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Convergence theorems of convex combination methods for treating d-accretive mappings in a Banach space and nonlinear equation

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Abstract

m-*d*-Accretive mappings, which are totally different from *m*-accretive mappings in non-Hilbertian Banach spaces, belong to another type of nonlinear mappings with practical backgrounds. The purpose of this paper is to present some new iterative schemes by means of convex combination methods to approximate the common zeros of finitely many *m*-*d*-accretive mappings. Some strong and weak convergence theorems are obtained in a real uniformly smooth and uniformly convex Banach space by using the techniques of the Lyapunov functional and retraction. The restrictions are weaker than in the previous corresponding works. Moreover, an example of *m*-*d*-accretive mapping is exemplified, from which we can see the connections between *m*-*d*-accretive mappings and the nonlinear elliptic equations.

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1 Introduction and preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* denote the dual space of E. We use '$\rightarrow$' and '$\rightharpoonup$' to denote strong and weak convergence, respectively. We denote the value of $f \in E^*$ at $x \in E$ by $\langle x, f\rangle$.

The normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx := \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E.$$

We call J weakly sequentially continuous if $\{x_n\}$ is sequence in E which converges weakly to x it follows that $\{Jx_n\}$ converges in weak* to Jx.

A mapping $T : D(T) = E \to E^*$ is said to be demi-continuous [1] on E if $Tx_n \rightharpoonup Tx$, as $n \to \infty$, for any sequence $\{x_n\}$ strongly convergent to x in E. A mapping $T : D(T) = E \to E^*$ is said to be hemi-continuous [1] on E if $w\text{-}\lim_{t\to 0} T(x + ty) = Tx$, for any $x, y \in E$. A mapping $T : E \to E$ is said to be non-expansive if $\|Tx - Ty\| \le \|x - y\|$, for $\forall x, y \in E$.

The Lyapunov functional $\varphi : E \times E \to R^+$ is defined as follows [2]:

$$\varphi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2,$$

for $\forall x, y \in E$.

It is obvious from the definition of φ that

$$\big(\|x\| - \|y\|\big)^2 \leq \varphi(x,y) \leq \big(\|x\| + \|y\|\big)^2, \tag{1.1}$$

for all $x, y \in E$. We also know that

$$\varphi(x,y) = \varphi(x,z) + \varphi(z,y) + 2\langle x - z, Jz - Jy\rangle, \tag{1.2}$$

for each $x, y, z \in E$; see [3, 4].

We use $\mathrm{Fix}(S)$ to denote the set of fixed points of a mapping $S : E \to E$. That is, $\mathrm{Fix}(S) := \{x \in E : Sx = x\}$. A mapping $S : E \to E$ is said to be generalized non-expansive [4] if $\mathrm{Fix}(S) \neq \emptyset$ and $\varphi(Sx, p) \leq \varphi(x, p)$, for $\forall x \in E$ and $p \in \mathrm{Fix}(S)$.

Let C be a nonempty closed subset of E and let Q be a mapping of E onto C. Then Q is said to be sunny [4] if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping $Q : E \to C$ is said to be a retraction [4] if $Q(z) = z$ for every $z \in C$. If E is a smooth and strictly convex Banach space, then a sunny generalized non-expansive retraction of E onto C is uniquely decided, which is denoted by R_C.

Let I denote the identity operator on E. A mapping $A : D(A) \subset E \to E$ is said to be accretive if $\langle Ax - Ay, J(x - y)\rangle \geq 0$, for $\forall x, y \in D(A)$ and it is called m-accretive if $R(I + \lambda A) = E$, for $\forall \lambda > 0$.

If A is accretive, we can define, for each $r > 0$, a single-valued mapping $J_r^A : R(I + rA) \to D(A)$ by $J_r^A = (I + rA)^{-1}$, which is called the resolvent of A. And, J_r^A is a non-expansive mapping [1]. In the process of constructing iterative schemes to approximate zeros of an accretive mapping A, the non-expansive property of J_r^A plays an important role.

A mapping $A : D(A) \subset E \to E$ is said to be d-accretive [5] if $\langle Ax - Ay, Jx - Jy\rangle \geq 0$, for $\forall x, y \in D(A)$. And it is called m-d-accretive if $R(I + \lambda A) = E$, for $\forall \lambda > 0$. However, the resolvent of an m-d-accretive mapping is not a non-expansive mapping.

An operator $B \subset E \times E^*$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$, for $\forall y_i \in Bx_i$, $i = 1, 2$. A monotone operator B is said to be maximal monotone if $R(J + \lambda B) = E^*$, for $\forall \lambda > 0$. An operator $B \subset E \times E^*$ is said to be strictly monotone if $\langle x_1 - x_2, y_1 - y_2\rangle > 0$, for $\forall x_1 \neq x_2$, $\forall y_i \in Bx_i$, $i = 1, 2$.

It is clear that in the frame of Hilbert spaces, (m-)accretive mappings, (m-)d-accretive mappings and (maximal) monotone operators are the same. But in the frame of non-Hilbertian Banach spaces, they belong to different classes of important nonlinear operators, which have practical backgrounds. During the past 50 years or so, a large number of researches have been done on the topics of constructing iterative schemes to approximate the zeros of m-accretive mappings and maximal monotone operators. However, rarely related work on d-accretive mappings can be found.

As we know, in 2000, Alber and Reich [5] presented the following iterative schemes for the d-accretive mapping T in a real uniformly smooth and uniformly convex Banach space:

$$x_{n+1} = x_n - \alpha_n Tx_n \tag{1.3}$$

and

$$x_{n+1} = x_n - \alpha_n \frac{Tx_n}{\|Tx_n\|}, \quad n \geq 0. \tag{1.4}$$

They proved that the iterative sequences $\{x_n\}$ generated by (1.3) and (1.4) converge weakly to the zero point of T under the assumptions that T is uniformly bounded and demi-continuous.

