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Fixed point solutions for variational inequalities in image restoration over q-uniformly smooth Banach spaces

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Abstract

In this paper, we introduce new implicit and explicit iterative methods for finding a common fixed point set of an infinite family of strict pseudo-contractions by the sunny nonexpansive retractions in a real q-uniformly and uniformly convex Banach space which admits a weakly sequentially continuous generalized duality mapping. Then we prove the strong convergence under mild conditions of the purposed iterative scheme to a common fixed point of an infinite family of strict pseudo-contractions which is a solution of some variational inequalities. Furthermore, we apply our results to study some strong convergence theorems in L_p and ℓ_p spaces with $1 < p < \infty$. Our results mainly improve and extend the results announced by Ceng *et al.* (Comput. Math. Appl. 61:2447-2455, 2011) and many authors from Hilbert spaces to Banach spaces. Finally, we give some numerical examples for support our main theorem in the end of the paper.

MSC: 47H09; 47H10; 47H17; 47J25; 49J40

Keywords: variational inequality; Banach space; strong convergence; iterative method; common fixed point; strongly accretive operator; inverse strongly accretive operator

1 Introduction

Let $C_1, C_2, \dots, C_n$ be nonempty, closed, and convex subsets of a real Hilbert space H such that $\bigcap_{i=1}^n C_i \neq \emptyset$. The problem of image recovery in a Hilbert space setting by using convex of metric projections P_{C_i}, may be stated as follows: the original unknown image z is known *a priori* to belong to the intersection of $\{C_i\}_{i=1}^n$; given only the metric projections P_{C_i} of H onto C_i for $i = 1, 2, \dots, n$ recover z by an iterative scheme. Youla and Webb [1] first used iterative methods for applied in image restoration. The problems of image recovery have been studied in a Banach space setting by Kitahara and Takahashi [2] (see also [3, 4]) by using convex combinations of sunny nonexpansive retractions in uniformly convex Banach spaces. On the other hand, Alber [5] studied the problem of image recovery by the products of generalized projections in a uniformly convex and uniformly smooth Banach space whose duality mapping is weakly sequentially continuous (see also [6, 7]). Nakajo *et al.* [8] and Kimura *et al.* [9] considered this problem by the sunny nonexpansive retractions and proved convergence of the iterative sequence to a common point of countable nonempty, closed, and convex subsets in a uniformly convex and smooth Banach space, and in a strictly convex, smooth and reflexive Banach space having the Kadec-Klee property, respectively. Some iterative methods have been studied in problem of image recovery by numerous authors (see [2–5, 10–12]).

The problems of image recovery are connected with the convex feasibility problem, convex minimization problems, multiple-set split feasibility problems, common fixed point problems, and variational inequalities. In particular, variational inequality theory has been studied widely in several branches of pure and applied sciences. This field is dynamics and is experiencing an explosive growth in both theory and applications. Indeed, applications of the variational inequalities span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, finance, and so on. Note that most of the variational problems, including minimization or maximization of functions, variational inequality problems, quasivariational inequality problems, decision and management sciences, and engineering sciences problems. Recently, some iterative methods have been developed for solving the fixed point problems and variational inequality problems in q-uniformly smooth Banach spaces by numerous authors (see [13–24]).

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \bar{\gamma}\|x\|^2 \quad \text{for all } x \in H. \tag{1.1}$$

Remark 1.1 From the definition of operator A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschitzian and η-strongly monotone operator.

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.2}$$

where C is the fixed point set of a nonexpansive mapping T on H and u is a given point in H.

In 2006, Marino and Xu [25] introduced and considered the following a general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \quad \forall n \geq 0, \tag{1.3}$$

where A is a strongly positive bounded linear operator on a real Hilbert space H. They proved that if the sequence $\{\alpha_n\}$ satisfies appropriate conditions, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to the unique solution of the variational inequality

$$\bigl\langle (\gamma f - A)x^*, x - x^*\bigr\rangle \leq 0, \quad \forall x \in \operatorname{Fix}(T), \tag{1.4}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.5}$$

where C is the fixed point set of a nonexpansive mapping T and h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for all $x \in H$).

On the other hand, Yamada [26] introduced a hybrid steepest descent method for a nonexpansive mapping T as follows:

$$x_{n+1} = Tx_n - \mu\lambda_n F(Tx_n), \quad \forall n \geq 0, \tag{1.6}$$

where F is a κ-Lipschitzian and η-strongly monotone operator on a real Hilbert space H with constants $\kappa, \eta > 0$ and $0 < \mu < \frac{2\eta}{\kappa^2}$. He proved that if $\{\lambda_n\}$ satisfy the appropriate conditions, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.7}$$

Tian [27] combined the iterative method (1.3) with the Yamada method (1.6) and considered a general iterative method for a nonexpansive mapping T on a real Hilbert space H as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \mu F)Tx_n, \quad \forall n \geq 0. \tag{1.8}$$

Then he proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to the unique solution of variational inequality

$$\langle (\gamma f - \mu F)x^*, x - x^* \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.9}$$

In 2011, Ceng *et al.* [28] combined the iterative method (1.3) with Tian's method (1.8) and consider the following a general composite iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n\big[Tx_n - \beta_n\big(\mu FTx_n - \gamma f(x_n)\big)\big], \quad \forall n \geq 0, \tag{1.10}$$

where A is a strongly positive bounded linear operator on H with coefficient $\bar{\gamma} \in (1,2)$, and $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1]$ satisfy appropriate conditions. Then they proved that the sequence $\{x_n\}$ generated by (1.10) converges strongly to the unique solution $x^* \in C$ of the variational inequality

$$\langle (I - A)x^*, x - x^* \rangle \leq 0, \quad \forall x \in C, \tag{1.11}$$

where $C = \mathrm{Fix}(T)$.

In this paper, motivated by the above facts, we introduce new implicit and explicit iterative methods for finding a common fixed point set of an infinite family of strict pseudo-contractions by the sunny nonexpansive retractions in a real q-uniformly and uniformly convex Banach space X which admits a weakly sequentially continuous generalized duality mapping. Consequently, we prove the strong convergence under mild conditions of the purposed iterative scheme to a common fixed point of an infinite family of strict pseudo-contractions of nonempty, closed, and convex subsets of X which is a solution of some variational inequalities. Furthermore, we apply our results to the study of some strong convergence theorems in L_p and ℓ_p spaces with $1 < p < \infty$. Our results extend the main result of Ceng *et al.* [28] in several aspects and the work of many authors from Hilbert spaces to Banach spaces. Finally, we give some numerical examples to support our main theorem in the end of the paper.

2 Preliminaries

Throughout this paper, we denote by X and X^* a real Banach space and the dual space of X, respectively. Let $q > 1$ be a real number. The *generalized duality mapping* $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\},$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the *normalized duality mapping* and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If $X := H$ is a real Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q (see [29]).

