
&= e^{(x-y)(0.952)} \\
&= \left(\frac{e^x}{e^y}\right)^{0.952} \\
&\leq \left(\frac{x}{y}\right)^{0.952} \\
&= \left(\left|\frac{x}{y}\right|^*\right)^{0.952} \\
&= \left[d^*(x, y)\right]^{\lambda}.
\end{aligned}$$

Similarly, for $x, y \in [0.4, 0.8]$ and $x < y$, we get

$$\begin{aligned}
\alpha(x)\beta(y) \cdot d^*(fx, fy) &= \left|\frac{fx}{fy}\right|^* \\
&= \left|\frac{e^{x-1-\frac{x^3}{10}}}{e^{y-1-\frac{y^3}{10}}}\right|^* \\
&= \left|e^{x-y-\frac{x^3-y^3}{10}}\right|^* \\
&= \left|e^{(x-y)-(x-y)(\frac{x^2+xy+y^2}{10})}\right|^* \\
&= \left|e^{(x-y)(1-\frac{x^2+xy+y^2}{10})}\right|^* \\
&= e^{(y-x)(1-\frac{x^2+xy+y^2}{10})} \\
&\leq e^{(y-x)(1-\frac{(0.4)^2+(0.4)^2+(0.4)^2}{10})} \\
&= e^{(y-x)(0.952)} \\
&= \left(\frac{e^y}{e^x}\right)^{0.952} \\
&\leq \left(\frac{y}{x}\right)^{0.952}
\end{aligned}$$

$$= \left(\left| \frac{x}{y} \right|^* \right)^{0.952}$$
$$= \left[d^*(x,y) \right]^{\lambda}.$$

In the other cases, it is easy to see that condition (2.1) holds since $\alpha(x)\beta(y) = 0$. Therefore,

$$\alpha(x)\beta(y) \cdot d^*(fx,fy) \leq d^*(x,y)^{\lambda}$$

for all $x, y \in X$, where $\lambda = 0.952$. Hence, f is a multiplicative (α, β)-Banach-contraction mapping.

Remark 2.3 In Example 2.2, f is not a multiplicative Banach-contraction mapping. Indeed, for $x = 5$ and $y = 10$, we get

$$d^*(fx,fy) = \left| \frac{fx}{fy} \right|^* = \left| \frac{14}{29} \right|^* > 2.07 > 2 = \left| \frac{5}{10} \right|^* = \left| \frac{x}{y} \right|^* = d^*(x,y) \geq \left[d^*(x,y) \right]^{\lambda}$$

for all $\lambda \in [0,1)$. Therefore, the class of multiplicative (α, β)-Banach-contraction mappings is a real wider class of Banach-contraction mappings.

Next, we give some fixed point result for multiplicative (α, β)-Banach-contraction mappings in complete multiplicative metric spaces.

Theorem 2.4 *Let (X,d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative (α, β)-Banach-contraction mapping. Suppose that the following conditions hold*:

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$*;
(2) *f is a cyclic (α, β)-admissible mapping*;
(3) *one of the following conditions holds*:
  (3.1) *f is continuous*;
  (3.2) *if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$.*

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Proof Starting from a point $x_0 \in X$ in condition (1), we get $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. We will construct the iterative sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a cyclic (α, β)-admissible mapping, we have

$$\alpha(x_0) \geq 1 \quad \Rightarrow \quad \beta(x_1) = \beta(fx_0) \geq 1 \tag{2.6}$$

and

$$\beta(x_0) \geq 1 \quad \Rightarrow \quad \alpha(x_1) = \alpha(fx_0) \geq 1. \tag{2.7}$$

By a similar method, we get

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. This implies that

$$\alpha(x_{n-1})\beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. From the (α, β)-Banach-contractive condition of f, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1})\beta(x_n) \cdot d(fx_{n-1}, fx_n) \\ &\leq d(x_{n-1}, x_n)^{\lambda} \\ &\vdots \\ &\leq d(x_0, x_1)^{\lambda^n} \end{aligned}$$

for all $n \in \mathbb{N}$. Let $m, n \in \mathbb{N}$ such that $m < n$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) \cdot d(x_{m+1}, x_{m+2}) \cdots d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1)^{\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}} \\ &\leq d(x_0, x_1)^{\frac{\lambda^m}{1-\lambda}}. \end{aligned}$$

Letting $m, n \to \infty$, we get $d(x_m, x_n) \to_* 1$ and so the sequence $\{x_n\}$ is multiplicative Cauchy. From the completeness of X, there exists $z \in X$ such that $x_n \to_* z$ as $n \to \infty$.