In 2007, Guan [6] presented the following projective method for the m-d-accretive mapping A in a real uniformly smooth and uniformly convex Banach space:

$$\begin{cases} x_1 \in D(A), \\ y_n = J_{r_n}^A x_n, \\ C_n = \{v \in D(A) : \varphi(v, y_n) \leq \varphi(v, x_n)\}, \\ Q_n = \{v \in D(A) : \langle x_n - v, Jx_1 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_1, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $J_{r_n}^A = (I + r_n A)^{-1}$, and $\Pi_{C_n \cap Q_n}$ is the generalized projection from $D(A)$ onto $C_n \cap Q_n$. It was shown that the iterative sequence $\{x_n\}$ generated by (1.5) converges strongly to the zero point of A under the assumptions that A is demi-continuous, the normalized duality mapping J is weakly sequentially continuous, and $J_{r_n}^A$ satisfies

$$\varphi(p, J_{r_n}^A x) \leq \varphi(p, x), \tag{1.6}$$

for $\forall x \in E$ and $p \in A^{-1}0$. The restrictions are extremely strong and it is hardly for us to find such an m-d-accretive mapping which both is demi-continuous and satisfies (1.6).

The so-called block iterative scheme for solving the problem of image recovery proposed by Aharoni and Censor [7] inspired us. In a finite-dimensional space H, the block iterative sequence $\{x_n\}$ is generated in the following way: $x_1 = x \in H$ and

$$x_{n+1} = \sum_{i=1}^{m} \omega_{n,i} \big(\alpha_{n,i} x_n + (1 - \alpha_{n,i}) P_i x_n\big), \tag{1.7}$$

where P_i is a non-expansive retraction from H onto C_i, and $\{C_i\}_{i=1}^m$ is a family of nonempty closed and convex subsets of H. $\{\omega_{n,i}\} \subset [0,1]$, $\sum_{i=1}^m \omega_{n,i} = 1$, and $\{\alpha_{n,i}\} \subset (-1,1)$, for $i = 1, 2, \ldots, m$ and $n \geq 1$.

In [8], Kikkawa and Takahashi applied the block iterative method to approximate the common fixed point of finite non-expansive mappings $\{T_i\}_{i=1}^m$ in Banach spaces in the following way and obtained the weak convergence theorems: $x_1 = x \in C$, and

$$x_{n+1} = \sum_{i=1}^{m} \omega_{n,i} \big(\alpha_{n,i} x_n + (1 - \alpha_{n,i}) T_i x_n\big), \tag{1.8}$$

where $\{\omega_{n,i}\} \subset [0,1]$, $\sum_{i=1}^m \omega_{n,i} = 1$, and $\{\alpha_{n,i}\} \subset [0,1]$, for $i = 1, 2, \ldots, m$ and $n \geq 1$.

In this paper, we shall borrow the idea of block iterative method which highlights the convex combination techniques. Our main work can be divided into three parts. In Section 2, we shall construct iterative schemes by convex combination techniques for approximating common zeros of m-d-accretive mappings. Some weak convergence theorems are obtained in a Banach space. In Section 3, we shall construct iterative schemes by convex combination and retraction techniques for approximating common zeros of m-d-accretive mappings. Some strong convergence theorems are obtained in a Banach space. In Section 4, we shall present a nonlinear elliptic equation from which we can define an m-d-accretive mapping. Our main contributions lie in the following aspects:

(i) The restrictions are weaker. The semi-continuity of the d-accretive mapping A and the inequality of (1.6) are no longer needed.
(ii) The Lyapunov functional is employed in the process of estimating the convergence of the iterative sequence. This is mainly because the resolvent of a d-accretive mapping is not non-expansive.
(iii) The connection between a nonlinear elliptic equation and an m-d-accretive mapping is set up, from which we cannot only find a good example of m-d-accretive mapping but also see the iterative construction of the solution of the nonlinear elliptic equation.

In order to prove our convergence theorems, we also need the following lemmas.

Lemma 1.1 [1, 9, 10] *The duality mapping $J : E \to 2^{E^*}$ has the following properties*:

(i) *If E is a real reflexive and smooth Banach space, then $J : E \to E^*$ is single-valued.*
(ii) *If E is reflexive, then J is a surjection.*
(iii) *If E is a real uniformly smooth and uniformly smooth Banach space, then $J^{-1} : E^* \to E$ is also a duality mapping. Moreover, J and J^{-1} are uniformly continuous on each bounded subset of E and E^*, respectively.*
(iv) *E is strictly convex if and only if J is strictly monotone.*

Lemma 1.2 [10] *Let E be a real smooth and uniformly convex Banach space, $B \subset E \times E^*$ be a maximal monotone operator, then $B^{-1}0$ is a closed and convex subset of E and the graph of B, $G(B)$ is demi-closed in the following sense*: *$\forall\{x_n\} \subset D(B)$ with $x_n \rightharpoonup x$ in E, and $\forall y_n \in Bx_n$ with $y_n \to y$ in E^* it follows that $x \in D(B)$ and $y \in Bx$.*

Lemma 1.3 [2] *Let E be a real reflexive, strictly convex, and smooth Banach space, let C be a nonempty closed subset of E, and let $R_C : E \to C$ be a sunny generalized non-expansive retraction. Then, for $\forall u \in C$ and $x \in E$, $\varphi(x, R_C x) + \varphi(R_C x, u) \le \varphi(x, u)$.*

Lemma 1.4 [3] *Let E be a real smooth and uniformly convex Banach space, and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\varphi(x_n, y_n) \to 0$, as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Lemma 1.5 [11] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers and $a_{n+1} \le a_n + b_n$, for $\forall n \ge 1$. If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.*

2 Weak convergence theorems

Theorem 2.1 *Let E be a real uniformly smooth and uniformly convex Banach space. Let $A_i : E \to E$ be a finite family of m-d-accretive mappings, $\{\omega_{n,i}\}, \{\eta_{n,i}\} \subset (0,1]$, $\{\alpha_{n,i}\}, \{\beta_{n,i}\} \subset$*

$[0,1)$, $\{r_{n,i}\},\{s_{n,i}\}\subset(0,+\infty)$, *for* $i=1,2,\ldots,m$. $\sum_{i=1}^m\omega_{n,i}=1$ *and* $\sum_{i=1}^m\eta_{n,i}=1$. *Let* $D:=\bigcap_{i=1}^m A_i^{-1}0\neq\emptyset$. *Suppose that the normalized duality mapping* $J:E\to E^*$ *is weakly sequentially continuous. Let* $\{x_n\}$ *be generated by the following iterative algorithm*:

$$\begin{cases} x_1\in E,\\ y_n=\sum_{i=1}^m\omega_{n,i}[\alpha_{n,i}x_n+(1-\alpha_{n,i})(I+r_{n,i}A_i)^{-1}x_n],\\ x_{n+1}=\sum_{i=1}^m\eta_{n,i}[\beta_{n,i}x_n+(1-\beta_{n,i})(I+s_{n,i}A_i)^{-1}y_n],\quad n\geq 1.\end{cases}\tag{2.1}$$

