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Strong convergence of iterative algorithms for the split equality problem

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Abstract

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$, $Q \subseteq H_2$ be two nonempty closed convex sets, and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The split equality problem (SEP) is finding $x \in C$, $y \in Q$ such that $Ax = By$. Recently, Moudafi has presented the ACQA algorithm and the RACQA algorithm to solve SEP. However, the two algorithms are weakly convergent. It is therefore the aim of this paper to construct new algorithms for SEP so that strong convergence is guaranteed. Firstly, we define the concept of the minimal norm solution of SEP. Using Tychonov regularization, we introduce two methods to get such a minimal norm solution. And then, we introduce two algorithms which are viewed as modifications of Moudafi's ACQA, RACQA algorithms and KM-CQ algorithm, respectively, and converge strongly to a solution of SEP. More importantly, the modifications of Moudafi's ACQA, RACQA algorithms converge strongly to the minimal norm solution of SEP. At last, we introduce some other algorithms which converge strongly to a solution of SEP.

Keywords: split equality problem; iterative algorithms; converge strongly

1 Introduction and preliminaries

Let C and Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and let $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is to find a point x satisfying the property

$$x \in C, \qquad Ax \in Q$$

if such a point exists. SFP was first introduced by Censor and Elfving [1], which has attracted many authors' attention due to its application in signal processing [1]. Various algorithms have been invented to solve it (see [2–7]).

Recently, Moudafi [8] proposed a new *split equality problem* (SEP): Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$, $Q \subseteq H_2$ be two nonempty closed convex sets, and let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Find $x \in C$, $y \in Q$ satisfying

$$Ax = By. \tag{1.1}$$

When $B = I$, SEP reduces to the well-known SFP. In the paper [8], Moudafi gave the following iterative algorithms for solving the split equality problem.

Alternating CQ-algorithm (ACQA):

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)). \end{cases}$$

Relaxed alternating CQ-algorithm (RACQA):

$$\begin{cases} x_{k+1} = P_{C_k}(x_k - \gamma A^*(Ax_k - By_k)); \\ y_{k+1} = P_{Q_k}(y_k + \beta B^*(Ax_{k+1} - By_k)). \end{cases}$$

However, the above algorithms converge weakly to a solution of SEP.

It is therefore the aim of this paper to construct a new algorithm for SEP so that strong convergence is guaranteed. The paper is organized as follows. In Section 2, we define the concept of the minimal norm solution of SEP (1.1). Using Tychonov regularization, we obtain a net of solutions for some minimization problem approximating such minimal norm solutions (see Theorem 2.4). In Section 3, we introduce an algorithm which is viewed as a modification of Moudafi's ACQA and RACQA algorithms; and we prove the strong convergence of the algorithm, more importantly, its limit is the minimum-norm solution of SEP (1.1) (see Theorem 3.2). In Section 4, we introduce a KM-CQ-like iterative algorithm which converges strongly to a solution of SEP (1.1) (see Theorem 4.3). In Section 5, we introduce some other iterative algorithms which converge strongly to a solution of SEP (1.1).

Throughout the rest of this paper, I denotes the identity operator on a Hilbert space H, $\mathrm{Fix}(T)$ is the set of the fixed points of an operator T and ∇f is the gradient of the functional $f : H \to R$. An operator T on a Hilbert space H is *nonexpansive* if, for each x and y in H, $\|Tx - Ty\| \le \|x - y\|$. T is said to be *averaged* if there exists $0 < \alpha < 1$ and a nonexpansive operator N such that $T = (1 - \alpha)I + \alpha N$.

Let P_S denote the projection from H onto a nonempty closed convex subset S of H; that is,

$$P_S(w) = \min_{x \in S} \|x - w\|.$$

It is well known that $P_S(w)$ is characterized by the inequality

$$\langle w - P_S(w), x - P_S(w) \rangle \le 0, \quad \forall x \in S,$$

and P_S is nonexpansive and averaged.

We now collect some elementary facts which will be used in the proofs of our main results.

Lemma 1.1 [9, 10] *Let X be a Banach space, C be a closed convex subset of X, and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Lemma 1.2 [11] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\{u_n\}$ be a sequence of nonnegative real numbers* *with* $\sum_{n=1}^{\infty} u_n < \infty$, *and* $\{t_n\}$ *be a sequence of real numbers with* $\limsup_n t_n \leq 0$. *Suppose that*

$$s_{n+1} = (1 - \alpha_n)s_n + \alpha_n t_n + u_n, \quad \forall n \in N.$$

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 1.3 [12] *Let* $\{w_n\}$, $\{z_n\}$ *be bounded sequences in a Banach space, and let* $\{\beta_n\}$ *be a sequence in* $[0,1]$ *which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose that $w_{n+1} = (1 - \beta_n)w_n + \beta_n z_n$ *and* $\limsup_{n\to\infty} \|z_{n+1} - z_n\| - \|w_{n+1} - w_n\| \leq 0$, *then* $\lim_{n\to\infty} \|z_n - w_n\| = 0$.

Lemma 1.4 [13] *Let* f *be a convex and differentiable functional, and let* C *be a closed convex subset of* H. *Then* $x \in C$ *is a solution of the problem*

$$\min_{x\in C} f(x)$$

if and only if $x \in C$ *satisfies the following optimality condition*:

$$\langle \nabla f(x), \nu - x \rangle \geq 0, \quad \forall \nu \in C.$$

Moreover, if f *is, in addition, strictly convex and coercive, then the minimization problem has a unique solution*.

Lemma 1.5 [3] *Let* A *and* B *be averaged operators and suppose that* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B)$ *is nonempty. Then* $\mathrm{Fix}(A) \cap \mathrm{Fix}(B) = \mathrm{Fix}(AB) = \mathrm{Fix}(BA)$.

2 Minimum-norm solution of SEP

In this section, we define the concept of the minimal norm solution of SEP (1.1). Using Tychonov regularization, we obtain a net of solutions for some minimization problem approximating such minimal norm solutions.

We use Γ to denote the solution set of SEP, *i.e.*,

$$\Gamma = \{(x, y) \in H_1 \times H_2, Ax = By, x \in C, y \in Q\}$$

and assume the consistency of SEP so that Γ is closed, convex and nonempty.

Let $S = C \times Q$ in $H = H_1 \times H_2$, define $G : H \to H_3$ by $G = [A, -B]$, then $G^*G : H \to H$ has the matrix form

$$G^*G = \begin{bmatrix} A^*A & -A^*B \\ -B^*A & B^*B \end{bmatrix}.$$

The original problem can now be reformulated as finding $w = (x, y) \in S$ with $Gw = 0$, or, more generally, minimizing the function $\|Gw\|$ over $w \in S$. Therefore solving SEP (1.1) is equivalent to solving the following minimization problem:

$$\min_{w\in S} f(w) = \frac{1}{2}\|Gw\|^2, \tag{2.1}$$

which is in general ill-posed. A classical way to deal with such a possibly ill-posed problem is the well-known Tychonov regularization, which approximates a solution of problem (2.1) by the unique minimizer of the regularized problem:

$$\min_{w\in S} f_\alpha(w) = \frac{1}{2}\|Gw\|^2 + \frac{1}{2}\alpha\|w\|^2, \tag{2.2}$$

where $\alpha > 0$ is the regularization parameter. Denote by $w_\alpha = (x_\alpha, y_\alpha)$ the unique solution of (2.2).