Now, we assume that f is continuous. Hence, we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\Bigl(\lim_{n\to\infty} x_n\Bigr) = fz.$$

Next, we will assume that condition (3.2) holds. Hence $\beta(z) \geq 1$. Then we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} d(fz, z) &\leq d(fz, fx_n) \cdot d(fx_n, z) \\ &= d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq \alpha(x_n)\beta(z) \cdot d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq d(x_n, z)^{\lambda} \cdot d(x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we get $d(fz, z) = 1$, that is, $fz = z$. This shows that z is a fixed point of f.

Finally, we show that z is the unique fixed point of f. Assume that y is another fixed point of f. From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(y) \geq 1$, and hence

$$\begin{aligned} d(z, y) &= d(fz, fy) \\ &\leq \alpha(z)\beta(y) \cdot d(fz, fy) \\ &\leq d(z, y)^{\lambda}. \end{aligned}$$

This shows that $d(z, y) = 1$ and then $z = y$. Therefore, z is the unique fixed point of f. This completes the proof. □

Now, we give some illustrative examples to the claim that our results properly generalize the results of Özavşar and Çevikel [10].

Example 2.5 Let $X = [0.1, 100]$ and $d^* : X \times X \to \mathbb{R}$ be defined as follows:

$$d^*(x,y) = \left|\frac{x}{y}\right|^*$$

for all $x, y \in X$, where $|\cdot|^* : X \to X$ is defined by

$$|a|^* = \begin{cases} a, & a \geq 1; \\ \frac{1}{a}, & a < 1. \end{cases} \tag{2.8}$$

It is easy to see that (X, d^*) is a complete multiplicative metric space.

Define mappings $\alpha, \beta : X \to [0, \infty)$ and $f : X \to X$ as follows:

$$\alpha(x) = \begin{cases} 1, & x \in [0.1, 0.8]; \\ 0, & \text{otherwise}, \end{cases} \tag{2.9}$$

$$\beta(x) = \begin{cases} 1, & x \in [0.4, 0.8]; \\ 0, & \text{otherwise}, \end{cases} \tag{2.10}$$

and

$$fx = \begin{cases} e^{x-1-\frac{x^3}{10}}, & x \in [0.1, 0.8]; \\ \frac{3x-1}{4}, & x \in (0.8, 100]. \end{cases} \tag{2.11}$$

From Remark 2.3, we can see that f is not a multiplicative Banach-contraction mapping. Therefore, the results of Özavşar and Çevikel [10] cannot be used for this case.

Here, we show that by Theorem 2.4 can be guaranteed the existence of a fixed point. From Example 2.2, we find that f is a multiplicative (α, β)-Banach-contraction with $\lambda = 0.952$. It is easy to see that there exists $x_0 = 0.4 \in X$ such that $\alpha(x_0) = \alpha(0.4) = 1$ and $\beta(x_0) = \beta(0.4) = 1$. This implies that condition (1) in Theorem 2.4 holds. Also, it is easy to prove that f is a cyclic (α, β)-admissible mapping. Furthermore, it is easy to see that condition (3.2) in Theorem 2.4 holds.

Therefore, all the conditions of Theorem 2.4 hold and so f has a unique fixed point, $0.7411317711 \in X$.

Corollary 2.6 ([10]) *Let (X, d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative Banach-contraction mapping. Then f has a unique fixed point. Moreover, for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 2.4, we get this result. □

Here we give the concepts of multiplicative (α, β)-Kannan-contraction and multiplicative (α, β)-Chatterjea-contraction mappings. Also, we establish the fixed point result for these mappings.