Suppose the following conditions are satisfied:

(i) $\limsup_{n\to\infty}\alpha_{n,i}<1$, $\limsup_{n\to\infty}\beta_{n,i}<1$, *for* $i=1,2,\ldots,m$;

(ii) $\liminf_{n\to\infty}\eta_{n,i}>0$, $\liminf_{n\to\infty}\omega_{n,i}>0$, *for* $i=1,2,\ldots,m$;

(iii) $\inf_{n\geq1}r_{n,i}>0$, $\inf_{n\geq1}s_{n,i}>0$, *for* $i=1,2,\ldots,m$.

Then $\{x_n\}$ *converges weakly to a point* $v_0\in D$.

Proof For $i=1,2,\ldots,m$, let $J_{r_{n,i}}^{A_i}=(I+r_{n,i}A_i)^{-1}$ and $J_{s_{n,i}}^{A_i}=(I+s_{n,i}A_i)^{-1}$.

We split the proof into the following six steps.

Step 1. For $p\in D$, $J_{r_{n,i}}^{A_i}$ and $J_{s_{n,i}}^{A_i}$ satisfy the following two inequalities, respectively:

$$\varphi\bigl(x_n,J_{r_{n,i}}^{A_i}x_n\bigr)+\varphi\bigl(J_{r_{n,i}}^{A_i}x_n,p\bigr)\leq\varphi(x_n,p),\tag{2.2}$$

$$\varphi\bigl(y_n,J_{s_{n,i}}^{A_i}y_n\bigr)+\varphi\bigl(J_{s_{n,i}}^{A_i}y_n,p\bigr)\leq\varphi(y_n,p).\tag{2.3}$$

In fact, using (1.2), we know that for $\forall p\in D$,

$$\varphi(x_n,p)=\varphi\bigl(x_n,J_{r_{n,i}}^{A_i}x_n\bigr)+\varphi\bigl(J_{r_{n,i}}^{A_i}x_n,p\bigr)+2\bigl\langle x_n-J_{r_{n,i}}^{A_i}x_n,JJ_{r_{n,i}}^{A_i}x_n-Jp\bigr\rangle.\tag{2.4}$$

Since A_i is d-accretive and $\frac{x_n-J_{r_{n,i}}^{A_i}x_n}{r_{n,i}}=A_iJ_{r_{n,i}}^{A_i}x_n$,

$$\left\langle\frac{x_n-J_{r_{n,i}}^{A_i}x_n}{r_{n,i}},JJ_{r_{n,i}}^{A_i}x_n-Jp\right\rangle\geq 0.$$

From (2.4) we know that (2.2) is true. So is (2.3).

Step 2. $\{x_n\}$ is bounded.

$\forall p\in D$, using (2.2) and (2.3), we have

$$\begin{aligned}\varphi(x_{n+1},p)&\leq\sum_{i=1}^m\eta_{n,i}\bigl[\beta_{n,i}\varphi(x_n,p)+(1-\beta_{n,i})\varphi\bigl(J_{s_{n,i}}^{A_i}y_n,p\bigr)\bigr]\\&\leq\sum_{i=1}^m\eta_{n,i}\bigl[\beta_{n,i}\varphi(x_n,p)+(1-\beta_{n,i})\varphi(y_n,p)\bigr]\\&\leq\sum_{i=1}^m\eta_{n,i}\beta_{n,i}\varphi(x_n,p)\\&\quad+\sum_{i=1}^m\eta_{n,i}(1-\beta_{n,i})\sum_{i=1}^m\omega_{n,i}\bigl[\alpha_{n,i}\varphi(x_n,p)+(1-\alpha_{n,i})\varphi\bigl(J_{r_{n,i}}^{A_i}x_n,p\bigr)\bigr]\\&\leq\varphi(x_n,p).\end{aligned}$$

Lemma 1.5 implies that $\lim_{n\to\infty}\varphi(x_n,p)$ exists. Then (1.1) ensures that $\{x_n\}$ is bounded.

Step 3. $A_iJ^{-1} \subset E^* \times E$ is maximal monotone, for each i, $1 \le i \le m$.

Since A_i is d-accretive, then $\forall x, y \in E^*$,

$$\langle x - y, A_iJ^{-1}x - A_iJ^{-1}y\rangle = \langle A_i(J^{-1}x) - A_i(J^{-1}y), J(J^{-1}x) - J(J^{-1}y)\rangle \ge 0.$$

Therefore, A_iJ^{-1} is monotone, for each i, $1 \le i \le m$.

Since $R(I + \lambda A_i) = E$, $\lambda > 0$, then $\forall y \in E$, there exists $x \in E$ satisfying $x + \lambda A_i x = y$, $\lambda > 0$. Using Lemma 1.1(ii), there exists $x^* \in E^*$ such that $J^{-1}x^* = x$. Thus $J^{-1}x^* + \lambda A_iJ^{-1}x^* = y$, which implies that $R(J^{-1} + \lambda A_iJ^{-1}) = E$, $\lambda > 0$. Thus A_iJ^{-1} is maximal monotone, for each i, $1 \le i \le m$.

Step 4. $(A_iJ^{-1})^{-1}0 \ne \emptyset$, for each i, $1 \le i \le m$.

Since $D \ne \emptyset$, then there exists $x \in E$ such that $A_ix = 0$, where $i = 1, 2, \ldots, m$. Using Lemma 1.1(ii) again, there exists $x^* \in E^*$ such that $J^{-1}x^* = x$. Thus $A_iJ^{-1}x^* = 0$, for each i, $1 \le i \le m$. That is, $x^* \in (A_iJ^{-1})^{-1}0$, for each i, $1 \le i \le m$.

Step 5. $\omega(x_n) \subset D$, where $\omega(x_n)$ denotes the set of all of the weak limit points of the weakly convergent subsequences of $\{x_n\}$.