Proposition 2.1 *For any $\alpha > 0$, the solution $w_\alpha = (x_\alpha, y_\alpha)$ of* (2.2) *is uniquely defined. Moreover, $w_\alpha = (x_\alpha, y_\alpha)$ is characterized by the inequality*

$$\langle G^*Gw_\alpha + \alpha w_\alpha, w - w_\alpha\rangle \geq 0, \quad \forall w \in S,$$

i.e.,

$$\langle A^*(Ax_\alpha - By_\alpha) + \alpha x_\alpha, x - x_\alpha\rangle \geq 0, \quad \forall x \in C;$$

and

$$\langle -B^*(Ax_\alpha - By_\alpha) + \alpha y_\alpha, y - y_\alpha\rangle \geq 0, \quad \forall y \in Q.$$

Proof It is well known that $f(w) = \frac{1}{2}\|Gw\|^2$ is convex and differentiable with gradient $\nabla f(w) = G^*Gw$, $f_\alpha(w) = f(w) + \frac{1}{2}\alpha\|w\|^2$. We can get that f_α is strictly convex, coercive, and differentiable with gradient

$$\nabla f_\alpha(w) = G^*Gw + \alpha w.$$

It follows from Lemma 1.4 that w_α is characterized by the inequality

$$\langle G^*Gw_\alpha + \alpha w_\alpha, w - w_\alpha\rangle \geq 0, \quad \forall w \in S. \tag{2.3}$$

Note that $\{(x,0), x \in C\} \subseteq S$, $\{(0,y), y \in Q\} \subseteq S$, adding up (2.3), we can get that

$$\langle A^*(Ax_\alpha - By_\alpha) + \alpha x_\alpha, x - x_\alpha\rangle \geq 0, \quad \forall x \in C;$$

and

$$\langle -B^*(Ax_\alpha - By_\alpha) + \alpha y_\alpha, y - y_\alpha\rangle \geq 0, \quad \forall y \in Q. \qquad \square$$

Definition 2.2 An element $\tilde{w} = (\tilde{x}, \tilde{y}) \in \Gamma$ is said to be the *minimal norm solution* of SEP (1.1) if $\|\tilde{w}\| = \inf_{w\in\Gamma} \|w\|$.

The next result collects some useful properties of $\{w_\alpha\}$, the unique solution of (2.2).

Proposition 2.3 *Let w_α be given as the unique solution of* (2.2). *Then the following assertions hold.*

(i) $\|w_\alpha\|$ *is decreasing for* $\alpha \in (0,\infty)$.

(ii) $\alpha \mapsto w_\alpha$ *defines a continuous curve from* $(0,\infty)$ *to* H.

Proof Let $\alpha > \beta > 0$; since w_α and w_β are the unique minimizers of f_α and f_β, respectively, we can get that

$$\frac{1}{2}\|Gw_\alpha\|^2 + \frac{1}{2}\alpha\|w_\alpha\|^2 \le \frac{1}{2}\|Gw_\beta\|^2 + \frac{1}{2}\alpha\|w_\beta\|^2,$$
$$\frac{1}{2}\|Gw_\beta\|^2 + \frac{1}{2}\beta\|w_\beta\|^2 \le \frac{1}{2}\|Gw_\alpha\|^2 + \frac{1}{2}\beta\|w_\alpha\|^2.$$

Hence we can obtain that $\|w_\alpha\| \le \|w_\beta\|$. That is to say, $\|w_\alpha\|$ is decreasing for $\alpha \in (0,\infty)$.

By Proposition 2.1, we have

$$\langle G^*Gw_\alpha + \alpha w_\alpha, w_\beta - w_\alpha\rangle \ge 0$$

and

$$\langle G^*Gw_\beta + \beta w_\beta, w_\alpha - w_\beta\rangle \ge 0.$$

It follows that

$$\langle w_\alpha - w_\beta, \alpha w_\alpha - \beta w_\beta\rangle \le \langle w_\alpha - w_\beta, G^*G(w_\beta - w_\alpha)\rangle \le 0.$$

Hence

$$\alpha\|w_\alpha - w_\beta\| \le (\alpha - \beta)\langle w_\alpha - w_\beta, w_\beta\rangle.$$

It turns out that

$$\|w_\alpha - w_\beta\|^2 \le \frac{|\alpha - \beta|}{\alpha}\|w_\beta\|.$$

Thus $\alpha \mapsto w_\alpha$ defines a continuous curve from $(0,\infty)$ to H. □

Theorem 2.4 *Let w_α be given as the unique solution of* (2.2). *Then w_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\tilde{w}$ of SEP* (1.1).

Proof For any $0 < \alpha < \infty$, w_α is given as (2.2), it follows that

$$\frac{1}{2}\|Gw_\alpha\|^2 + \frac{1}{2}\alpha\|w_\alpha\|^2 \le \frac{1}{2}\|G\tilde{w}\|^2 + \frac{1}{2}\alpha\|\tilde{w}\|^2.$$

Since $\tilde{w} \in \Gamma$ is a solution for SEP, we get

$$\frac{1}{2}\|Gw_\alpha\|^2 + \frac{1}{2}\alpha\|w_\alpha\|^2 \le \frac{1}{2}\alpha\|\tilde{w}\|^2.$$

Hence, $\|w_\alpha\| \le \|\tilde{w}\|$ for all $\alpha > 0$. That is to say, $\{w_\alpha\}$ is a bounded net in $H = H_1 \times H_2$.

For any sequence $\{\alpha_n\}$ such that $\lim_n \alpha_n = 0$, let w_{α_n} be abbreviated as w_n. All we need to prove is that $\{w_n\}$ contains a subsequence converging strongly to $\tilde{w}$.

Indeed $\{w_n\}$ is bounded and S is bounded convex. By passing to a subsequence if necessary, we may assume that $\{w_n\}$ converges weakly to a point $\hat{w} \in S$. By Proposition 2.1, we get that

$$\left\langle G^*Gw_n + \alpha_n w_n, \tilde{w} - w_n \right\rangle \geq 0.$$

It follows that

$$\langle Gw_n, G\tilde{w} - Gw_n \rangle \geq \alpha_n \langle w_n, w_n - \tilde{w} \rangle.$$

Since $\tilde{w} \in \Gamma$, it turns out that

$$\langle Gw_n, -Gw_n \rangle \geq \alpha_n \langle w_n, w_n - \tilde{w} \rangle.$$

Using $\|w_n\| \leq \|\tilde{w}\|$, we can get that

$$\|Gw_n\| \leq 2\alpha_n \|\tilde{w}\| \to 0.$$

Furthermore, note that $\{w_n\}$ converges weakly to a point $\hat{w} \in S$, then $\{Gw_n\}$ converges weakly to $G\hat{w}$. It follows that $G\hat{w} = 0$, *i.e.*, $\hat{w} \in \Gamma$.

At last, we prove that $\hat{w} = \tilde{w}$ and this finishes the proof.

Since $\{w_n\}$ converges weakly to $\hat{w}$ and $\|w_n\| \leq \|\tilde{w}\|$, we can get that

$$\hat{w} \leq \liminf_n \|w_n\| \leq \|\tilde{w}\| = \min\bigl\{\|w\| : w \in \Gamma\bigr\}.$$

This shows that $\hat{w}$ is also a point in Γ which assumes a minimum norm. Due to the uniqueness of a minimum-norm element, we obtain $\hat{w} = \tilde{w}$. □

Finally, we introduce another method to get the minimum-norm solution of SEP.