A Banach space X is said to be *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. A Banach space X is said to be *uniformly convex* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\frac{\|x+y\|}{2} \leq 1 - \delta$ holds. Let $S(X) = \{x \in X : \|x\| = 1\}$. The norm of X is said to be *Gâteaux differentiable* (or X is said to be smooth) if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. The norm of X is said to be *uniformly Gâteaux differentiable*, if, for each $y \in S(X)$, the limit is attained uniformly for $x \in S(X)$.

Let $\rho_X : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of X defined by

$$\rho_X(\tau) = \sup\left\{\frac{1}{2}\big(\|x + y\| + \|x - y\|\big) - 1 : x \in S(X), \|y\| \leq \tau\right\}.$$

A Banach space X is said to be *uniformly smooth* if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. Suppose that $q > 1$, then X is said to be *q-uniformly smooth* if there exists $c > 0$ such that $\rho_X(t) \leq ct^q$ for all $t > 0$. It is shown in [30] (see also [31]) that there is no Banach space which is q-uniformly smooth with $q > 2$. If X is q-uniformly smooth, then X is uniformly smooth. It is well known that each uniformly convex Banach space X is reflexive and strictly convex and every uniformly smooth Banach space X is a reflexive Banach space with uniformly Gâteaux differentiable norm (see [29]). Typical examples of both uniformly convex and uniformly smooth Banach spaces are L_p, where $p > 1$. More precisely, L_p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Let C be a nonempty, closed, and convex subset of X and T be a self-mapping on C. We denote the fixed points set of the mapping T by $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$.

Definition 2.1 A mapping $T : C \to C$ is said to be:

(i) *λ-strictly pseudo-contractive* [32] if, for all $x, y \in C$, there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\big\langle Tx - Ty, j_q(x - y)\big\rangle \leq \|x - y\|^q - \lambda\big\|(I - T)x - (I - T)y\big\|^q, \tag{2.1}$$

or equivalently

$$\bigl\langle (I-T)x-(I-T)y, j_q(x-y)\bigr\rangle \geq \lambda \bigl\|(I-T)x-(I-T)y\bigr\|^q. \tag{2.2}$$

(ii) *L-Lipschitzian* if, for all $x, y \in C$, there exists a constant $L > 0$ such that

$$\|Tx-Ty\| \leq L\|x-y\|.$$

If $0 < L < 1$, then T is a contraction and if $L = 1$, then T is a nonexpansive mapping. By the definition, we know that every λ-strictly pseudo-contractive mapping is $(\frac{1+\lambda}{\lambda})$-Lipschitzian (see [33]).

Remark 2.2 Let C be a nonempty subset of a real Hilbert space H and $T : C \to C$ be a mapping. Then T is said to be *k-strictly pseudo-contractive* [32] if, for all $x, y \in C$, there exists $k \in [0, 1)$ such that

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\bigl\|(I-T)x-(I-T)y\bigr\|^2. \tag{2.3}$$

It is well known that (2.3) is equivalent to the following inequality:

$$\langle Tx-Ty, x-y\rangle \leq \|x-y\|^2 - \frac{1-k}{2}\bigl\|(I-T)x-(I-T)y\bigr\|^2.$$

A mapping $F : C \to X$ is said to be *accretive* if, for all $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\bigl\langle Fx-Fy, j_q(x-y)\bigr\rangle \geq 0.$$

For some $\eta > 0$, $F : C \to X$ is said to be *strongly accretive* if, for all $x, y \in C$, there exists $j_q(x-y) \in J_q(x-y)$ such that

$$\bigl\langle Fx-Fy, j_q(x-y)\bigr\rangle \geq \eta\|x-y\|^q.$$

Remark 2.3 If $X := H$ is a real Hilbert space, accretive and strongly accretive mappings coincide with monotone and strongly monotone mappings, respectively.

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be *sunny* [34] if

$$Q\bigl(Qx + t(x-Qx)\bigr) = Qx,$$

whenever $Qx + t(x-Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \to D$ is said to be *retraction* if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a *sunny nonexpansive retraction* of C if there exists a sunny nonexpansive retraction from C onto D. It is well known that if $X := H$ is a real Hilbert space, then a sunny nonexpansive retraction Q is coincident with the metric projection from X onto C.

Lemma 2.4 ([14]) *Let C be a closed and convex subset of a smooth Banach space X. Let D be a nonempty subset of C. Let $Q : C \to D$ be a retraction and let j, j_q be the normalized* *duality mapping and generalized duality mapping on X, respectively. Then the following are equivalent*:

(a) *Q is sunny and nonexpansive.*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, j(Qx - Qy)\rangle$ *for all* $x, y \in C$.
(c) $\langle x - Qx, j(y - Qx)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in D$.
(d) $\langle x - Qx, j_q(y - Qx)\rangle \leq 0$ *for all* $x \in C$ *and* $y \in D$.

Lemma 2.5 ([35]) *Suppose that $q > 1$. Then the following inequality holds*:

$$ab \leq \frac{1}{q}a^q + \left(\frac{q-1}{q}\right)b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b.

In a real q-uniformly smooth Banach space, Xu [36] proved the following important inequality:

Lemma 2.6 ([36]) *Let X be a real q-uniformly smooth Banach space. Then the following inequality holds*:

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q \tag{2.4}$$

for all $x, y \in X$ and for some $C_q > 0$.

Remark 2.7 The constant C_q satisfying (2.4) is called the *best q-uniform smoothness constant.*

Lemma 2.8 ([21]) *Let C be a nonempty and convex subset of a real q-uniformly smooth Banach space X and $T : C \to C$ be a λ-strict pseudo-contraction. For $\gamma \in (0,1)$, define $Sx = (1-\gamma)x + \gamma Tx$. Then, as $\gamma \in (0,\nu)$, $\nu = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, $S : C \to C$ is nonexpansive and $\operatorname{Fix}(S) = \operatorname{Fix}(T)$, where C_q is the best q-uniform smoothness constant.*

Definition 2.9 ([37]) Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X. Let $T_{n,k} = \theta_{n,k}S_k + (1-\theta_{n,k})I$, where $S_k : C \to C$ is λ_k-strict pseudo-contraction and $\{t_n\}$ be a nonnegative real sequence with $0 \leq t_n \leq 1$, $\forall n \in \mathbb{N}$. For $n \geq 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = t_nT_{n,n}U_{n,n+1} + (1-t_n)I, \\
&\vdots \\
&U_{n,k} = t_kT_{n,k}U_{n,k+1} + (1-t_k)I, \\
&U_{n,k-1} = t_{k-1}T_{n,k-1}U_{n,k} + (1-t_{k-1})I, \\
&\vdots \\
&U_{n,2} = t_2T_{n,2}U_{n,3} + (1-t_2)I, \\
&W_n = U_{n,1} = t_1T_{n,1}U_{n,2} + (1-t_1)I.
\end{aligned} \tag{2.5}$$

Such a mapping W_n is called the W-mapping generated by $T_{n,n}, T_{n,n-1}, \dots, T_{n,1}$ and $t_n, t_{n-1}, \dots, t_1$.