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$, for simplicity, we still denote it by $\{x_n\}$ such that $x_n \rightharpoonup x$, as $n \to \infty$.

For $\forall p \in D$, using (2.2) and (2.3), we have

$$\begin{aligned}
\varphi(x_{n+1}, p) &\le \sum_{i=1}^m \eta_{n,i}\bigl[\beta_{n,i}\varphi(x_n, p) + (1 - \beta_{n,i})\varphi(y_n, p)\bigr] \\
&\le \sum_{i=1}^m \eta_{n,i}\beta_{n,i}\varphi(x_n, p) \\
&\quad + \sum_{i=1}^m \eta_{n,i}(1 - \beta_{n,i}) \sum_{i=1}^m \omega_{n,i}\bigl[\alpha_{n,i}\varphi(x_n, p) + (1 - \alpha_{n,i})\varphi\bigl(J_{r_{n,i}}^{A_i}x_n, p\bigr)\bigr] \\
&\le \sum_{i=1}^m \eta_{n,i}\beta_{n,i}\varphi(x_n, p) + \sum_{i=1}^m \eta_{n,i}(1 - \beta_{n,i}) \sum_{i=1}^m \omega_{n,i}\alpha_{n,i}\varphi(x_n, p) \\
&\quad + \sum_{i=1}^m \eta_{n,i}(1 - \beta_{n,i}) \sum_{i=1}^m \omega_{n,i}(1 - \alpha_{n,i})\bigl[\varphi(x_n, p) - \varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr)\bigr] \\
&= \varphi(x_n, p) - \sum_{i=1}^m \eta_{n,i}(1 - \beta_{n,i}) \sum_{i=1}^m \omega_{n,i}(1 - \alpha_{n,i})\varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr),
\end{aligned}$$

which implies that

$$\sum_{i=1}^m \eta_{n,i}(1 - \beta_{n,i}) \sum_{i=1}^m \omega_{n,i}(1 - \alpha_{n,i})\varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr) \le \varphi(x_n, p) - \varphi(x_{n+1}, p).$$

Using the assumptions and the result of Step 2, we know that $\varphi(x_n, J_{r_{n,i}}^{A_i}x_n) \to 0$, as $n \to \infty$, for $i = 1, 2, \ldots, m$. Then Lemma 1.4 ensures that $x_n - J_{r_{n,i}}^{A_i}x_n \to 0$, as $n \to \infty$, for $i = 1, 2, \ldots, m$.

Let $u_i = A_i v$, since A_i is d-accretive, then

$$\left\langle u_i - \frac{x_n - J_{r_{n,i}}^{A_i}x_n}{r_{n,i}}, Jv - JJ_{r_{n,i}}^{A_i}x_n\right\rangle \ge 0.$$

Since both $\{x_n\}$ and $\{J_{r_{n,i}}^{A_i} x_n\}$ are bounded, then letting $n \to \infty$ and using Lemma 1.1(iii), we have

$$\langle u_i, Jv - Jx\rangle \geq 0,$$

$i = 1, 2, \ldots, m$. That is, $\langle A_i J^{-1}(Jv), Jv - Jx\rangle \geq 0$, for $i = 1, 2, \ldots, m$. From Step 3 and Lemma 1.2, we know that $Jx \in (A_i J^{-1})^{-1}0$, which implies that $x \in A_i^{-1}0$. And then $x \in D$.

Step 6. $x_n \rightharpoonup v_0$, as $n \to \infty$, where v_0 is the unique element in D.

From Steps 2 and 5, we know that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_0 \in D$, as $i \to \infty$. If there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup v_1 \in D$, as $j \to \infty$, then from Step 2, we know that

$$\begin{aligned}\lim_{n\to\infty}\bigl[\varphi(x_n, v_0) - \varphi(x_n, v_1)\bigr] &= \lim_{i\to\infty}\bigl[\varphi(x_{n_i}, v_0) - \varphi(x_{n_i}, v_1)\bigr] \\ &= \lim_{i\to\infty}\bigl[\|v_0\|^2 - \|v_1\|^2 + 2\langle x_{n_i}, Jv_1 - Jv_0\rangle\bigr] \\ &= \|v_0\|^2 - \|v_1\|^2 + 2\langle v_0, Jv_1 - Jv_0\rangle. \end{aligned} \tag{2.5}$$

Similarly,

$$\begin{aligned}\lim_{n\to\infty}\bigl[\varphi(x_n, v_0) - \varphi(x_n, v_1)\bigr] &= \lim_{j\to\infty}\bigl[\varphi(x_{n_j}, v_0) - \varphi(x_{n_j}, v_1)\bigr] \\ &= \lim_{j\to\infty}\bigl[\|v_0\|^2 - \|v_1\|^2 + 2\langle x_{n_j}, Jv_1 - Jv_0\rangle\bigr] \\ &= \|v_0\|^2 - \|v_1\|^2 + 2\langle v_1, Jv_1 - Jv_0\rangle. \end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have $\langle v_1 - v_0, Jv_1 - Jv_0\rangle = 0$, which implies that $v_0 = v_1$ since J is strictly monotone.

This completes the proof. □

Remark 2.1 If E reduces to the Hilbert space H, then (2.1) becomes the iterative scheme for approximating common zeros of m-accretive mappings.

Remark 2.2 The iterative scheme (2.1) can be regarded as two-step block iterative scheme.