Lemma 2.5 *Let $T = I - \gamma G^*G$, where $0 < \gamma < 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G on H. Then we have the following*:

(1) $\|T\| \leq 1$ (*i.e.*, *T is nonexpansive*) *and averaged*;

(2) $\mathrm{Fix}(T) = \{(x,y) \in H, Ax = By\}$, $\mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma$;

(3) $w \in \mathrm{Fix}(P_S T)$ *if and only if w is a solution of the variational inequality* $\langle G^*Gw, v - w\rangle \geq 0$, $\forall v \in S$.

Proof (1) It is easily proved that $\|T\| \leq 1$, we only prove that $T = I - \gamma G^*G$ is averaged. Indeed, choose $0 < \beta < 1$ such that $\gamma/(1-\beta) < 2/\rho(G^*G)$, then $T = I - \gamma G^*G = \beta I + (1-\beta)V$, where $V = I - \gamma/(1-\beta)G^*G$ is a nonexpansive mapping. That is to say, T is averaged.

(2) If $w \in \{(x,y) \in H, Ax = By\}$, it is obvious that $w \in \mathrm{Fix}(T)$. Conversely, assume that $w \in \mathrm{Fix}(T)$, we have $w = w - \gamma G^*Gw$, hence $\gamma G^*Gw = 0$, then $\|Gw\|^2 = \langle G^*Gw, w\rangle = 0$, we get that $w \in \{(x,y) \in H, Ax = By\}$. This leads to $\mathrm{Fix}(T) = \{(x,y) \in H, Ax = By\}$.

Now we prove $\mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma$. By $\mathrm{Fix}(T) = \{(x,y) \in H, Ax = By\}$, $\mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma$ is obvious. On the other hand, since $\mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma \neq \emptyset$, and both P_S and T are averaged, from Lemma 1.5, we have $\mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T)$.

(3)

$$\begin{aligned} \langle G^*Gw, v - w\rangle \geq 0, \quad \forall v \in S \quad &\Leftrightarrow \quad \langle w - (w - \gamma G^*Gw), v - w\rangle \geq 0, \quad \forall v \in S \\ &\Leftrightarrow \quad w = P_S(w - \gamma G^*Gw) \\ &\Leftrightarrow \quad w \in \mathrm{Fix}(P_S T). \end{aligned}$$

□

Remark 2.6 Take a constant γ such that $0 < \gamma < 2/\rho(G^*G)$ with $\rho(G^*G)$ being the spectral radius of the self-adjoint operator G^*G. For $\alpha \in (0, \frac{2-\gamma\|G^*G\|}{2\gamma})$, we define a mapping

$$W_\alpha(w) := P_S[(1-\alpha\gamma)I - \gamma G^*G]w.$$

It is easy to check that W_α is contractive. So, W_α has a unique fixed point denoted by w_α, that is,

$$w_\alpha = P_S[(1-\alpha\gamma)I - \gamma G^*G]w_\alpha. \tag{2.4}$$

Theorem 2.7 *Let w_α be given as* (2.4). *Then w_α converges strongly as $\alpha \to 0$ to the minimum-norm solution $\tilde{w}$ of SEP* (1.1).

Proof Let $\check{w}$ be a point in Γ. Since $\alpha \in (0, \frac{2-\gamma\|G^*G\|}{2\gamma})$, $I - \frac{\gamma}{(1-\alpha\gamma)}G^*G$ is nonexpansive. It follows that

$$\begin{aligned} \|w_\alpha - \check{w}\| &= \|P_S[(1-\alpha\gamma)I - \gamma G^*G]w_\alpha - P_S[\check{w} - \gamma G^*G\check{w}]\| \\ &\leq \|[(1-\alpha\gamma)I - \gamma G^*G]w_\alpha - [\check{w} - \gamma G^*G\check{w}]\| \\ &= \left\|(1-\alpha\gamma)\left[w_\alpha - \frac{\gamma}{1-\alpha\gamma}G^*Gw_\alpha\right] - (1-\alpha\gamma)\left[\check{w} - \frac{\gamma}{1-\alpha\gamma}G^*G\check{w}\right] - \alpha\gamma\check{w}\right\| \\ &\leq (1-\alpha\gamma)\left\|\left(w_\alpha - \frac{\gamma}{1-\alpha\gamma}G^*Gw_\alpha\right) - \left(\check{w} - \frac{\gamma}{1-\alpha\gamma}G^*G\check{w}\right)\right\| + \alpha\gamma\|\check{w}\| \\ &\leq (1-\alpha\gamma)\|w_\alpha - \check{w}\| + \alpha\gamma\|\check{w}\|. \end{aligned}$$

Hence,

$$\|w_\alpha - \check{w}\| \leq \|\check{w}\|.$$

Then $\{w_\alpha\}$ is bounded.

From (2.4), we have

$$\|w_\alpha - P_S[I - \gamma G^*G]w_\alpha\| \leq \alpha\|\gamma w_\alpha\| \to 0.$$

Next we show that $\{w_\alpha\}$ is relatively norm compact as $\alpha \to 0^+$. In fact, assume that $\{\beta_n\} \subseteq (0, \frac{2-\gamma\|G^*G\|}{2\gamma})$ is such that $\alpha_n \to 0^+$ as $n \to \infty$. Put $w_n := w_{\alpha_n}$, we have the following:

$$\|w_n - P_S[I - \gamma G^*G]w_n\| \leq \alpha_n\|\gamma w_n\| \to 0.$$

By the property of the projection, we deduce that

$$\begin{aligned}
\|w_\alpha - \check{w}\|^2 &= \bigl\|P_S\bigl[(1-\alpha\gamma)I - \gamma G^*G\bigr]w_\alpha - P_S\bigl[\check{w} - \gamma G^*G\check{w}\bigr]\bigr\|^2 \\
&\le \bigl\langle\bigl[(1-\alpha\gamma)I - \gamma G^*G\bigr]w_\alpha - \bigl[\check{w} - \gamma G^*G\check{w}\bigr], w_\alpha - \check{w}\bigr\rangle \\
&= \left\langle(1-\alpha\gamma)\left[w_\alpha - \frac{\gamma}{1-\alpha\gamma}G^*Gw_\alpha\right] - (1-\alpha\gamma)\left[\check{w} - \frac{\gamma}{1-\alpha\gamma}G^*G\check{w}\right], w_\alpha - \check{w}\right\rangle \\
&\quad - \alpha\gamma\langle\check{w}, w_\alpha - \check{w}\rangle \\
&\le (1-\alpha\gamma)\|w_\alpha - \check{w}\|^2 - \alpha\gamma\langle\check{w}, w_\alpha - \check{w}\rangle.
\end{aligned}$$

Therefore,

$$\|w_\alpha - \check{w}\|^2 \le \langle -\check{w}, w_\alpha - \check{w}\rangle.$$

In particular,

$$\|w_n - \check{w}\|^2 \le \langle -\check{w}, w_n - \check{w}\rangle, \quad \forall\check{w} \in \Gamma.$$

Since $\{w_n\}$ is bounded, there exists a subsequence of $\{w_n\}$ which converges weakly to a point $\tilde{w}$. Without loss of generality, we may assume that $\{w_n\}$ converges weakly to $\tilde{w}$. Notice that

$$\bigl\|w_n - P_S\bigl[I - \gamma G^*G\bigr]w_n\bigr\| \le \alpha_n\|\gamma w_n\| \to 0,$$

and by Lemma 1.1 we can get that $\tilde{w} \in \mathrm{Fix}(P_S[I - \gamma G^*G]) = \Gamma$.