Throughout this paper, we will assume that $\{\theta_{n,k}\}$ satisfies the following conditions:
(H1) $\theta_{n,k} \in (0, \nu]$, $\nu = \min\{1, (\frac{q\bar{\lambda}}{C_q})^{\frac{1}{q-1}}\}$ with $\bar{\lambda} = \inf \lambda_k > 0$, $\forall n, k \in \mathbb{N}$;
(H2) $|\theta_{n+1,k} - \theta_{n,k}| \leq a_n$, $\forall n \in \mathbb{N}$ and $1 \leq k \leq n$ with $\sum_{n=1}^{\infty} a_n < \infty$;
The hypothesis (H2) secures the existence of $\lim_{n\to\infty} \theta_{n,k}$, $\forall k \in \mathbb{N}$. Set $\theta_{1,k} := \lim_{n\to\infty} \theta_{n,k}$, $\forall n \in \mathbb{N}$. Furthermore, we assume
(H3) $\theta_{1,k} > 0$, $\forall k \in \mathbb{N}$.
It is obvious that $\theta_{1,k}$ satisfies (H1). Using condition (H3), from $T_{n,k} = \theta_{n,k} S_k + (1 - \theta_{n,k})I$, we define mappings $T_{1,k}x := \lim_{n\to\infty} T_{n,k}x = \theta_{1,k} S_k x + (1 - \theta_{1,k})x$, $\forall x \in C$.

Lemma 2.10 ([37]) *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth and strictly convex Banach space X. Let $T_{n,i} = \theta_{n,i} S_i + (1 - \theta_{n,i})I$, where $S_i : C \to C$ $(i = 1, 2, \dots)$ is λ_i-strict pseudo-contraction with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$ and $\inf \lambda_i > 0$. Let $t_1, t_2, \dots$ be nonnegative real numbers such that $0 < t_n \leq b < 1$, $\forall n \geq 1$. Assume the sequence $\{\theta_{n,k}\}$ satisfies* (H1)-(H3). *Then*
(1) *W_n is nonexpansive and $\mathrm{Fix}(W_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$ for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists;*
(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad \forall x \in C,$$

is a nonexpansive mapping satisfying $\mathrm{Fix}(W) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$ and it is called the W-mapping generated by $S_1, S_2, \dots$ and $t_1, t_2, \dots$ and $\theta_{n,k}$, $\forall n \in \mathbb{N}$ and $1 \leq k \leq n$.

Lemma 2.11 ([37]) *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth and strictly convex Banach space X. Let $T_{n,i} = \theta_{n,i} S_i + (1 - \theta_{n,i})I$, where $S_i : C \to C$ $(i = 1, 2, \dots)$ is λ_i-strict pseudo-contraction with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \neq \emptyset$ and $\inf \lambda_i > 0$. Let $t_1, t_2, \dots$ be nonnegative real numbers such that $0 < t_n \leq b < 1$, $\forall n \geq 1$. Assume the sequence $\{\theta_{n,k}\}$ satisfies* (H1)-(H3). *If $\{\omega_n\}$ is a bounded sequence in C, then*

$$\lim_{n\to\infty} \|W\omega_n - W_n\omega_n\| = 0.$$

In the following, the notation $\rightharpoonup$ and $\to$ denote the weak and strong convergence, respectively. The duality mapping J_q from a smooth Banach space X into X^* is said to be *weakly sequentially continuous generalized duality mapping* if, for all $\{x_n\} \subset X$, $x_n \rightharpoonup x$ implies $J_q(x_n) \rightharpoonup^* J_q(x)$.

A Banach space X is said to be satisfy *Opial's condition* [38], that is, for any sequence $\{x_n\}$ in X, $x_n \rightharpoonup x$ implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X \text{ with } x \neq y.$$

By Theorem 3.2.8 in [39], it is well known that if X admits a weakly sequentially continuous generalized duality mapping, then X satisfies Opial's condition.

Lemma 2.12 ([13]) *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X which admits weakly sequentially continuous generalized duality* *mapping j_q from X into X^*. Let $T : C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 2.13 ([40]) *Let $\{a_n\}$, $\{\mu_n\}$, and $\{\delta_n\}$ be real sequences of nonnegative numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \mu_n + \delta_n, \quad \forall n \geq 1,$$

where $\sigma_n \in (0,1)$, $\sum_{n=1}^{\infty} \sigma_n = \infty$, $\mu_n = \circ(\sigma_n)$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

In order to prove our main result, the following lemma is needed.

Lemma 3.1 *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X with the best q-uniform smoothness constant $C_q > 0$. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$. Let $0 < \mu < (\frac{q\eta}{C_q\kappa^q})^{\frac{1}{q-1}}$ and $\tau = \mu(\eta - \frac{C_q\mu^{q-1}\kappa^q}{q})$. Then for $t \in (0, \min\{1, \frac{1}{q\tau}\})$, the mapping $S : C \to X$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

Proof Since $0 < \mu < (\frac{q\eta}{C_q\kappa^q})^{\frac{1}{q-1}}$ and $t \in (0, \min\{1, \frac{1}{q\tau}\})$. This implies that $1 - t\tau \in (0,1)$. From Lemma 2.6, for all $x, y \in C$, we have

$$\begin{aligned}
\|Sx - Sy\|^q &= \big\|(I - t\mu F)x - (I - t\mu F)y\big\|^q \\
&= \big\|(x - y) - t\mu(Fx - Fy)\big\|^q \\
&\leq \|x - y\|^q - qt\mu\langle Fx - Fy, j_q(x - y)\rangle + C_q t^q \mu^q \|Fx - Fy\|^q \\
&\leq \|x - y\|^q - qt\mu\eta\|x - y\|^q + C_q t^q \mu^q \kappa^q \|x - y\|^q \\
&\leq \big[1 - t\mu\big(q\eta - C_q\mu^{q-1}\kappa^q\big)\big]\|x - y\|^q \\
&= \left[1 - t\mu q\left(\eta - \frac{C_q\mu^{q-1}\kappa^q}{q}\right)\right]\|x - y\|^q \\
&\leq \left[1 - t\mu\left(\eta - \frac{C_q\mu^{q-1}\kappa^q}{q}\right)\right]^q \|x - y\|^q \\
&= (1 - t\tau)^q\|x - y\|^q.
\end{aligned}$$

It follows that

$$\|Sx - Sy\| \leq (1 - t\tau)\|x - y\|.$$

Hence, we have $S := I - t\mu F$ is a contraction with constant $1 - t\tau$. □

Lemma 3.2 *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X and $G : C \to X$ be a mapping.*

(i) *If G is a δ-strongly accretive and λ-strictly pseudo-contractive mapping wit $\delta + \lambda > 1$, then $I - G$ is a contraction with constant $L_{\delta,\lambda} := (\frac{1-\delta}{\lambda})^{\frac{1}{q}}$.*

(ii) *If G is a δ-strongly accretive and λ-strictly pseudo-contractive mapping with $\delta + \lambda > 1$. For a fixed number $t \in (0,1)$, then $I - tG$ is a contraction with constant $1 - (1 - L_{\delta,\lambda})t$.*

Proof (i) For all $x, y \in C$, from (2.2), we have

$$\begin{aligned}\lambda\big\|(I-G)x-(I-G)y\big\|^q &\le \|x-y\|^q-\big\langle Gx-Gy, j_q(x-y)\big\rangle \\ &\le (1-\delta)\|x-y\|^q.\end{aligned}$$

Observe that

$$\delta+\lambda>1 \quad\Longleftrightarrow\quad \left(\frac{1-\delta}{\lambda}\right)^{\frac{1}{q}} \in (0,1).$$

It follows that

$$\big\|(I-G)x-(I-G)y\big\| \le \left(\frac{1-\delta}{\lambda}\right)^{\frac{1}{q}}\|x-y\| := L_{\delta,\lambda}\|x-y\|.$$

Hence, $I-G$ is a contraction with constant $L_{\delta,\lambda}$.