Remark 2.3 If $m = 1$, then (2.1) becomes to the following one approximating the zero point of an m-d-accretive mapping A:

$$\begin{cases} x_1 \in E, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(I + r_n A)^{-1} x_n, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(I + s_n A)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{2.7}$$

If, moreover, $\alpha_n \equiv 0$, $\beta_n \equiv 0$, then (2.7) becomes the so-called double-backward iterative scheme for the m-d-accretive mapping A:

$$\begin{cases} x_1 \in E, \\ y_n = (I + r_n A)^{-1} x_n, \\ x_{n+1} = (I + s_n A)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{2.8}$$

3 Strong convergence theorems

Theorem 3.1 *Let E be a real uniformly smooth and uniformly convex Banach space. Let $A_i : E \to E$ be a finite family of m-d-accretive mappings, $\{\omega_{n,i}\}, \{\eta_{n,i}\} \subset (0,1]$, $\{\alpha_{n,i}\}, \{\beta_{n,i}\} \subset [0,1)$, $\{r_{n,i}\}, \{s_{n,i}\} \subset (0,+\infty)$, for $i = 1,2,\ldots,m$. $\sum_{i=1}^m \omega_{n,i} = 1$ and $\sum_{i=1}^m \eta_{n,i} = 1$. Let $D := \bigcap_{i=1}^m A_i^{-1}0 \neq \emptyset$. Suppose the normalized duality mapping $J : E \to E^*$ is weakly sequentially continuous. Let $\{x_n\}$ be generated by the iterative scheme*:

$$\begin{cases} x_0 \in E, \\ u_n = \sum_{i=1}^m \omega_{n,i}[\alpha_{n,i}x_n + (1-\alpha_{n,i})(I + r_{n,i}A_i)^{-1}x_n], \\ v_n = \sum_{i=1}^m \eta_{n,i}[\beta_{n,i}x_n + (1-\beta_{n,i})(I + s_{n,i}A_i)^{-1}u_n], \\ H_0 = E, \\ H_{n+1} = \{z \in H_n : \varphi(u_n,z) \leq \varphi(x_n,z)\}, \\ x_{n+1} = R_{H_{n+1}}x_0, \quad n \geq 0. \end{cases} \tag{3.1}$$

Suppose the following conditions are satisfied:

(i) $\limsup_{n\to\infty} \alpha_{n,i} < 1$, $\limsup_{n\to\infty} \beta_{n,i} < 1$, *for* $i = 1,2,\ldots,m$;

(ii) $\liminf_{n\to\infty} \eta_{n,i} > 0$, $\liminf_{n\to\infty} \omega_{n,i} > 0$, *for* $i = 1,2,\ldots,m$;

(iii) $\inf_{n\geq 0} r_{n,i} > 0$, $\inf_{n\geq 0} s_{n,i} > 0$, *for* $i = 1,2,\ldots,m$.

Then $\{x_n\}$ converges strongly to $p_0 = R_D x_0$, where R_D is the sunny generalized non-expansive retraction from E onto D, as $n \to \infty$.

Proof We split the proof into six steps.

Step 1. $\{x_n\}$ is well defined.

Noticing that

$$\varphi(u_n,z) \leq \varphi(x_n,z) \quad \Longleftrightarrow \quad \|u_n\|^2 - \|x_n\|^2 \leq 2\langle u_n - x_n, Jz\rangle,$$

then from Lemma 1.1(iii), we can easily know that H_n ($n \geq 0$) is a closed subset of E.

For $\forall p \in D$, using (2.2), we know that

$$\begin{aligned} \varphi(u_n,p) &\leq \sum_{i=1}^m \omega_{n,i}\bigl[\alpha_{n,i}\varphi(x_n,p) + (1-\alpha_{n,i})\varphi\bigl(J_{r_{n,i}}^{A_i}x_n,p\bigr)\bigr] \\ &\leq \sum_{i=1}^m \omega_{n,i}\bigl[\alpha_{n,i}\varphi(x_n,p) + (1-\alpha_{n,i})\varphi(x_n,p)\bigr] = \varphi(x_n,p), \end{aligned}$$

which implies that $p \in H_n$. Thus $\emptyset \neq D \subset H_n$, for $n \geq 0$.

Since H_n is a nonempty closed subset of E, there exists a unique sunny generalized non-expansive retraction from E onto H_n, which is denoted by R_{H_n}. Therefore, $\{x_n\}$ is well defined.

Step 2. $\{x_n\}$ is bounded.

Using Lemma 1.3, $\varphi(x_{n+1},p) \leq \varphi(x_0,p)$, $\forall p \in D \subset H_{n+1}$. Thus $\{\varphi(x_n,p)\}$ is bounded and then (1.1) ensures that $\{x_n\}$ is bounded.

Step 3. $\omega(x_n) \subset D$, where $\omega(x_n)$ denotes the set of all of the weak limit points of the weakly convergent subsequences of $\{x_n\}$.

Since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$, for simplicity, we still denote it by $\{x_n\}$ such that $x_n \rightharpoonup x$, as $n \to \infty$.

Since $x_{n+1} \in H_{n+1} \subset H_n$, using Lemma 1.3, we have

$$\varphi(x_n, x_{n+1}) + \varphi(x_0, x_n) \leq \varphi(x_0, x_{n+1}),$$

which implies that $\lim_{n\to\infty} \varphi(x_0, x_n)$ exists. Thus $\varphi(x_n, x_{n+1}) \to 0$. Lemma 1.4 implies that $x_n - x_{n+1} \to 0$, as $n \to \infty$.

Since $x_{n+1} \in H_{n+1} \subset H_n$, then $\varphi(u_n, x_{n+1}) \leq \varphi(x_n, x_{n+1}) \to 0$, which implies that $x_n - u_n \to 0$, as $n \to \infty$.

$\forall p \in D$, using (2.2) again, we have

$$\begin{aligned}
\varphi(u_n, p) &\leq \sum_{i=1}^{m} \omega_{n,i}\bigl[\alpha_{n,i}\varphi(x_n, p) + (1-\alpha_{n,i})\varphi\bigl(J_{r_{n,i}}^{A_i}x_n, p\bigr)\bigr] \\
&\leq \sum_{i=1}^{m} \omega_{n,i}\alpha_{n,i}\varphi(x_n, p) + \sum_{i=1}^{m} \omega_{n,i}(1-\alpha_{n,i})\bigl[\varphi(x_n, p) - \varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr)\bigr] \\
&= \varphi(x_n, p) - \sum_{i=1}^{m} \omega_{n,i}(1-\alpha_{n,i})\varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i=1}^{m} \omega_{n,i}(1-\alpha_{n,i})\varphi\bigl(x_n, J_{r_{n,i}}^{A_i}x_n\bigr) \\
&\quad\leq \varphi(x_n, p) - \varphi(u_n, p) = \|x_n\|^2 - \|u_n\|^2 - 2\langle x_n - u_n, Jp\rangle \to 0,
\end{aligned}$$

as $n \to \infty$. Lemma 1.4 implies that $x_n - J_{r_{n,i}}^{A_i}x_n \to 0$, as $n \to \infty$, where $i = 1, 2, \ldots, m$.