By

$$\|w_n - \check{w}\|^2 \le \langle -\check{w}, w_n - \check{w}\rangle, \quad \forall\check{w} \in \Gamma,$$

we have

$$\|w_n - \tilde{w}\|^2 \le \langle -\tilde{w}, w_n - \tilde{w}\rangle.$$

Consequently, $\{w_n\}$ converges weakly to $\tilde{w}$ actually implies that $\{w_n\}$ converges strongly to $\tilde{w}$. That is to say, $\{w_\alpha\}$ is relatively norm compact as $\alpha \to 0^+$.

On the other hand, by

$$\|w_n - \check{w}\|^2 \le \langle -\check{w}, w_n - \check{w}\rangle, \quad \forall\check{w} \in \Gamma,$$

let $n \to \infty$, we have

$$\|\tilde{w} - \check{w}\|^2 \le \langle -\check{w}, \tilde{w} - \check{w}\rangle, \quad \forall\check{w} \in \Gamma.$$

This implies that

$$\langle -\check{w}, \check{w} - \tilde{w}\rangle \le 0, \quad \forall\check{w} \in \Gamma,$$

which is equivalent to

$$\langle -\tilde{w}, \check{w} - \tilde{w} \rangle \le 0, \quad \forall \check{w} \in \Gamma.$$

It follows that $\tilde{w} \in P_S(0)$. Therefore, each cluster point of w_α equals $\tilde{w}$. So $w_\alpha \to \tilde{w}$ ($\alpha \to 0$) the minimum-norm solution of SEP. □

3 Modification of Moudafi's ACQA and RACQA algorithms

In this section, we introduce the following algorithm which is viewed as a modification of Moudafi's ACQA and RACQA algorithms. The purpose for such a modification lies in the hope of strong convergence.

Algorithm 3.1 For an arbitrary point $w_0 = (x_0, y_0) \in H = H_1 \times H_2$, the sequence $\{w_n\} = \{(x_n, y_n)\}$ is generated by the iterative algorithm

$$w_{n+1} = P_S\big\{(1-\alpha_n)\big[I - \gamma G^* G\big] w_n\big\}, \tag{3.1}$$

i.e.,

$$\begin{cases} x_{n+1} = P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, & n \ge 0; \\ y_{n+1} = P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, & n \ge 0, \end{cases}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_n \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_n |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Now, we prove the strong convergence of the iterative algorithm.

Theorem 3.2 *The sequence $\{w_n\}$ generated by algorithm* (3.1) *converges strongly to the minimum-norm solution $\tilde{w}$ of SEP* (1.1).

Proof Let R_n and R be defined by

$$R_n w := P_S\big\{(1-\alpha_n)\big[I - \gamma G^* G\big]\big\} w = P_S\big[(1-\alpha_n) T w\big],$$
$$R w := P_S\big(I - \gamma G^* G\big) w = P_S(Tw),$$

where $T = I - \gamma G^* G$. By Lemma 2.5 it is easy to see that R_n is a contraction with contractive constant $1 - \alpha_n$; and algorithm (3.1) can be written as $w_{n+1} = R_n w_n$.

For any $\hat{w} \in \Gamma$, we have

$$\begin{aligned} \|R_n \hat{w} - \hat{w}\| &= \big\|P_S\big[(1-\alpha_n) T\hat{w}\big] - \hat{w}\big\| \\ &= \big\|P_S\big[(1-\alpha_n) T\hat{w}\big] - P_S(T\hat{w})\big\| \\ &\le \big\|(1-\alpha_n) T\hat{w} - T\hat{w}\big\| \\ &= \alpha_n \|T\hat{w}\| \le \alpha_n \|\hat{w}\|. \end{aligned}$$

Hence,

$$\begin{aligned}\|w_{n+1}-\hat{w}\| &= \|R_n w_n-\hat{w}\| \le \|R_n w_n - R_n\hat{w}\| + \|R_n\hat{w}-\hat{w}\| \\ &\le \left\|P_S\left[(1-\alpha_n)T\hat{w}\right]-P_S(T\hat{w})\right\| \\ &\le (1-\alpha_n)\|w_n-\hat{w}\| + \alpha_n\|\hat{w}\| \\ &\le \max\left\{\|w_n-\hat{w}\|, \|\hat{w}\|\right\}.\end{aligned}$$

It follows that $\|w_n-\hat{w}\| \le \max\{\|w_0-\hat{w}\|, \|\hat{w}\|\}$. So $\{w_n\}$ is bounded.

Next we prove that $\lim_n \|w_{n+1}-w_n\| = 0$.

Indeed,

$$\begin{aligned}\|w_{n+1}-w_n\| &= \|R_n w_n - R_{n-1}w_{n-1}\| \\ &\le \|R_n w_n - R_n w_{n-1}\| + \|R_n w_{n-1} - R_{n-1}w_{n-1}\| \\ &\le (1-\alpha_n)\|w_n - w_{n-1}\| + \|R_n w_{n-1} - R_{n-1}w_{n-1}\|.\end{aligned}$$

Notice that

$$\begin{aligned}\|R_n w_{n-1} - R_{n-1}w_{n-1}\| &= \left\|P_S\left[(1-\alpha_n)Tw_{n-1}\right] - P_S\left[(1-\alpha_{n-1})Tw_{n-1}\right]\right\| \\ &\le \left\|(1-\alpha_n)Tw_{n-1} - (1-\alpha_{n-1})Tw_{n-1}\right\| \\ &= |\alpha_n - \alpha_{n-1}|\|Tw_{n-1}\| \\ &\le |\alpha_n - \alpha_{n-1}|\|w_{n-1}\|.\end{aligned}$$

Hence,

$$\|w_{n+1}-w_n\| \le (1-\alpha_n)\|w_n-w_{n-1}\| + |\alpha_n-\alpha_{n-1}|\|w_{n-1}\|.$$

By virtue of assumptions (1)-(3) and Lemma 1.2, we have

$$\lim_n \|w_{n+1}-w_n\| = 0.$$

Therefore,

$$\begin{aligned}\|w_n - Rw_n\| &\le \|w_{n+1}-w_n\| + \|R_n w_n - Rw_n\| \\ &\le \|w_{n+1}-w_n\| + \left\|(1-\alpha_n)Tw_n - Tw_n\right\| \\ &\le \|w_{n+1}-w_n\| + \alpha_n\|w_n\| \to 0.\end{aligned}$$

The demiclosedness principle ensures that each weak limit point of $\{w_n\}$ is a fixed point of the nonexpansive mapping $R = P_S T$, that is, a point of the solution set Γ of SEP (1.1).

At last, we will prove that $\lim_n \|w_{n+1}-\tilde{w}\| = 0$.