(ii) Since $I-G$ is a contraction with constant $L_{\delta,\lambda}$. For all $t \in (0,1)$, we have

$$\begin{aligned}\big\|(I-tG)x-(I-tG)y\big\| &= \big\|(x-y)-t(Gx-Gy)\big\| \\ &= \big\|(1-t)(x-y)+t\big[(I-G)x-(I-G)y\big]\big\| \\ &\le (1-t)\|x-y\|+t\big\|(I-G)x-(I-G)y\big\| \\ &\le \big(1-(1-L_{\delta,\lambda})t\big)\|x-y\|.\end{aligned}$$

Hence, $I-tG$ is a contraction with constant $1-(1-L_{\delta,\lambda})t$. This completes the proof. □

3.1 Implicit iteration scheme

Let C be a nonempty, closed, and convex subset of a real reflexive and q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F: C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$, $G: C \to X$ be a δ-strongly accretive and λ-strictly pseudo-contractive mapping with $\delta+\lambda>1$, $V: C \to X$ be an L-Lipschitzian mapping with constant $L \ge 0$ and $T: C \to C$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{C_q\kappa^q})^{\frac{1}{q-1}}$ and $0 \le \gamma L < \tau$, where $\tau = \mu(\eta - \frac{C_q\mu^{q-1}\kappa^q}{q})$. For each $\sigma \in (\frac{L_{\delta,\lambda}}{\tau-\gamma L}, \min\{1, \frac{1}{q\tau}, \frac{1+L_{\delta,\lambda}}{\tau-\gamma L}\})$ and $t \in (0,1)$, we define a mapping $S_t: C \to C$ defined by

$$Sx := Q_C\big[(I-tG)Tx+t\big(Tx-\sigma(\mu FTx-\gamma Vx)\big)\big], \quad \forall x \in C.$$

It is easy to see immediately that S_t is a contraction. Indeed, for all $x, y \in C$, from Lemmas 3.1 and 3.2(ii), we have

$$\begin{aligned}\|S_tx-S_ty\| &= \big\|Q_C\big[(I-tG)Tx+t\big(Tx-\sigma(\mu FTx-\gamma Vx)\big)\big] \\ &\quad - Q_C\big[(I-tG)Ty+t\big(Ty-\sigma(\mu FTy-\gamma Vy)\big)\big]\big\| \\ &\le \big\|(I-tG)(Tx-Ty)+t\big[(I-\sigma\mu F)(Tx-Ty)+\sigma\gamma(Vx-Vy)\big]\big\|\end{aligned}$$

$$\begin{aligned}
&\le \bigl(1 - t(1 - L_{\delta,\lambda})\bigr)\|x - y\| + t\bigl[\sigma\gamma\|Vx - Vy\| + \bigl\|(I - \sigma\mu F)(Tx - Ty)\bigr\|\bigr] \\
&\le \bigl(1 - t(1 - L_{\delta,\lambda})\bigr)\|x - y\| + t\bigl(1 - \sigma(\tau - \gamma L)\bigr)\|x - y\| \\
&= \bigl[1 - t\bigl(\sigma(\tau - \gamma L) - L_{\delta,\lambda}\bigr)\bigr]\|x - y\| \\
&= (1 - t\theta)\|x - y\|,
\end{aligned} \tag{3.1}$$

where $\theta := \sigma(\tau - \gamma L) - L_{\delta,\lambda}$. Since $\tau - \gamma L > 0$ and $L_{\delta,\lambda} \in (0,1)$, observe that

$$\frac{L_{\delta,\lambda}}{\tau - \gamma L} < \sigma < \min\left\{1, \frac{1}{q\tau}, \frac{1 + L_{\delta,\lambda}}{\tau - \gamma L}\right\} \le \frac{1 + L_{\delta,\lambda}}{\tau - \gamma L}.$$

It follows that

$$\sigma < \frac{1 + L_{\delta,\lambda}}{\tau - \gamma L} \quad \Longleftrightarrow \quad \theta = \sigma(\tau - \gamma L) - L_{\delta,\lambda} < 1$$

and

$$\frac{L_{\delta,\lambda}}{\tau - \gamma L} < \sigma \quad \Longleftrightarrow \quad \theta = \sigma(\tau - \gamma L) - L_{\delta,\lambda} > 0.$$

This implies that $\theta = \sigma(\tau - \gamma L) - L_{\delta,\lambda} \in (0,1)$, which together with $t \in (0,1)$ gives

$$1 - t\bigl(\sigma(\tau - \gamma L) - L_{\delta,\lambda}\bigr) \in (0,1).$$

Hence S_t is a contraction. By the Banach contraction principle, S_t has a unique fixed point, denote by x_t, which uniquely solves the fixed point equation

$$x_t = Q_C\bigl[(I - tG)Tx_t + t\bigl(Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr)\bigr]. \tag{3.2}$$

The following proposition summarizes the properties of the net $\{x_t\}$.

Proposition 3.3 *Let* $\{x_t\}$ *be defined by* (3.2). *Then the following hold*:
- (i) $\{x_t\}$ *is bounded for each* $t \in (0,1)$;
- (ii) $\lim_{t\to 0}\|x_t - Tx_t\| = 0$;
- (iii) $\{x_t\}$ *defines a continuous curve from* $(0,1)$ *into* C.

Proof (i) Take $p \in \mathrm{Fix}(T)$, and denote a mapping $S_t : C \to C$ by

$$S_t x := Q_C\bigl[(I - tG)Tx + t\bigl(Tx - \sigma(\mu FTx - \gamma Vx)\bigr)\bigr], \quad \forall x \in C.$$

From (3.1), we have

$$\begin{aligned}
\|x_t - p\| &\le \|S_t x_t - S_t p\| + \|S_t p - p\| \\
&\le (1 - t\theta)\|x_t - p\| + \bigl\|Q_C\bigl[(I - tG)Tp + t\bigl(Tp - \sigma(\mu FTp - \gamma Vp)\bigr)\bigr] - Q_C p\bigr\| \\
&\le (1 - t\theta)\|x_t - p\| + t\bigl\|-Gp + p - \sigma(\mu Fp - \gamma Vp)\bigr\| \\
&\le (1 - t\theta)\|x_t - p\| + t\bigl[\|I - G\|\|p\| + \sigma\mu\|Fp\| + \sigma\gamma\|Vp\|\bigr],
\end{aligned}$$

where $\theta := \sigma(\tau - \gamma L) - L_{\delta,\lambda}$. It follows that

$$\|x_t - p\| \le \frac{\|I - G\|\|p\| + \sigma\mu\|Fp\| + \sigma\gamma\|Vp\|}{\theta}.$$

Hence $\{x_t\}$ is bounded, so are $\{Vx_t\}$, $\{FTx_t\}$, and $\{GTx_t\}$.