Definition 2.7 Let (X,d) be a multiplicative metric space and let $\alpha,\beta : X \to [0,\infty)$ be two mappings. The mapping $f : X \to X$ is said to be a multiplicative (α,β)-Kannan-contraction if

$$\alpha(x)\beta(y) \cdot d(fx,fy) \leq \bigl(d(fx,x) \cdot d(fy,y)\bigr)^{\lambda}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{2})$.

Definition 2.8 Let (X,d) be a multiplicative metric spaces and let $\alpha,\beta : X \to [0,\infty)$ be two mappings. The mapping $f : X \to X$ is said to be a multiplicative (α,β)-Chatterjea-contraction if

$$\alpha(x)\beta(y) \cdot d(fx,fy) \leq \bigl(d(fx,y) \cdot d(fy,x)\bigr)^{\lambda}$$

for all $x,y \in X$, where $\lambda \in [0,\frac{1}{2})$.

Theorem 2.9 *Let (X,d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative (α,β)-Kannan-contraction mapping. Suppose that the following conditions hold*:

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$*;
(2) *f is a cyclic (α,β)-admissible*;
(3) *one of the following conditions holds*:
  (3.1) *f is continuous*;
  (3.2) *if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$.*

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Proof Starting from a point $x_0 \in X$ in condition (1), we get $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. We will construct the iterative sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a cyclic (α,β)-admissible mapping, we have

$$\alpha(x_0) \geq 1 \quad \Rightarrow \quad \beta(x_1) = \beta(fx_0) \geq 1 \tag{2.12}$$

and

$$\beta(x_0) \geq 1 \quad \Rightarrow \quad \alpha(x_1) = \alpha(fx_0) \geq 1. \tag{2.13}$$

By a similar method, we get

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. This implies that

$$\alpha(x_{n-1})\beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. From the (α, β)-Kannan-multiplicative contractive condition of f, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1})\beta(x_n) \cdot d(fx_{n-1}, fx_n) \\ &\leq \bigl(d(fx_{n-1}, x_{n-1}) \cdot d(fx_n, x_n)\bigr)^{\lambda}, \end{aligned}$$

for all $n \in \mathbb{N}$. Thus we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^h$$

for all $n \in \mathbb{N}$, where $h = \frac{\lambda}{1-\lambda} < 1$. Let $m, n \in \mathbb{N}$ such that $m < n$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) \cdot d(x_{m+1}, x_{m+2}) \cdots d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1)^{h^m + h^{m+1} + \cdots + h^{n-1}} \\ &\leq d(x_0, x_1)^{\frac{h^m}{1-h}}. \end{aligned}$$

Letting $m, n \to \infty$, we get $d(x_m, x_n) \to_* 1$ and so the sequence $\{x_n\}$ is multiplicative Cauchy. From the completeness of X, there exists $z \in X$ such that $x_n \to_* z$ as $n \to \infty$.

Now, we assume that f is continuous. Hence, we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\Bigl(\lim_{n\to\infty} x_n\Bigr) = fz.$$

Next, we will assume that condition (3.2) holds. Hence $\beta(z) \geq 1$. Then we have

$$\begin{aligned} d(fz, z) &\leq d(fz, fx_n) \cdot d(fx_n, z) \\ &= d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq \alpha(x_n)\beta(z) \cdot d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq d(x_n, z)^{\lambda} \cdot d(x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we get $d(fz, z) = 1$, that is, $fz = z$. This shows that z is a fixed point of f.

Now we show that z is the unique fixed point of f. Assume that y is another fixed point of f. From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(y) \geq 1$, and hence

$$\begin{aligned} d(z, y) &= d(fz, fy) \\ &\leq \alpha(z)\beta(y) \cdot d(fz, fy) \\ &\leq d(z, y)^{\lambda}. \end{aligned}$$

This shows that $d(z, y) = 1$ and then $z = y$. Therefore, z is the unique fixed point of f. This completes the proof. □

Corollary 2.10 ([10]) *Let (X, d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative Kannan-contraction mapping. Then f has a unique fixed point. Moreover, for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 2.9, we get this result. □

Theorem 2.11 *Let (X, d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative (α, β)-Chatterjea-contraction mapping. Suppose that the following conditions hold*:

(1) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$*;
(2) *f is a cyclic (α, β)-admissible*;
(3) *one of the following conditions holds*:
  (3.1) *f is continuous*;
  (3.2) *if $\{x_n\}$ is a sequence in X such that $x_n \to_* x \in X$ as $n \to \infty$ and $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$, then $\beta(x) \geq 1$.*

Then f has a fixed point. Furthermore, if $\alpha(x) \geq 1$ and $\beta(x) \geq 1$ for all fixed point $x \in X$, then f has a unique fixed point.