The remaining part is similar to that of Step 5 in Theorem 2.1, then we have $\omega(x_n) \subset D$.

Step 4. $\{x_n\}$ is a Cauchy sequence.

If, on the contrary, there exist two subsequences $\{n_k\}$ and $\{m_k\}$ of $\{n\}$ such that $\|x_{n_k+m_k} - x_{n_k}\| \geq \varepsilon_0$, $\forall k \geq 1$.

Since $\lim_{n\to\infty} \varphi(x_0, x_n)$ exists, using Lemma 1.3 again,

$$\begin{aligned}
\varphi(x_{n_k}, x_{n_k+m_k}) &\leq \varphi(x_0, x_{n_k+m_k}) - \varphi(x_0, x_{n_k}) \\
&= \varphi(x_0, x_{n_k+m_k}) - \lim_{k\to\infty} \varphi(x_0, x_{n_k+m_k}) \\
&\quad + \lim_{k\to\infty} \varphi(x_0, x_{n_k}) - \varphi(x_0, x_{n_k}) \to 0,
\end{aligned}$$

as $k \to \infty$. Lemma 1.4 implies that $\lim_{k\to\infty} \|x_{n_k+m_k} - x_{n_k}\| = 0$, which makes a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

Step 5. D is a closed subset of E.

Let $z_n \in D$ with $z_n \to z$, as $n \to \infty$. Then $A_i z_n = 0$, for $i = 1, 2, \ldots, m$. In view of Lemma 1.1(ii), there exists $z_n^* \in E^*$ such that $z_n = J^{-1}z_n^*$. Using Lemma 1.1(iii), $z_n^* \to Jz$, as $n \to \infty$. Since $A_i J^{-1} z_n^* = 0$, $z_n^* \to Jz$ and $A_i J^{-1}$ is maximal monotone, Lemma 1.2 ensures that $Jz \in (A_i J^{-1})^{-1}0$. Thus, $z \in A_i^{-1}0$, for $i = 1, 2, \ldots, m$. And then $z \in D$. Therefore, D is closed subset of E, which ensures there exists a unique sunny generalized non-expansive retraction R_D from E onto D.

Step 6. $x_n \to q_0 = R_D x_0$, as $n \to \infty$.

Since $\{x_n\}$ is a Cauchy sequence, there exists $q_0 \in E$ such that $x_n \to q$, as $n \to \infty$. From Step 5, $q_0 \in D$.

Now, we prove that $q_0 = R_D x_0$.

Using Lemma 1.3, we have the following two inequalities:

$$\varphi(x_0, R_D x_0) + \varphi(R_D x_0, q_0) \leq \varphi(x_0, q_0) \tag{3.2}$$

and

$$\varphi(x_0, x_n) + \varphi(x_n, R_D x_0) \leq \varphi(x_0, R_D x_0). \tag{3.3}$$

Letting $n \to +\infty$, from (3.3), we know that

$$\varphi(x_0, q_0) + \varphi(q_0, R_D x_0) \leq \varphi(x_0, R_D x_0). \tag{3.4}$$

From (3.2) and (3.4), $0 \leq \varphi(R_D x_0, q_0) \leq -\varphi(q_0, R_D x_0) \leq 0$. Thus $\varphi(R_D x_0, q_0) = 0$. So in view of Lemma 1.4, $q = R_D x_0$.

This completes the proof. □

Remark 3.1 Combining the techniques of convex combination and the retraction, the strong convergence of iterative scheme (3.1) is obtained.

Remark 3.2 It is obvious that the restrictions in Theorems 2.1 and 3.1 are weaker.

4 Connection between nonlinear mappings and nonlinear elliptic equations

We have mentioned that in a Hilbert space, m-d-accretive mappings and m-accretive mappings are the same, while in a non-Hilbertian Banach space, they are different. So, there are many examples of m-d-accretive mappings in Hilbert spaces. Can we find one mapping that is (m-)d-accretive but not (m-)accretive?

In Section 4.1, we shall review the work done in [12], where an m-accretive mapping is constructed for discussing the existence of solution of one kind nonlinear elliptic equations.

In Section 4.2, we shall construct an m-d-accretive mapping based on the same nonlinear elliptic equation presented in Section 4.1, from which we can see that it is quite different from the m-accretive mapping defined in Section 4.1.

4.1 *m*-Accretive mappings and nonlinear elliptic equations

The following nonlinear elliptic boundary value problem is extensively studied in [12, 13]:

$$\begin{cases} -\operatorname{div}(\alpha(\operatorname{grad} u)) + |u|^{p-2}u + g(x, u(x)) = f(x), & \text{a.e. in } \Omega, \\ -\langle \vartheta, \alpha(\operatorname{grad} u)\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma. \end{cases} \tag{4.1}$$

In (4.1), Ω is a bounded conical domain of a Euclidean space R^N with its boundary $\Gamma \in C^1$ (see [14]). $f \in L^s(\Omega)$ is a given function, ϑ is the exterior normal derivative of Γ, $g: \Omega \times R \to R$ is a given function satisfying Carathéodory's conditions such that the mapping $u \in L^s(\Omega) \to g(x, u(x)) \in L^s(\Omega)$ is defined and there exists a function $T(x) \in L^s(\Omega)$ such that $g(x,t)t \geq 0$, for $|t| \geq T(x)$ and $x \in \Omega$. β_x is the subdifferential of a proper, convex, and semi-lower-continuous function. $\alpha : R^N \to R^N$ is a monotone and continuous function, and there exist positive constants k_1, k_2, and k_3 such that, for $\forall \xi, \xi' \in R^N$, the following conditions are satisfied:

(i) $|\alpha(\xi)| \leq k_1 |\xi|^{p-1}$;
(ii) $|\alpha(\xi) - \alpha(\xi')| \leq k_2 ||\xi|^{p-2}\xi - |\xi'|^{p-2}\xi'|$;
(iii) $\langle\langle \alpha(\xi), \xi \rangle\rangle \geq k_3 |\xi|^p$,

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the inner product in R^N.

In [12], they first present the following definitions.

Definition 4.1 [12] Define the mapping $B_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ by

$$(v, B_p u) = \int_\Omega \langle\langle \alpha(\operatorname{grad} u), \operatorname{grad} v \rangle\rangle \, dx + \int_\Omega |u(x)|^{p-2} u(x) v(x) \, dx,$$

for any $u, v \in W^{1,p}(\Omega)$.