Choose $0 < \beta < 1$ such that $\gamma/(1-\beta) < 2/\rho(G^*G)$, then $T = I - \gamma G^*G = \beta I + (1-\beta)V$, where $V = I - \gamma/(1-\beta)G^*G$ is a nonexpansive mapping. Taking $z \in \Gamma$, we deduce that

$$\begin{aligned}
\|w_{n+1} - z\|^2 &= \bigl\|P_S\bigl[(1-\alpha_n)Tw_n\bigr] - z\bigr\|^2 \\
&\leq \bigl\|(1-\alpha_n)Tw_n - z\bigr\|^2 \\
&\leq (1-\alpha_n)\|Tw_n - z\|^2 + \alpha_n\|z\|^2 \\
&\leq \bigl\|\beta(w_n - z) + (1-\beta)(Vw_n - z)\bigr\|^2 + \alpha_n\|z\|^2 \\
&\leq \beta\bigl\|(w_n - z)\bigr\|^2 + (1-\beta)\bigl\|(Vw_n - z)\bigr\|^2 - \beta(1-\beta)\|w_n - Vw_n\|^2 + \alpha_n\|z\|^2 \\
&\leq \bigl\|(w_n - z)\bigr\|^2 - \beta(1-\beta)\|w_n - Vw_n\|^2 + \alpha_n\|z\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
\beta(1-\beta)\|w_n - Vw_n\| &\leq \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \alpha_n\|z\|^2 \\
&\leq \bigl(\|w_n - z\| + \|w_{n+1} - z\|\bigr)\bigl(\|w_n - z\| - \|w_{n+1} - z\|\bigr)\alpha_n\|z\|^2 \\
&\leq \bigl(\|w_n - z\| + \|w_{n+1} - z\|\bigr)\bigl(\|w_n - w_{n+1}\|\bigr)\alpha_n\|z\|^2 \to 0.
\end{aligned}$$

Note that $T = I - \gamma G^*G = \beta I + (1-\beta)V$, it follows that $\lim_n \|Tw_n - w_n\| = 0$.

Take a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\limsup_n\langle w_n - \tilde{w}, -\tilde{w}\rangle = \lim_k\langle w_{n_k} - \tilde{w}, -\tilde{w}\rangle$.

By virtue of the boundedness of w_n, we may further assume, with no loss of generality, that w_{n_k} converges weakly to a point $\check{w}$. Since $\|Rw_n - w_n\| \to 0$, using the demiclosedness principle, we know that $\check{w} \in \mathrm{Fix}(R) = \mathrm{Fix}(P_S T) = \Gamma$. Noticing that $\tilde{w}$ is the projection of the origin onto Γ, we get that

$$\limsup_n\langle w_n - \tilde{w}, -\tilde{w}\rangle = \lim_k\langle w_{n_k} - \tilde{w}, -\tilde{w}\rangle = \langle \check{w} - \tilde{w}, -\tilde{w}\rangle \leq 0.$$

Finally, we compute

$$\begin{aligned}
\|w_{n+1} - \tilde{w}\|^2 &= \bigl\|P_S\bigl[(1-\alpha_n)Tw_n\bigr] - \tilde{w}\bigr\|^2 \\
&= \bigl\|P_S\bigl[(1-\alpha_n)Tw_n\bigr] - P_S T\tilde{w}\bigr\|^2 \\
&\leq \bigl\|(1-\alpha_n)Tw_n - T\tilde{w}\bigr\|^2 \\
&= \bigl\|(1-\alpha_n)Tw_n - \tilde{w}\bigr\|^2 \\
&= \bigl\|(1-\alpha_n)(Tw_n - \tilde{w}) + \alpha_n(-\tilde{w})\bigr\|^2 \\
&= (1-\alpha_n)^2\bigl\|(Tw_n - \tilde{w})\bigr\|^2 + \alpha_n^2\|\tilde{w}\|^2 + 2\alpha_n(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle \\
&\leq (1-\alpha_n)\bigl\|(Tw_n - \tilde{w})\bigr\|^2 + \alpha_n\bigl[\alpha_n\|\tilde{w}\|^2 + 2(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle\bigr].
\end{aligned}$$

Since $\limsup_n\langle w_n - \tilde{w}, -\tilde{w}\rangle \leq 0$, $\|w_n - Tw_n\| \to 0$, we know that $\limsup_n(\alpha_n\|\tilde{w}\|^2 + 2(1-\alpha_n)\langle Tw_n - \tilde{w}, -\tilde{w}\rangle) \leq 0$. By Lemma 1.2, we conclude that $\lim_n \|w_{n+1} - \tilde{w}\| = 0$. This completes the proof. □

Remark 3.3 When $B = I$, the iteration algorithm (3.1) becomes

$$x_{n+1} = P_C\bigl\{(1-\alpha_n)\bigl[x_n - \gamma A^*(Ax_n - y_n)\bigr]\bigr\};$$

$$y_{n+1} = P_Q\{(1-\alpha_n)[y_n + \gamma(Ax_n - y_n)]\}.$$

By Theorem 3.2, we can get the following result.

Corollary 3.4 *For an arbitrary point $w_0 = (x_0, y_0) \in H = H_1 \times H_2$, the sequence $\{w_n\} = \{(x_n, y_n)\}$ is generated by the iterative algorithm*

$$\begin{cases} x_{n+1} = P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - y_n)]\}, & n \geq 0; \\ y_{n+1} = P_Q\{(1-\alpha_n)[y_n + \gamma(Ax_n - y_n)]\}, & n \geq 0, \end{cases}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_n \alpha_n = 0$;
(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(iii) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_n |\alpha_{n+1} - \alpha_n|/\alpha_n = 0$.

Then x_n converges strongly to the minimum-norm solution of SFP.

4 KM-CQ-like iterative algorithm for SEP

In this section, we establish a KM-CQ-like algorithm converging strongly to a solution of SEP.

Algorithm 4.1 For an arbitrary initial point $w_0 = (x_0, y_0)$, the sequence $\{w_n = (x_n, y_n)\}$ is generated by the iteration

$$w_{n+1} = (1-\beta_n)w_n + \beta_n P_S[(1-\alpha_n)(I - \gamma G^* G)]w_n, \tag{4.1}$$

i.e.,

$$\begin{cases} x_{n+1} = (1-\beta_n)x_n + \beta_n P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}, & n \geq 0; \\ y_{n+1} = (1-\beta_n)y_n + \beta_n P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, & n \geq 0, \end{cases}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} |\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Lemma 4.2 *If $z \in \mathrm{Fix}(T) = \mathrm{Fix}(I - \gamma G^* G)$, then for any w we have $\|Tw - z\|^2 \leq \|w - z\|^2 - \beta(1-\beta)\|Vw - w\|^2$, where β and V are the same as in Lemma* 2.5(1).

Proof According to Lemma 2.5(1), we know that $T = \beta I + (1-\beta)V$, where $0 < \beta < 1$ and V is nonexpansive. It is easy to check that $z \in \mathrm{Fix}(T) = \mathrm{Fix}(V)$, and

$$\begin{aligned} \|Tw - z\|^2 &= \|\beta w + (1-\beta)Vw - z\|^2 \\ &\leq \beta\|w - z\|^2 + (1-\beta)\|Vw - z\|^2 - \beta(1-\beta)\|Vw - w\|^2 \\ &\leq \beta\|w - z\|^2 + (1-\beta)\|w - z\|^2 - \beta(1-\beta)\|Vw - w\|^2 \\ &= \|w - z\|^2 - \beta(1-\beta)\|Vw - w\|^2. \end{aligned}$$

□

Theorem 4.3 *The sequence* $\{w_n\}$ *generated by algorithm* (4.1) *converges strongly to a solution of SEP* (1.1).