Proof Starting from a point $x_0 \in X$ in condition (1), we get $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. We will construct the iterative sequence $\{x_n\}$, where $x_n = fx_{n-1}$ for all $n \in \mathbb{N}$. Since f is a cyclic (α, β)-admissible mapping, we have

$$\alpha(x_0) \geq 1 \quad \Rightarrow \quad \beta(x_1) = \beta(fx_0) \geq 1 \tag{2.14}$$

and

$$\beta(x_0) \geq 1 \quad \Rightarrow \quad \alpha(x_1) = \alpha(fx_0) \geq 1. \tag{2.15}$$

By a similar method, we get

$$\alpha(x_n) \geq 1 \quad \text{and} \quad \beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. This implies that

$$\alpha(x_{n-1})\beta(x_n) \geq 1$$

for all $n \in \mathbb{N}$. From the multiplicative (α, β)-Chatterjea-contraction condition of f, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \alpha(x_{n-1})\beta(x_n) \cdot d(fx_{n-1}, fx_n) \\ &\leq \bigl(d(fx_{n-1}, x_n) \cdot d(fx_n, x_{n-1})\bigr)^{\lambda} \\ &= d(x_{n-1}, x_{n+1})^{\lambda} \\ &\leq \bigl(d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1})\bigr)^{\lambda} \end{aligned}$$

for each $n \in \mathbb{N}$. Thus we have

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)^{h}$$

for all $n \in \mathbb{N}$, where $h = \frac{\lambda}{1-\lambda} < 1$. Let $m, n \in \mathbb{N}$ such that $m < n$, then we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) \cdot d(x_{m+1}, x_{m+2}) \cdots d(x_{n-1}, x_n) \\ &\leq d(x_0, x_1)^{h^m + h^{m+1} + \cdots + h^{n-1}} \\ &\leq d(x_0, x_1)^{\frac{h^m}{1-h}}. \end{aligned}$$

Letting $m, n \to \infty$, we get $d(x_m, x_n) \to_* 1$ and so the sequence $\{x_n\}$ is multiplicative Cauchy. From the completeness of X, there exists $z \in X$ such that $x_n \to_* z$ as $n \to \infty$.

Now, we assume that f is continuous. Hence, we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = f\Big(\lim_{n\to\infty} x_n\Big) = fz.$$

Next, we will assume that condition (3.2) holds. Hence $\beta(z) \geq 1$. Then we have

$$\begin{aligned} d(fz, z) &\leq d(fz, fx_n) \cdot d(fx_n, z) \\ &= d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq \alpha(x_n)\beta(z) \cdot d(fx_n, fz) \cdot d(fx_n, z) \\ &\leq d(x_n, z)^{\lambda} \cdot d(x_{n+1}, z). \end{aligned}$$

Letting $n \to \infty$, we get $d(fz, z) = 1$, that is, $fz = z$. This shows that z is a fixed point of f.

Now we show that z is the unique fixed point of f. Assume that y is another fixed point of f. From the hypothesis, we find that $\alpha(z) \geq 1$ and $\beta(y) \geq 1$, and hence

$$\begin{aligned} d(z, y) &= d(fz, fy) \\ &\leq \alpha(z)\beta(y) \cdot d(fz, fy) \\ &\leq d(z, y)^{\lambda}. \end{aligned}$$

This shows that $d(z, y) = 1$ and then $z = y$. Therefore, z is the unique fixed point of f. This completes the proof. □

Corollary 2.12 ([10]) *Let (X, d) be a complete multiplicative metric space and $f : X \to X$ be a multiplicative Chatterjea-contraction mapping. Then f has a unique fixed point. Moreover, for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.*

Proof Setting $\alpha(x) = 1$ and $\beta(x) = 1$ for all $x \in X$ in Theorem 2.11, we get this result. □

3 Some cyclic contractions via cyclic (α, β)-admissible mapping

In 2003, Kirk *et al.* [12] introduced the concept of cyclic mappings and cyclic contractions.