(ii) By definition of $\{x_t\}$, we have

$$\begin{aligned}\|x_t - Tx_t\| &= \bigl\|Q_C\bigl[(I - tG)Tx_t + t\bigl(Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr)\bigr] - Q_C Tx_t\bigr\| \\ &\le t\bigl\|(I - G)Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr\| \to 0 \quad \text{as } t \to 0.\end{aligned}$$

(iii) Take $t, t_0 \in (0,1)$ and calculate

$$\begin{aligned}\|x_t - x_{t_0}\| &= \bigl\|Q_C\bigl[(I - tG)Tx_t + t\bigl(Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr)\bigr] \\ &\quad - Q_C\bigl[(I - t_0 G)Tx_{t_0} + t\bigl(Tx_{t_0} - \sigma(\mu FTx_{t_0} - \gamma Vx_{t_0})\bigr)\bigr]\bigr\| \\ &\le \bigl\|(t_0 - t)GTx_t + (I - t_0 G)(Tx_t - Tx_{t_0}) + (t - t_0)\bigl[Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr] \\ &\quad + t_0\bigl[Tx_t - \sigma(\mu FTx_{t_0} - \gamma Vx_{t_0}) - \bigl[Tx_{t_0} - \sigma(\mu FTx_{t_0} - \gamma Vx_{t_0})\bigr]\bigr]\bigr\| \\ &= \bigl\|(t_0 - t)GTx_t + (I - t_0 G)(Tx_t - Tx_{t_0}) + (t - t_0)\bigl[Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr] \\ &\quad + t_0\bigl[\sigma\gamma(Vx_t - Vx_{t_0}) + (I - \sigma\mu F)(Tx_t - Tx_{t_0})\bigr]\bigr\| \\ &\le |t - t_0|\|GTx_t\| + \bigl(1 - (1 - L_{\delta,\lambda})\bigr)\|x_t - x_{t_0}\| \\ &\quad + |t - t_0|\bigl\|Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr\| \\ &\quad + t_0\bigl(1 - \sigma(\tau - \gamma L)\bigr)\|x_t - x_{t_0}\|.\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\|GTx_t\| + \|Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\|}{t_0(\sigma(\tau - \gamma L) - L_{\delta,\lambda})}|t - t_0|.$$

Since $\{Vx_t\}$, $\{FTx_t\}$, and $\{GTx_t\}$ are bounded. Hence $\{x_t\}$ defines a continuous curve from $(0,1)$ into C. □

Theorem 3.4 *Assume that* $\{x_t\}$ *is defined by* (3.2), *then* $\{x_t\}$ *converges strongly to* $x^* \in \operatorname{Fix}(T)$ *as* $t \to 0$, *where* x^* *is the unique solution of the variational inequality*

$$\bigl\langle\bigl(G - I + \sigma(\mu F - \gamma V)\bigr)x^*, j_q\bigl(x^* - v\bigr)\bigr\rangle \le 0, \quad \forall v \in \operatorname{Fix}(T). \tag{3.3}$$

Proof We observe that

$$\begin{aligned}\frac{C_q\mu^{q-1}\kappa^q}{q} > 0 \quad &\Longleftrightarrow \quad \eta - \frac{C_q\mu^{q-1}\kappa^q}{q} < \eta \\ &\Longleftrightarrow \quad \mu\left(\eta - \frac{C_q\mu^{q-1}\kappa^q}{q}\right) < \mu\eta \\ &\Longleftrightarrow \quad \tau < \mu\eta.\end{aligned} \tag{3.4}$$

It follows that

$$0 \le \gamma L < \tau < \mu\eta. \tag{3.5}$$

First, we show the uniqueness of solution of the variational inequality. Suppose that $\tilde{x}, x^* \in \operatorname{Fix}(T)$ are solutions of (3.3), then

$$\langle (G - I + \sigma(\mu F - \gamma V))x^*, j_q(x^* - \tilde{x})\rangle \le 0 \tag{3.6}$$

and

$$\langle (G - I + \sigma(\mu F - \gamma V))\tilde{x}, j_q(\tilde{x} - x^*)\rangle \le 0. \tag{3.7}$$

Adding up (3.6) and (3.7), and from Lemma 3.2(i), we obtain

$$\begin{aligned}
0 &\ge \langle (G - I + \sigma(\mu F - \gamma V))x^* - (G - I + \sigma(\mu F - \gamma V))\tilde{x}, j_q(x^* - \tilde{x})\rangle \\
&= \langle (G - I)x^* - (G - I)\tilde{x}, j_q(x^* - \tilde{x})\rangle + \sigma\langle (\mu F - \gamma V)x^* - (\mu F - \gamma V)\tilde{x}, j_q(x^* - \tilde{x})\rangle \\
&= -\langle (I - G)x^* - (I - G)\tilde{x}, j_q(x^* - \tilde{x})\rangle + \sigma\mu\langle Fx^* - F\tilde{x}, j_q(x^* - \tilde{x})\rangle \\
&\quad - \sigma\gamma\langle Vx^* - V\tilde{x}, j_q(x^* - \tilde{x})\rangle \\
&\ge -L_{\delta,\lambda}\|x^* - \tilde{x}\|^q + \sigma\mu\eta\|x^* - \tilde{x}\|^q - \sigma\gamma\|Vx^* - V\tilde{x}\|\|x^* - \tilde{x}\|^{q-1} \\
&\ge (\sigma(\mu\eta - \gamma L) - L_{\delta,\lambda})\|x^* - \tilde{x}\|^q.
\end{aligned}$$

On the other hand, we observe from (3.5) that

$$\begin{aligned}
\frac{L_{\delta,\lambda}}{\tau - \gamma L} < \sigma \quad &\Longleftrightarrow \quad L_{\delta,\lambda} < \sigma(\tau - \gamma L) \\
&\Longleftrightarrow \quad L_{\delta,\lambda} < \sigma(\mu\eta - \gamma L) \\
&\Longleftrightarrow \quad 0 < \sigma(\mu\eta - \gamma L) - L_{\delta,\lambda}.
\end{aligned} \tag{3.8}$$

Note that (3.8) implies that $x^* = \tilde{x}$ and the uniqueness is proved. Below, we use $\tilde{x}$ to denote the unique solution of the variational inequality (3.3).