Proof For any solution of SEP $\hat{w}$, according to Lemma 2.5, $\hat{w} \in \mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T)$, where $T = I - \gamma G^* G$, and

$$\begin{aligned}
\|w_{n+1} - \hat{w}\| &= \bigl\|(1-\beta_n)w_n + \beta_n P_S\bigl[(1-\alpha_n)T\bigr]w_n - \hat{w}\bigr\| \\
&= \bigl\|(1-\beta_n)(w_n - \hat{w}) + \beta_n\bigl(P_S\bigl[(1-\alpha_n)T\bigr]w_n - \hat{w}\bigr)\bigr\| \\
&\leq (1-\beta_n)\|w_n - \hat{w}\| + \beta_n\bigl\|P_S\bigl[(1-\alpha_n)T\bigr]w_n - \hat{w}\bigr\| \\
&\leq (1-\beta_n)\|w_n - \hat{w}\| \\
&\quad + \beta_n\bigl\|P_S\bigl[(1-\alpha_n)T\bigr]w_n - P_S\bigl[(1-\alpha_n)T\bigr]\hat{w}\bigr\| \\
&\quad + \beta_n\bigl\|P_S\bigl[(1-\alpha_n)T\bigr]\hat{w} - \hat{w}\bigr\| \\
&\leq (1-\beta_n)\|w_n - \hat{w}\| + \beta_n(1-\alpha_n)\|w_n - \hat{w}\| + \beta_n\alpha_n\|\hat{w}\| \\
&= (1-\beta_n\alpha_n)\|w_n - \hat{w}\| + \beta_n\alpha_n\|\hat{w}\| \\
&\leq \max\bigl\{\|w_n - \hat{w}\|, \|\hat{w}\|\bigr\}.
\end{aligned}$$

By induction,

$$\|w_n - \hat{w}\| \leq \max\bigl\{\|w_0 - \hat{w}\|, \|\hat{w}\|\bigr\}.$$

Hence, $\{w_n\}$ is bounded and so is $\{Tw_n\}$. Moreover,

$$\begin{aligned}
\bigl\|P_S\bigl[(1-\alpha_n)T\bigr]w_n - \hat{w}\bigr\| &\leq \bigl\|(1-\alpha_n)Tw_n - \hat{w}\bigr\| \\
&= \bigl\|(1-\alpha_n)[Tw_n - \hat{w}] - \alpha_n\hat{w}\bigr\| \\
&\leq (1-\alpha_n)\|w_n - \hat{w}\| + \alpha_n\|\hat{w}\| \\
&\leq \max\bigl\{\|w_n - \hat{w}\|, \|\hat{w}\|\bigr\}.
\end{aligned}$$

Since $\{w_n\}$ is bounded, we have that $\{Tw_n\}$, $(1-\alpha_n)Tw_n$ and $\{P_S[(1-\alpha_n)T]w_n\}$ are also bounded.

Let $z_n = P_S[(1-\alpha_n)T]w_n$, and $M > 0$ such that $M = \sup_{n\geq 1}\{Tw_n\}$. We observe that

$$\begin{aligned}
\bigl\|P_S\bigl[(1-\alpha_{n+1})T\bigr]w_n - P_S\bigl[(1-\alpha_n)T\bigr]w_n\bigr\| &\leq \bigl\|(1-\alpha_{n+1})Tw_n - (1-\alpha_n)Tw_n\bigr\| \\
&= \bigl\|(\alpha_n - \alpha_{n+1})Tw_n\bigr\| \\
&\leq M|\alpha_n - \alpha_{n+1}|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \bigl\|P_S\bigl[(1-\alpha_{n+1})T\bigr]w_{n+1} - P_S\bigl[(1-\alpha_n)T\bigr]w_n\bigr\| \\
&\leq \bigl\|P_S\bigl[(1-\alpha_{n+1})T\bigr]w_{n+1} - P_S\bigl[(1-\alpha_{n+1})T\bigr]w_n\bigr\| \\
&\quad + \bigl\|P_S\bigl[(1-\alpha_{n+1})T\bigr]w_n - P_S\bigl[(1-\alpha_n)T\bigr]w_n\bigr\|
\end{aligned}$$

$$\begin{aligned}&\le (1-\alpha_{n+1})\|w_{n+1}-w_n\| + \big\|P_S\big[(1-\alpha_{n+1})T\big]w_n - P_S\big[(1-\alpha_n)T\big]w_n\big\| \\ &\le (1-\alpha_{n+1})\|w_{n+1}-w_n\| + M|\alpha_n-\alpha_{n+1}|.\end{aligned}$$

Since $0 < \alpha_n < 1$ and $\lim_{n\to\infty}|\alpha_{n+1}-\alpha_n| = 0$, we obtain that

$$\|z_{n+1}-z_n\| - \|w_{n+1}-w_n\| \le M|\alpha_n-\alpha_{n+1}|$$

and

$$\limsup_{n\to\infty} \|z_{n+1}-z_n\| - \|w_{n+1}-w_n\| \le 0.$$

Using Lemma 1.3, we get that

$$\lim_{n\to\infty}\big\|P_S\big[(1-\alpha_n)T\big]w_n - w_n\big\| = \lim_{n\to\infty}\|z_n-w_n\| = 0.$$

Therefore,

$$\begin{aligned}\|w_{n+1}-w_n\| &= \big\|(1-\beta_n)w_n + \beta_n P_S\big[(1-\alpha_n)T\big]w_n - w_n\big\| \\ &= \beta_n\big\|P_S\big[(1-\alpha_n)T\big]w_n - w_n\big\| \to 0.\end{aligned}$$

Let R_n and R be defined by

$$\begin{aligned}&R_n w := P_S\big\{(1-\alpha_n)\big[I-\gamma G^*G\big]\big\}w = P_S\big[(1-\alpha_n)Tw\big], \\ &Rw := P_S\big(I-\gamma G^*G\big)w = P_S(Tw).\end{aligned}$$

We find

$$\begin{aligned}\|w_n - Rw_n\| &\le \|w_n - w_{n+1}\| + \|w_{n+1} - Rw_n\| \\ &= \|w_n - w_{n+1}\| + \big\|(1-\beta_n)w_n + \beta_n R_n w_n - Rw_n\big\| \\ &\le \|w_n - w_{n+1}\| + (1-\beta_n)\|w_n - Rw_n\| + \beta_n\|R_n w_n - Rw_n\|.\end{aligned}$$

So, we have

$$\begin{aligned}\|w_n - Rw_n\| &\le \|w_n - w_{n+1}\|/\beta_n + \|R_n w_n - Rw_n\| \\ &= \|w_n - w_{n+1}\|/\beta_n + \big\|P_S\big[(1-\alpha_n)T\big]w_n - P_S Tw_n\big\| \\ &\le \|w_n - w_{n+1}\|/\beta_n + \big\|(1-\alpha_n)Tw_n - Tw_n\big\| \\ &\le \|w_n - w_{n+1}\|/\beta_n + M\alpha_n.\end{aligned}$$

By assumption, we have

$$\lim_{n\to\infty}\|w_n - Rw_n\| = 0.$$

On the other hand, $\{w_n\}$ is bounded, there exists a subsequence of $\{w_n\}$ which converges weakly to a point $\check{w}$. Without loss of generality, we may assume that $\{w_n\}$ converges weakly to $\check{w}$. Since $\|Rw_n - w_n\| \to 0$, using the demiclosedness principle we know that $\check{w} \in \mathrm{Fix}(R) = \mathrm{Fix}(P_S T) = \mathrm{Fix}(P_S) \cap \mathrm{Fix}(T) = \Gamma$.