Definition 3.1 ([12]) Let A and B be nonempty subsets of a metric space (X, d). A mapping $f : A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.

Definition 3.2 ([12]) Let A and B be nonempty subsets of a metric space (X, d). A mapping $f : A \cup B \to A \cup B$ is called a cyclic contraction if there exists $k \in [0, 1)$ such that $d(fx, fy) \leq kd(x, y)$ for all $x \in A$ and $y \in B$.

Notice that although a Banach-contraction is continuous, a cyclic contraction need not to be. This is one of the important gains of fixed point results for cyclic mappings. Following [12], a number of fixed point theorems on a cyclic mappings have appeared (see, *e.g.*, [13–21]).

In this section, we apply our main results for proving a fixed point theorems involving a cyclic mapping in multiplicative metric spaces.

Definition 3.3 Let A and B be nonempty subsets of a multiplicative metric space (X,d). A mapping $f: A \cup B \to A \cup B$ is called cyclic if $f(A) \subseteq B$ and $f(B) \subseteq A$.

Theorem 3.4 *Let A and B be two closed subsets of a complete multiplicative metric space (X,d) such that $A \cap B \neq \emptyset$ and $f: A \cup B \to A \cup B$ be a cyclic mapping. Assume that*

$$d(fx,fy) \leq d(x,y)^{\lambda} \tag{3.1}$$

for all $x \in A$ and $y \in B$, where $\lambda \in [0,1)$. Then f has a unique fixed point in $A \cap B$.

Proof Define mappings $\alpha, \beta: X \to [0,\infty)$ by

$$\alpha(x) = \begin{cases} 1, & x \in A; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 1, & x \in B; \\ 0, & \text{otherwise.} \end{cases}$$

For $x \in A$ and $y \in B$, we get

$$\alpha(x)\beta(y) \cdot d(fx,fy) \leq d(x,y)^{\lambda}. \tag{3.2}$$

In other cases, we see that the contractive condition (3.2) holds. Therefore, f is a multiplicative (α,β)-Banach-contraction mapping. It is easy to see that f is a cyclic (α,β)-admissible mapping. Since $A \cap B \neq \emptyset$, there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

Next, we show that condition (3.2) in Theorem 2.4 holds. Let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$. Then we have $x_n \in B$ for all $n \in \mathbb{N}$. Since B is closed subset of X, we get $x \in B$ and then $\beta(x) \geq 1$. Now, the conditions (1), (2), and (3.2) of Theorem 2.4 hold. So, f has a unique fixed point in $A \cup B$, say z. If $z \in A$, then $z = fz \in B$. Similarly, if $z \in B$, then we have $z \in A$. Therefore $z \in A \cap B$. This completes the proof. □

Similarly, we can prove the following theorems.

Theorem 3.5 *Let A and B be two closed subsets of a complete multiplicative metric space (X,d) such that $A \cap B \neq \emptyset$ and $f: A \cup B \to A \cup B$ be a cyclic mapping. Assume that*

$$d(fx,fy) \leq \bigl(d(fx,x) \cdot d(fy,y)\bigr)^{\lambda} \tag{3.3}$$

for all $x \in A$ and $y \in B$, where $\lambda \in [0,\frac{1}{2})$. Then f has a unique fixed point in $A \cap B$.

Theorem 3.6 *Let A and B be two closed subsets of a complete multiplicative metric space (X, d) such that $A \cap B \neq \emptyset$ and $f : A \cup B \to A \cup B$ be a cyclic mapping. Assume that*

$$d(fx, fy) \leq \big(d(fx, y) \cdot d(fy, x)\big)^{\lambda} \tag{3.4}$$

for all $x \in A$ and $y \in B$, where $\lambda \in [0, \frac{1}{2})$. Then f has a unique fixed point in $A \cap B$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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