Next, we show that $x_t \to x^*$ as $t \to 0$. Set $x_t = Q_C y_t$, where $y_t = (I - tG)Tx_t + t(Tx_t - \sigma(\mu FTx_t - \gamma Vx_t))$. Assume that $\{t_n\} \subset (0,1)$ is a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$ and $y_n := y_{t_n}$. For $z \in \operatorname{Fix}(T)$, we note that

$$\begin{aligned}
x_n - z &= Q_C y_n - y_n + y_n - z \\
&= Q_C y_n - y_n + (I - t_n G)(Tx_n - z) + t_n(Tx_n - \sigma(\mu FTx_n - \gamma Vx_n) - Gz) \\
&= Q_C y_n - y_n + (I - t_n G)(Tx_n - z) + t_n[(I - \sigma\mu F)Tx_n + \sigma\gamma Vx_n - Gz] \\
&= Q_C y_n - y_n + (I - t_n G)(Tx_n - z) + t_n[(I - \sigma\mu F)(Tx_n - z) + \sigma\gamma(Vx_n - Vz)] \\
&\quad + t_n[(I - \sigma\mu F)z + \sigma\gamma Vz - Gz].
\end{aligned} \tag{3.9}$$

By Lemma 2.4, we have

$$\langle Q_C y_n - y_n, j_q(Q_C y_n - z)\rangle \le 0. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\begin{aligned}\|x_n - z\|^q &= \langle Q_C y_n - y_n, j_q(Q_C y_n - z)\rangle + \langle y_n - z, j_q(x_n - z)\rangle \\ &\leq \langle (I - t_n G)(Tx_n - z), j_q(x_n - z)\rangle + t_n\langle (I - \sigma\mu F)(Tx_n - z), j_q(x_n - z)\rangle \\ &\quad + t_n\sigma\gamma\langle Vx_n - Vz, j_q(x_n - z)\rangle + t_n\langle (I - \sigma\mu F)z + \sigma\gamma Vz - Gz, j_q(x_n - z)\rangle \\ &\leq \bigl[1 - t_n\bigl(\sigma(\tau - \gamma L) - L_{\delta,\lambda}\bigr)\bigr]\|x_n - z\|^q \\ &\quad + t_n\langle (I - \sigma\mu F)z + \sigma\gamma Vz - Gz, j_q(x_n - z)\rangle,\end{aligned}$$

which implies that

$$\|x_n - z\|^q \leq \frac{1}{\sigma(\tau - \gamma L) - L_{\delta,\lambda}}\langle (I - \sigma\mu F)z + \sigma\gamma Vz - Gz, j_q(x_n - z)\rangle.$$

In particular, we have

$$\|x_{n_i} - z\|^q \leq \frac{1}{\sigma(\tau - \gamma L) - L_{\delta,\lambda}}\langle (I - \sigma\mu F)z + \sigma\gamma Vz - Gz, j_q(x_{n_i} - z)\rangle. \tag{3.11}$$

By reflexivity of a Banach space X and boundedness of $\{x_n\}$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \tilde{x}$ as $i \to \infty$. Since a Banach space X has a weakly sequentially continuous generalized duality mapping and by (3.11), we obtain $x_{n_i} \to \tilde{x}$. By Proposition 3.3(ii), we have $x_{n_i} - Tx_{n_i} \to 0$ as $i \to \infty$. Hence, it follows from Lemma 2.12 that $\tilde{x} \in \operatorname{Fix}(T)$.

Next, we show that $\tilde{x}$ solves the variational inequality (3.3). We note that

$$x_t = Q_C y_t = Q_C y_t - y_t + (I - tG)Tx + t\bigl(Tx_t - \sigma(\mu FTx_t - \gamma Vx_t)\bigr),$$

we derive

$$\bigl(G - I + \sigma(\mu F - \gamma V)\bigr)x_t = \frac{1}{t}(Q_C y_t - y_t) - \frac{1}{t}\bigl((I - tG)(I - T)x_t + t(I - \sigma\mu F)(I - T)x_t\bigr). \tag{3.12}$$

Since $I - T$ is accretive (*i.e.*, $\langle (I - T)x - (I - T)y, j_q(x - y)\rangle \geq 0$ for $x, y \in C$). For all $v \in \operatorname{Fix}(T)$, it follows from (3.10) and (3.12) that

$$\begin{aligned}&\langle \bigl(G - I + \sigma(\mu F - \gamma V)\bigr)x_t, j_q(x_t - v)\rangle \\ &\quad = \frac{1}{t}\langle Q_C y_t - y_t, j_q(Q_C y_t - v)\rangle - \frac{1}{t}\langle (I - tG)(I - T)x_t, j_q(x_t - v)\rangle \\ &\qquad - \langle (I - \sigma\mu F)(I - T)x_t, j_q(x_t - v)\rangle \\ &\quad \leq -\frac{1}{t}\langle (I - T)x_t - (I - T)v, j_q(x_t - v)\rangle + \langle G(I - T)x_t, j_q(x_t - v)\rangle \\ &\qquad - \langle (I - T)x_t - (I - T)v, j_q(x_t - v)\rangle + \sigma\mu\langle F(I - T)x_t, j_q(x_t - v)\rangle \\ &\quad \leq \langle G(I - T)x_t, j_q(x_t - v)\rangle + \sigma\mu\langle F(I - T)x_t, j_q(x_t - v)\rangle \\ &\quad \leq \|G\|\|x_t - Tx_t\|\|x_t - v\|^{q-1} + \sigma\mu\|F\|\|x_t - Tx_t\|\|x_t - v\|^{q-1} \\ &\quad \leq \|x_t - Tx_t\|M_1,\end{aligned} \tag{3.13}$$

where $M_1 > 0$ is an appropriate constant such that $M_1 = \sup_{t\in(0,1)}\{\|G\|\|x_t - v\|^{q-1}, \sigma\mu\|F\| \times \|x_t - v\|^{q-1}\}$. Now, replacing t in (3.13) with t_n and taking the limit as $n \to \infty$, we notice that $x_{t_n} - Tx_{t_n} \to \tilde{x} - T\tilde{x} = 0$, we obtain

$$\langle (G - I + \sigma(\mu F - \gamma V))\tilde{x}, j_q(\tilde{x} - v)\rangle \leq 0.$$

That is, $\tilde{x} \in \operatorname{Fix}(T)$ is the solution of the variational inequality (3.3). Consequently, $\tilde{x} = x^*$ by uniqueness. In a summary, we have shown that each cluster point of $\{x_t\}$ is equal to x^*. Therefore $x_t \to x^*$ as $t \to 0$. This completes the proof. □

3.2 Explicit iteration scheme

Theorem 3.5 *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth and uniformly convex Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let Q_C be a sunny nonexpansive retraction such that X onto C. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$, $G : C \to X$ be a δ-strongly accretive and λ-strictly pseudo-contractive mapping with $\delta + \lambda > 1$, $V : C \to X$ be an L-Lipschitzian mapping with constant $L \geq 0$. Let $\{S_i\}_{i=1}^{\infty}$ be an infinite family of λ_i-strictly pseudo-contractive mapping from C into itself such that $\mathcal{F} := \bigcap_{i=1}^{\infty} \operatorname{Fix}(S_i) \neq \emptyset$. For given $x_1 \in C$, define the sequence $\{x_n\}$ by*

$$x_{n+1} = Q_C\big[(I - \alpha_n G)W_n x_n + \alpha_n\big(W_n x_n - \sigma(\mu F W_n x_n - \gamma V x_n)\big)\big], \quad \forall n \geq 1, \tag{3.14}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ which satisfies the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $|\alpha_{n+1} - \alpha_n| \leq \circ(\alpha_n) + \sigma_n$ *with* $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Suppose in addition that $\{\theta_{n,k}\}$ satisfies (H1)-(H3). *Then the sequence $\{x_n\}$ defined by* (3.14) *converges strongly to $x^* \in \mathcal{F}$ as $n \to \infty$, where x^* is the unique solution of the variational inequality*

$$\langle (G - I + \sigma(\mu F - \gamma V))x^*, j_q(x^* - v)\rangle \leq 0, \quad \forall v \in \mathcal{F}. \tag{3.15}$$