At last, we will prove that $\lim_n \|w_{n+1} - \check{w}\| = 0$. To do this, we calculate

$$\begin{aligned}
\|w_{n+1} - \check{w}\|^2 &= \bigl\|(1-\beta_n)w_n + \beta_n P_S\bigl[(1-\alpha_n)T\bigr]w_n - P_S T\check{w}\bigr\|^2 \\
&\le (1-\beta_n)\|w_n - \check{w}\|^2 + \beta_n\bigl\|P_S\bigl[(1-\alpha_n)T\bigr]w_n - P_S T\check{w}\bigr\|^2 \\
&\le (1-\beta_n)\|w_n - \check{w}\|^2 + \beta_n\bigl\|(1-\alpha_n)Tw_n - \check{w}\bigr\|^2 \\
&= (1-\beta_n)\|w_n - \check{w}\|^2 + \beta_n\bigl\|(1-\alpha_n)(Tw_n - \check{w}) + \alpha_n\check{w}\bigr\|^2 \\
&= (1-\beta_n)\|w_n - \check{w}\|^2 + \beta_n\bigl[(1-\alpha_n)^2\|Tw_n - \check{w}\|^2 + \alpha_n^2\|\check{w}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle Tw_n - \check{w}, -\check{w}\rangle\bigr] \\
&\le (1-\beta_n)\|w_n - \check{w}\|^2 + \beta_n\bigl[(1-\alpha_n)\|w_n - \check{w}\|^2 + \alpha_n^2\|\check{w}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle Tw_n - \check{w}, -\check{w}\rangle\bigr] \\
&= (1-\alpha_n\beta_n)\|w_n - \check{w}\|^2 + \alpha_n\beta_n\bigl[2(1-\alpha_n)\langle Tw_n - \check{w}, -\check{w}\rangle + \alpha_n\|\check{w}\|^2\bigr].
\end{aligned}$$

By Lemma 1.2, we only need to prove that

$$\limsup_{n\to\infty}\langle Tw_n - \check{w}, -\check{w}\rangle \le 0.$$

By Lemma 2.5, T is averaged, that is, $T = \beta I + (1-\beta)V$, where $0 < \beta < 1$ and V is nonexpansive. Then, for $z \in \mathrm{Fix}(P_S T)$, we have

$$\begin{aligned}
\|w_{n+1} - z\|^2 &= \bigl\|(1-\beta_n)w_n + \beta_n P_S\bigl[(1-\alpha_n)T\bigr]w_n - z\bigr\|^2 \\
&\le (1-\beta_n)\|w_n - z\|^2 + \beta_n\bigl\|(1-\alpha_n)Tw_n - z\bigr\|^2 \\
&= (1-\beta_n)\|w_n - z\|^2 + \beta_n\bigl\|(1-\alpha_n)(Tw_n - z) - \alpha_n z\bigr\|^2 \\
&\le (1-\beta_n)\|w_n - z\|^2 + \beta_n\bigl[(1-\alpha_n)\|Tw_n - z\|^2 + \alpha_n\|z\|^2\bigr] \\
&\le (1-\beta_n)\|w_n - z\|^2 + \beta_n\bigl[\|Tw_n - z\|^2 + \alpha_n\|z\|^2\bigr].
\end{aligned}$$

By Lemma 4.2, we can get

$$\begin{aligned}
\|w_{n+1} - z\|^2 &\le (1-\beta_n)\|w_n - z\|^2 \\
&\quad + \beta_n\bigl[\|w_n - z\|^2 - \beta(1-\beta)\|Vw_n - w_n\|^2 + \alpha_n\|z\|^2\bigr] \\
&\le \|w_n - z\|^2 - \beta_n\beta(1-\beta)\|Vw_n - w_n\|^2 + \beta_n\alpha_n\|z\|^2.
\end{aligned}$$

Let $K > 0$ such that $\|w_n - z\| \le K$ for all n, then we have

$$\begin{aligned}
\beta_n\beta(1-\beta)\|Vw_n - w_n\|^2 &\le \|w_n - z\|^2 - \|w_{n+1} - z\|^2 + \beta_n\alpha_n\|z\|^2 \\
&\le 2N\bigl|\|w_n - z\| - \|w_{n+1} - z\|\bigr| + \beta_n\alpha_n\|z\|^2 \\
&\le 2N\|w_n - w_{n+1}\| + \beta_n\alpha_n\|z\|^2.
\end{aligned}$$

Hence,

$$\beta(1-\beta)\|Vw_n - w_n\|^2 \le \frac{2N\|w_n - w_{n+1}\|}{\beta_n} + \alpha_n\|z\|^2.$$

Since $\|w_n - w_{n+1}\| \to 0$, we can get that

$$\|Vw_n - w_n\| \to 0.$$

Therefore,

$$\|Tw_n - w_n\| \to 0.$$

It follows that

$$\limsup_{n\to\infty}\langle Tw_n - \check{w}, -\check{w}\rangle = \limsup_{n\to\infty}\langle w_n - \check{w}, -\check{w}\rangle.$$

Since $\{w_n\}$ converges weakly to $\check{w}$, it follows that

$$\limsup_{n\to\infty}\langle Tw_n - \check{w}, -\check{w}\rangle \le 0.$$ □

Similar to the proof of Theorem 4.3, we can get that the following iterative algorithm converges strongly to a solution of SEP also. Since the proof is similar to Theorem 4.3, we omit it.

Algorithm 4.4 For an arbitrary initial point $w_0 = (x_0, y_0)$, the sequence $\{w_n = (x_n, y_n)\}$ is generated by the iteration

$$w_{n+1} = (1-\beta_n)(1-\alpha_n)\big(I - \gamma G^*G\big)w_n + \beta_n P_S\big[(1-\alpha_n)\big(I-\gamma G^*G\big)\big]w_n, \tag{4.2}$$

i.e.,

$$\begin{cases} x_{n+1} = (1-\beta_n)(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)] \\ \qquad\quad + \beta_n P_C\{(1-\alpha_n)[x_n - \gamma A^*(Ax_n - By_n)]\}; \\ y_{n+1} = (1-\beta_n)(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)] \\ \qquad\quad + \beta_n P_Q\{(1-\alpha_n)[y_n + \gamma B^*(Ax_n - By_n)]\}, \end{cases}$$

where $\alpha_n > 0$ is a sequence in $(0,1)$ such that

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$;
(ii) $\lim_{n\to\infty}|\alpha_{n+1} - \alpha_n| = 0$;
(iii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$.

5 Other iterative methods

In this section, we introduce some other iterative algorithms which converge strongly to a solution of SEP.

According to Lemma 2.5, we know that $w = (x, y)$ belongs to the solution set Γ of SEP (1.1) if and only if $w \in \mathrm{Fix}(P_S(I - \gamma G^*G))$. Moreover, $P_S(I - \gamma G^*G)$ is a nonexpansive mapping.

That is to say, the essence of SEP is to find a fixed point for the nonexpansive mapping $P_S(I - \gamma G^* G)$.