Definition 4.2 [12] Define the mapping $\Phi_p : W^{1,p}(\Omega) \to R$ by $\Phi_p(u) = \int_\Gamma \varphi_x(u|_\Gamma(x)) \, d\Gamma(x)$, for any $u \in W^{1,p}(\Omega)$.

Definition 4.3 [12] Define a mapping $A : L^2(\Omega) \to 2^{L^2(\Omega)}$ as follows:

$$D(A) = \{ u \in L^2(\Omega) \mid \text{there exists an } f \in L^2(\Omega) \text{ such that } f \in B_p u + \partial\Phi_p(u) \}.$$

For $u \in D(A)$, $Au = \{ f \in L^2(\Omega) \mid f \in B_p u + \partial\Phi_p(u) \}$.

Definition 4.4 [12] Define a mapping $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ as follows:

(i) If $s \geq 2$, then

$$D(A_s) = \{ u \in L^s(\Omega) \mid \text{there exists an } f \in L^s(\Omega) \text{ such that } f \in B_p u + \partial\Phi_p(u) \}.$$

For $u \in D(A_s)$, we set $A_s u = \{ f \in L^s(\Omega) \mid f \in B_p u + \partial\Phi_p(u) \}$.

(ii) If $1 < s < 2$, then define $A_s : L^s(\Omega) \to 2^{L^s(\Omega)}$ as the L^s-closure of $A : L^2(\Omega) \to 2^{L^2(\Omega)}$ defined in Definition 4.3.

Then they get the following important result in [12].

Proposition 4.1 [12] *Both A and A_s are m-accretive mapping.*

Later, by using the perturbations on ranges of m-accretive mappings, the sufficient condition on the existence of solution of (4.1) is discussed.

Theorem 4.1 [12] *If $f \in L^s(\Omega)$ ($\frac{2N}{N+1} < p \leq s < +\infty$) satisfies the following condition*:

$$\int_\Gamma \beta_-(x) \, d\Gamma(x) + \int_\Omega g_-(x) \, dx < \int_\Omega f(x) \, dx < \int_\Gamma \beta_+(x) \, d\Gamma(x) + \int_\Omega g_+(x) \, dx,$$

then (4.1) *has a solution in $L^s(\Omega)$.*

The meaning of $\beta_-(x)$, $\beta_+(x)$, $g_-(x)$, and $g_+(x)$ can be found in the following two definitions.

Definition 4.5 [12, 14] For $t \in R$ and $x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \emptyset$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in the case $\beta_x(t) = \emptyset$. Finally, let $\beta_\pm(x) = \lim_{t\to\pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. Then $\beta_\pm(x)$ define measurable functions on Γ.

Definition 4.6 [12, 14] Define $g_+(x) = \liminf_{t\to+\infty} g(x,t)$ and $g_-(x) = \limsup_{t\to-\infty} g(x,t)$.

4.2 Examples of *m-d*-accretive mappings

Now, based on nonlinear elliptic problem (4.1), we are ready to give the example of *m-d*-accretive mapping in the sequel.

Lemma 4.1 [10] *Let E be a Banach space, if $B : E \to 2^{E^*}$ is an everywhere defined, monotone, and hemi-continuous mapping, then B is maximal monotone.*

Definition 4.7 Let $1 < p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Define the mapping $B : W^{1,p'}(\Omega) \to (W^{1,p'}(\Omega))^*$ by

$$\langle v, Bu\rangle = \int_\Omega \bigl\langle\!\bigl\langle \alpha\bigl(\operatorname{grad}\bigl(|u|^{p'-1}\operatorname{sgn} u\|u\|_{p'}^{2-p'}\bigr)\bigr), \operatorname{grad}\bigl(|v|^{p'-1}\operatorname{sgn} v\|v\|_{p'}^{2-p'}\bigr)\bigr\rangle\!\bigr\rangle\,dx,$$

for any $u, v \in W^{1,p'}(\Omega)$.

Proposition 4.2 $B : W^{1,p'}(\Omega) \to (W^{1,p'}(\Omega))^*$ $(1 < p \leq 2)$ *is maximal monotone.*

Proof We split the proof into four steps.

Step 1. B is everywhere defined.

In fact, for $u, v \in W^{1,p'}(\Omega)$, from the property (i) of α, we have

$$\begin{aligned}
\bigl|\langle v, Bu\rangle\bigr| &\leq k_1 \int_\Omega \bigl|\operatorname{grad}\bigl(|u|^{p'-1}\operatorname{sgn} u\|u\|_{p'}^{2-p'}\bigr)\bigr|^{p-1}\bigl|\operatorname{grad}\bigl(|v|^{p'-1}\operatorname{sgn} v\|v\|_{p'}^{2-p'}\bigr)\bigr|\,dx \\
&= k_1\bigl(p'-1\bigr)^p \|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'} \int_\Omega |\operatorname{grad} u|^{p-1}|u|^{2-p}|\operatorname{grad} v||v|^{p'-2}\,dx \\
&\leq k_1\bigl(p'-1\bigr)^p \|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'} \left(\int_\Omega |\operatorname{grad} u|^p |u|^{p'-p}\,dx\right)^{\frac{1}{p'}} \\
&\quad \times \left(\int_\Omega |\operatorname{grad} v|^p |v|^{p'-p}\,dx\right)^{\frac{1}{p}} \\
&\leq k_1\bigl(p'-1\bigr)^p \|u\|_{p'}^{(p-1)(2-p')}\|v\|_{p'}^{2-p'} \left(\int_\Omega |\operatorname{grad} u|^{p'}\,dx\right)^{\frac{p}{(p')^2}} \\
&\quad \times \left(\int_\Omega |u|^{p'}\,dx\right)^{\frac{p'-p}{(p')^2}} \left(\int_\Omega |\operatorname{grad} v|^{p'}\,dx\right)^{\frac{1}{p'}} \left(\int_\Omega |v|^{p'}\,dx\right)^{\frac{p'-p}{p'p}} \\
&\leq \text{const.}\|u\|_{1,p'}^{p-1}\|v\|_{1,p'}.
\end{aligned}$$

Thus B is everywhere defined.