Proof From the condition (C1), we may assume, without loss of generality, that $\alpha_n \leq \min\{1, \frac{1}{q\tau}\}$ for all $n \in \mathbb{N}$. First, we show that $\{x_n\}$ is bounded. Take $p \in \mathcal{F}$, and denote a mapping $S_n^{\alpha_n} : C \to C$ by

$$S_n^{\alpha_n} x := Q_C\big[(I - \alpha_n G)W_n x + \alpha_n\big(W_n x - \sigma(\mu F W_n x - \gamma V x)\big)\big], \quad \forall x \in C.$$

Then we have

$$S_n^{\alpha_n} p = Q_C\big[(I - \alpha_n G)W_n p + \alpha_n\big(W_n p - \sigma(\mu F W_n p - \gamma V p)\big)\big].$$

From (3.1), we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \big\|S_n^{\alpha_n} x_n - S_n^{\alpha_n} p\big\| + \big\|S_n^{\alpha_n} p - p\big\| \\ &\leq (1 - \alpha_n\theta)\|x_n - p\| + \big\|Q_C\big[(I - \alpha_n G)p + \alpha_n\big(p - \sigma(\mu F p - \gamma V p)\big)\big] - Q_C p\big\|\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n\theta)\|x_n-p\| + \alpha_n\bigl\|-Gp+p-\sigma(\mu Fp-\gamma Vp)\bigr\| \\
&\leq (1-\alpha_n\theta)\|x_n-p\| + \alpha_n\bigl(\|I-G\|\|p\| + \sigma\|\mu Fp-\gamma Vp\|\bigr) \\
&\leq (1-\alpha_n\theta)\|x_n-p\| + \alpha_n\theta\frac{\|I-G\|\|p\| + \sigma\mu\|Fp\| + \gamma\|Vp\|}{\theta} \\
&\leq \max\biggl\{\|x_n-p\|, \frac{\|I-G\|\|p\| + \sigma\mu\|Fp\| + \gamma\|Vp\|}{\theta}\biggr\},
\end{aligned}$$

where $\theta := \sigma(\tau - \gamma L) - L_{\delta,\lambda}$. By induction, we obtain

$$\|x_n-p\| \leq \max\biggl\{\|x_1-p\|, \frac{\|I-G\|\|p\| + \sigma\mu\|Fp\| + \gamma\|Vp\|}{\theta}\biggr\}, \quad \forall n \geq 1.$$

Hence, $\{x_n\}$ is bounded, so are $\{Vx_n\}$, $\{FW_nx_n\}$, and $\{GW_nx_n\}$.

Next, we show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. Set $S_n^{\alpha_n}x_n = Q_C y_n$, where $y_n = (I - \alpha_n G)W_nx_n + \alpha_n(W_nx_n - \sigma(\mu FW_nx_n - \gamma Vx_n))$. From (2.5), we have

$$\begin{aligned}
&\|W_{n+1}x_n - W_nx_n\| \\
&\quad= \bigl\|t_1T_{n+1,1}U_{n+1,2}x_n + (1-t_1)x_n - t_1T_{n,1}U_{n,2} - (1-t_1)x_n\bigr\| \\
&\quad= t_1\|T_{n+1,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| \\
&\quad= t_1\bigl\|\bigl(\theta_{n+1,1}S_1 + (1-\theta_{n+1,1})\bigr)U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\bigr\| \\
&\quad= t_1\bigl\|\bigl(\theta_{n,1}S_1 + (1-\theta_{n,1})\bigr)U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n + (\theta_{n+1,1}-\theta_{n,1})(S_1U_{n+1,2}x_n - U_{n+1,2}x_n)\bigr\| \\
&\quad\leq t_1\|T_{n,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| + t_1|\theta_{n+1,2}-\theta_{n,1}|\|S_1U_{n+1,2}x_n - U_{n+1,2}x_n\| \\
&\quad\leq t_1\|T_{n,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| + t_1|\theta_{n+1,2}-\theta_{n,1}|M^* \\
&\quad\leq t_1\|T_{n,1}U_{n+1,2}x_n - T_{n,1}U_{n,2}x_n\| + t_1a_nM_1 \\
&\qquad\vdots \\
&\quad\leq \prod_{i=1}^{n} t_i\|U_{n+1,n+1}x_n - U_{n,n+1}x_n\| + \Biggl(a_n\sum_{j=1}^{n}\sum_{i=1}^{j} t_i\Biggr)M_1 \\
&\quad\leq \prod_{i=1}^{n} t_i\bigl\|t_{n+1}T_{n+1,n+1}x_n + (1-t_{n+1})x_n - x_n\bigr\| + \frac{b}{1-b}a_nM_1 \\
&\quad\leq \prod_{i=1}^{n+1} t_i\|T_{n+1,n+1}x_n - x_n\| + \frac{b}{1-b}a_nM_1 \\
&\quad\leq \biggl(b^{n+1} + \frac{b}{1-b}a_n\biggr)M_1,
\end{aligned} \tag{3.16}$$

where $M_1 = \inf_{i=1,2,\dots}(\frac{1+2\lambda_i^{q-1}}{\lambda_i^{q-1}})\sup_{n\geq 1}\{\|x_n-p\|\}$ with $p \in \mathcal{F}$.

On the other hand, we note that

$$\begin{aligned}
y_{n+1} - y_n = {}& (I-\alpha_{n+1}G)W_{n+1}x_n + \alpha_{n+1}\bigl[W_{n+1}x_n - \sigma(\mu FW_{n+1}x_n - \gamma Vx_n)\bigr] \\
&- (I-\alpha_nG)W_nx_n - \alpha_n\bigl[W_nx_n - \sigma(\mu FW_nx_n - \gamma Vx_n)\bigr]
\end{aligned}$$