For the fixed point of a nonexpansive mapping, the following results have been obtained.

In 1974, Ishikawa [14] gave the Ishikawa iteration as follows:

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0, \end{cases}$$

where $x_0 \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$. He proved that if $0 \leq \alpha_n \leq \beta_n \leq 1$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.

In 2004, Xu [15] gave the viscosity iteration for nonexpansive mappings. He considered the iteration process

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n, \quad n \geq 0,$$

where f is a contraction on C and x_0 is an arbitrary (but fixed) element in C. He proved that if $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$, then $\{x_n\}$ converges strongly to a fixed point of T.

Halpern's iteration is as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n, \quad n \geq 0,$$

where $u \in C$ is an arbitrary (but fixed) element in C.

Mann's iteration method that produces a sequence $\{x_n\}$ via the recursive manner is as follows:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0,$$

where the initial guess $x_0 \in C$ is chosen arbitrarily. However, this scheme has only weak convergence even in a Hilbert space.

In 2005, Kim and Xu [16] modified Mann's iteration scheme and the modified iteration method still works in a Banach space. Let C be a closed convex subset of a Banach space and $T : C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Define $\{x_n\}$ in the following way:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad n \geq 0, \\ x_{n+1} = \beta_n u + (1 - \beta_n) T y_n, \quad n \geq 0, \end{cases}$$

where $u \in C$ is an arbitrary (but fixed) element in C, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$. They proved that if $\alpha_n \to 0, \beta_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \beta_n = \infty$, and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, then $\{x_n\}$ converges strongly to a fixed point of T.

Therefore, we have the following iterative algorithms which converge strongly to a solution of SEP.

Algorithm 5.1

$$\begin{cases} w_0 = (x_0, y_0) \in H = H_1 \times H_2, \\ v_n = (1-\beta_n)w_n + \beta_n P_S T w_n, \quad n \geq 0, \\ w_{n+1} = (1-\alpha_n)w_n + \alpha_n P_S T v_n, \quad n \geq 0, \end{cases}$$

particulars:

$$\begin{cases} x_0 \in H_1, y_0 \in H_2, \\ z_n = x_n - \gamma A^*(Ax_n - By_n), \\ h_n = y_n + \gamma B^*(Ax_n - By_n), \\ j_n = x_n - A^*(\gamma Ax_n - By_n), \\ k_n = y_n + B^*(Ax_n - \gamma By_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C[(1-\beta_n)j_n + \beta_n(I - \gamma A^*A)P_C z_n + \beta_n A^*BP_Q h_n], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_Q[(1-\beta_n)k_n + \beta_n B^*AP_C z_n + \beta_n(I - \gamma B^*B)P_Q h_n], \end{cases}$$

where $w_0 = (x_0, y_0)$ is an arbitrary (but fixed) element in H, $T = I - \gamma G^*G$ and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$. If $0 \leq \alpha_n \leq \beta_n \leq 1$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$, then $\{w_n\}$ converges strongly to a solution of SEP.

Algorithm 5.2

$$w_{n+1} = \alpha_n f(w_n) + (1-\alpha_n)P_S T w_n, \quad n \geq 0,$$

particulars:

$$\begin{cases} x_{n+1} = \alpha_n P_{H_1} f(x_n, y_n) + (1-\alpha_n)P_C[x_n - \gamma A^*(Ax_n - By_n)], \\ y_{n+1} = \alpha_n P_{H_2} f(x_n, y_n) + (1-\alpha_n)P_Q[y_n + \gamma B^*(Ax_n - By_n)], \end{cases}$$

where f is a contraction on $H = H_1 \times H_2$ and $w_0 = (x_0, y_0)$ is an arbitrary (but fixed) element in H, and $T = I - \gamma G^*G$. If $\alpha_n \to 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty}(\alpha_{n+1}/\alpha_n) = 1$, then $\{w_n\}$ converges strongly to a solution of SEP.

Algorithm 5.3

$$\begin{cases} w_0 = (x_0, y_0), u = (x_1, y_1) \in H = H_1 \times H_2, \\ v_n = \alpha_n w_n + (1-\alpha_n)P_S T w_n, \quad n \geq 0, \\ w_{n+1} = \beta_n u + (1-\beta_n)P_S T v_n, \quad n \geq 0, \end{cases}$$

particulars:

$$\begin{cases} x_0, x_1 \in H_1, y_0, y_1 \in H_2, \\ z_n = x_n - \gamma A^*(Ax_n - By_n), \\ h_n = y_n + \gamma B^*(Ax_n - By_n), \\ j_n = x_n - A^*(\gamma Ax_n - By_n), \\ k_n = y_n + B^*(Ax_n - \gamma By_n), \\ x_{n+1} = \alpha_n x_1 + (1-\alpha_n)P_C[\beta_n j_n + (1-\beta_n)(I - \gamma A^*A)P_C z_n + (1-\beta_n)A^*BP_Q h_n], \\ y_{n+1} = \alpha_n y_1 + (1-\alpha_n)P_Q[\beta_n k_n + (1-\beta_n)B^*AP_C z_n + (1-\beta_n)(I - \gamma B^*B)P_Q h_n], \end{cases}$$

where u, w_0 are arbitrary (but fixed) elements in H, $T = I - \gamma G^*G$, and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $(0,1)$. They proved that if $\alpha_n \to 0$, $\beta_n \to 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, then $\{w_n\}$ converges strongly to a solution of SEP.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by LS, RC and YW prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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References

1. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**(2-4), 221-239 (1994)
2. Byrne, C: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Problems **18**(2), 441-453 (2002)
3. Byrne, C: A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Problems **20**(1), 103-120 (2004)
4. Qu, B, Xiu, N: A note on the CQ algorithm for the split feasibility problem. Inverse Problems **21**(5), 1655-1665 (2005)
5. Xu, H-K: A variable Krasnosel'skii-Mann algorithm and the multiple-set split feasibility problem. Inverse Problems **22**(6), 2021-2034 (2006)
6. Yang, Q: The relaxed CQ algorithm solving the split feasibility problem. Inverse Problems **20**(4), 1261-1266 (2004)
7. Yang, Q, Zhao, J: Generalized KM theorems and their applications. Inverse Problems **22**(3), 833-844 (2006)
8. Moudafi, A: A relaxed alternating CQ-algorithms for convex feasibility problems. Nonlinear Anal., Theory Methods Appl. **79**, 117-121 (2013)
9. Geobel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge Studies in Advanced Mathematics, vol. 28. Cambridge University Press, Cambridge (1990)
10. Geobel, K, Reich, S: Uniform Convexity, Nonexpansive Mappings, and Hyperbolic Geometry. Dekker, New York (1984)
11. Aoyama, K, Kimura, Y, Takahashi, W, Toyoda, M: Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal., Theory Methods Appl. **67**(8), 2350-2360 (2007)
12. Suzuki, T: Strong convergence theorems for infinite families of nonexpansive mappings in general Banach space. Fixed Point Theory Appl. **1**, 103-123 (2005)
13. Engl, HW, Hanke, M, Neubauer, A: Regularization of Inverse Problems. Mathematics and Its Applications, vol. 375. Kluwer Academic, Dordrecht (1996)
14. Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)
15. Xu, H-K: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279-291 (2004)
16. Kim, TH, Xu, H-K: Strong convergence of modified Mann iterations. Nonlinear Anal. **61**, 51-60 (2005)

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