Step 2. B is monotone.

Since α is monotone, then, for $u, v \in D(B)$,

$$\begin{aligned}
&\langle u - v, Bu - Bv\rangle \\
&\quad = \int_\Omega \bigl\langle \bigl\langle \alpha\bigl(\operatorname{grad}\bigl(|u|^{p'-1} \operatorname{sgn} u \|u\|_{p'}^{2-p'}\bigr)\bigr) - \alpha\bigl(\operatorname{grad}\bigl(|v|^{p'-1} \operatorname{sgn} v \|v\|_{p'}^{2-p'}\bigr)\bigr), \\
&\qquad \operatorname{grad}\bigl(|u|^{p'-1} \operatorname{sgn} u \|u\|_{p'}^{2-p'}\bigr) - \operatorname{grad}\bigl(|v|^{p'-1} \operatorname{sgn} v \|v\|_{p'}^{2-p'}\bigr)\bigr\rangle\bigr\rangle \, dx \geq 0,
\end{aligned}$$

which implies that B is monotone.

Step 3. B is hemi-continuous.

To show that B is hemi-continuous. It suffices to prove that for $u, v, w \in W^{1,p'}(\Omega)$ and $t \in [0,1]$, $\langle w, B(u + tv) - Bu\rangle$ as $t \to 0$.

In fact, since α is continuous,

$$\begin{aligned}
&\bigl|\bigl\langle w, B(u + tv) - Bu\bigr\rangle\bigr| \\
&\quad \leq \int_\Omega \bigl|\alpha\bigl(\operatorname{grad}\bigl(|u + tv|^{p'-1} \operatorname{sgn}(u + tv) \|u + tv\|_{p'}^{2-p'}\bigr)\bigr) - \alpha\bigl(\operatorname{grad}\bigl(|u|^{p'-1} \operatorname{sgn} u \|u\|_{p'}^{2-p'}\bigr)\bigr)\bigr| \\
&\qquad \times \bigl|\operatorname{grad}\bigl(|w|^{p'-1} \operatorname{sgn} w \|w\|_{p'}^{2-p'}\bigr)\bigr| \, dx \to 0,
\end{aligned}$$

as $t \to 0$.

Step 4. B is maximal monotone.

Lemma 4.1 implies that B is maximal monotone.

This completes the proof. □

Remark 4.1 [10] There exists a maximal monotone extension of B from $L^{p'}(\Omega)$ to $L^p(\Omega)$, which is denoted by $\widetilde{B}$.

Definition 4.8 For $1 < p \leq 2$, the normalized duality mapping $J : L^{p'}(\Omega) \to L^p(\Omega)$ is defined by

$$Ju = |u|^{p'-1} \operatorname{sgn} u \|u\|_{p'}^{2-p'},$$

for $u \in L^{p'}(\Omega)$.

Define the mapping $A : L^p(\Omega) \to L^p(\Omega)$ $(1 < p \leq 2)$ as follows:

$$Au = \widetilde{B}J^{-1}u, \quad u \in L^p(\Omega).$$

Proposition 4.3 *The mapping $A : L^p(\Omega) \to L^p(\Omega)$ $(1 < p \leq 2)$ is m-d-accretive.*

Proof Since $\widetilde{B}$ is monotone, for $\forall u, v \in D(A)$,

$$\bigl\langle Au - Av, J^{-1}u - J^{-1}v\bigr\rangle = \bigl\langle \widetilde{B}J^{-1}u - \widetilde{B}J^{-1}v, J^{-1}u - J^{-1}v\bigr\rangle \geq 0.$$

Thus A is d-accretive.

In view of Remark 4.1, $\widetilde{B}$ is maximal monotone, then $R(J + \lambda\widetilde{B}) = L^p(\Omega)$, for $\forall\lambda > 0$.

For $\forall f \in L^p(\Omega)$, there exists $u \in L^{p'}(\Omega)$ such that $Ju + \lambda\widetilde{B}u = f$. Using Lemma 1.1 again, there exists $u^* \in L^p(\Omega)$ such that $u = J^{-1}u^*$. Then $u^* + \lambda\widetilde{B}J^{-1}u^* = f$. Thus $f \in R(I + \lambda A)$ and then $R(I + \lambda A) = L^p(\Omega)$, for $\lambda > 0$.

Thus A is m-d-accretive.

This completes the proof. □

Proposition 4.4 $A^{-1}0 = \{u \in L^p(\Omega) : u(x) \equiv \text{const.}\}$.

Proof It is obvious that $\{u \in L^p(\Omega) : u(x) \equiv \text{const.}\} \subset A^{-1}0$.

On the other hand, if $u(x) \in A^{-1}0$, then $Au(x) \equiv 0$. Let $u^* \in L^{p'}(\Omega)$ be such that $u = Ju^*$. From the property (iii) of α, we have

$$0 = \langle u^*, AJu^* \rangle \geq k_3 \int_\Omega \left|\operatorname{grad}\left(|u^*|^{p'-1} \operatorname{sgn} u^* \|u^*\|_{p'}^{2-p'}\right)\right|^p dx = k_3 \int_\Omega \left|\operatorname{grad}\left(Ju^*\right)\right|^p dx,$$

which implies that $u = Ju^* \equiv \text{const.}$

Thus $A^{-1}0 \subset \{u \in L^p(\Omega) : u(x) \equiv \text{const.}\}$.

This completes the proof. □

Remark 4.2 From Propositions 4.2 and 4.3, we know that the restriction on the m-d-accretive mapping in Theorem 2.1 or 3.1 that $A^{-1}0 \neq \emptyset$ is valid.

Remark 4.3 If (4.1) is reduced to the following:

$$-\operatorname{div}\left(\alpha(\operatorname{grad} u)\right) = 0, \quad \text{a.e. in } \Omega, \tag{4.2}$$

then it is not difficult to see that $u \in A^{-1}0$ is exactly the solution of (4.2), from which we cannot only see the connections between the zeros of an m-d-accretive mapping and the nonlinear equation but also see that the work on designing the iterative schemes to approximate zeros of nonlinear mappings is meaningful.

Competing interests

The authors declare that there is no conflict of interests regarding the publication of this article.

Authors' contributions

The main idea was proposed by LW, and YL and RPA participated in the research. All authors read and approved the final manuscript.

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