$$
\begin{aligned}
&= (I-\alpha_{n+1}G)(W_{n+1}x_n - W_nx_n) + (\alpha_n-\alpha_{n+1})GW_nx_n + (\alpha_{n+1}-\alpha_n)W_{n+1}x_n \\
&\quad + \alpha_n(W_{n+1}x_n - W_nx_n) + \sigma(\alpha_n-\alpha_{n+1})(\mu FW_{n+1}x_n - \gamma Vx_n) \\
&\quad - \sigma\alpha_n\bigl[\mu FW_{n+1}x_n - \gamma Vx_n - (\mu FW_nx_n - \gamma Vx_n)\bigr] \\
&= \bigl[(1+\alpha_n)I - \alpha_{n+1}G\bigr](W_{n+1}x_n - W_nx_n) + (\alpha_{n+1}-\alpha_n)[W_{n+1}x_n - GW_nx_n] \\
&\quad + \sigma(\alpha_n-\alpha_{n+1})[\mu FW_{n+1}x_n - \gamma Vx_n] - \sigma\alpha_n\mu F[W_{n+1}x_n - W_nx_n] \\
&= \bigl[(1+\alpha_n)I - \alpha_{n+1}G - \sigma\alpha_n\mu F\bigr](W_{n+1}x_n - W_nx_n) \\
&\quad + (\alpha_{n+1}-\alpha_n)[W_{n+1}x_n - GW_nx_n] + \sigma(\alpha_n-\alpha_{n+1})[\mu FW_{n+1}x_n - \gamma Vx_n].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\bigl\|S_{n+1}^{\alpha_{n+1}}x_n - S_n^{\alpha_n}x_n\bigr\| &= \|Q_Cy_{n+1} - Q_Cy_n\| \\
&\leq \|y_{n+1} - y_n\| \\
&\leq \bigl\|(1+\alpha_n)I - \alpha_{n+1}G - \sigma\alpha_n\mu F\bigr\|\|W_{n+1}x_n - W_nx_n\| \\
&\quad + |\alpha_{n+1}-\alpha_n|\|W_{n+1}x_n - GW_nx_n\| \\
&\quad + \sigma|\alpha_{n+1}-\alpha_n|\|\mu FW_{n+1}x_n - \gamma Vx_n\| \\
&\leq \bigl(\|W_{n+1}x_n - W_nx_n\| + |\alpha_{n+1}-\alpha_n|\bigr)M_2,
\end{aligned}
$$

where $M_2 = \sup_{n\geq 1}\{\|(1+\alpha_n)I - \alpha_{n+1}G - \sigma\alpha_n\mu F\|, \|W_{n+1}x_n - GW_nx_n\|, \sigma\|\mu FW_{n+1}x_n - \gamma Vx_n\|\}$. It follows from (3.1) and (3.16) that

$$
\begin{aligned}
\|x_{n+2} - x_{n+1}\| &\leq \bigl\|S_{n+1}^{\alpha_{n+1}}x_{n+1} - S_{n+1}^{\alpha_{n+1}}x_n\bigr\| + \bigl\|S_{n+1}^{\alpha_{n+1}}x_n - S_n^{\alpha_n}x_n\bigr\| \\
&\leq (1-\alpha_{n+1}\theta)\|x_{n+1} - x_n\| + \bigl(|\alpha_{n+1}-\alpha_n| + \|W_{n+1}x_n - W_nx_n\|\bigr)M_2 \\
&\leq (1-\alpha_{n+1}\theta)\|x_{n+1} - x_n\| + \bigl(\circ(\alpha_n) + \sigma_n\bigr)M_2 + \|W_{n+1}x_n - W_nx_n\|M_2 \\
&\leq (1-\alpha_{n+1}\theta)\|x_{n+1} - x_n\| + \circ(\alpha_n)M_2 + \left(\sigma_n + b^{n+1} + \frac{b}{1-b}a_n\right)M,
\end{aligned}
\tag{3.17}
$$

where $M = \max\{M_1, M_2\}$. Then, by Lemma 2.13, we have

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.18}
$$

Next, we show that $\lim_{n\to\infty}\|x_n - Wx_n\| = 0$. Since

$$
\begin{aligned}
\|x_n - W_nx_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - W_nx_n\| \\
&= \|x_n - x_{n+1}\| + \bigl\|Q_C\bigl[(I-\alpha_nG)W_nx_n + \alpha_n\bigl(W_nx_n - \sigma(\mu FW_nx_n - \gamma Vx_n)\bigr)\bigr] \\
&\quad - Q_CW_nx_n\bigr\| \\
&\leq \|x_n - x_{n+1}\| + \alpha_n\bigl\|(I-G)W_nx_n - \sigma(\mu FW_nx_n - \gamma Vx_n)\bigr\|.
\end{aligned}
$$

From (3.18) and the condition (C1), we obtain

$$
\lim_{n\to\infty}\|x_n - W_nx_n\| = 0. \tag{3.19}
$$

At the same time, observe that

$$\|x_n - Wx_n\| \le \|x_n - W_nx_n\| + \|W_nx_n - Wx_n\|.$$

It follows from (3.19) and Lemma 2.11, we have

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{3.20}$$

Next, we show that

$$\limsup_{n\to\infty} \langle [I - G + \sigma(\gamma V - \mu F)]x^*, j_q(x_n - x^*)\rangle \le 0,$$

where x^* is the same as in Theorem 3.4. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle [I - G + \sigma(\gamma V - \mu F)]x^*, j_q(x_n - x^*)\rangle \\
&\quad = \lim_{i\to\infty} \langle [I - G + \sigma(\gamma V - \mu F)]x^*, j_q(x_{n_i} - x^*)\rangle.
\end{aligned}$$

By reflexivity of a Banach space X and boundedness of $\{x_n\}$, without loss of generality, we may assume that $x_{n_i} \rightharpoonup v$ as $i \to \infty$. It follows from (3.20) and Lemma 2.12 that $v \in \mathcal{F}$. Since a Banach space X has a weakly sequentially continuous generalized duality mapping, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \langle [I - G + \sigma(\gamma V - \mu F)]x^*, j_q(x_n - x^*)\rangle \\
&\quad = \lim_{i\to\infty} \langle [I - G + \sigma(\gamma V - \mu F)]x^*, j_q(x_{n_i} - x^*)\rangle \\
&\quad = \langle [I - G + \sigma(\gamma V - \mu F)]x^* j_q(v - x^*)\rangle \le 0.
\end{aligned} \tag{3.21}$$

Finally, we show that $x_n \to x^*$ as $n \to \infty$. Set $x_{n+1} = Q_C y_n$, where $y_n = (I - \alpha_n G)W_nx_n + \alpha_n(W_nx_n - \sigma(\mu W_nx_n - \gamma Vx_n))$. From Lemmas 2.4 and 2.5, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad = \langle y_n - x^*, j_q(x_{n+1} - x^*)\rangle + \langle Q_C y_n - y_n, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le \langle y_n - x^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad = \langle (I - \alpha_n G)(W_nx_n - x^*), j_q(x_{n+1} - x^*)\rangle + \alpha_n\langle (I - \sigma\mu F)(W_nx_n - x^*), j_q(x_{n+1} - x^*)\rangle \\
&\qquad + \alpha_n\sigma\gamma\langle Vx_n - Vx^*, j_q(x_{n+1} - x^*)\rangle + \alpha_n\langle (I - \sigma\mu F)x^* + \sigma\gamma Vx^* - Gx^*, j_q(x_{n+1} - x^*)\rangle \\
&\quad \le (1 - \alpha_n(1 - L_{\delta,\lambda}))\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \alpha_n(1 - \sigma\tau)\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} \\
&\qquad + \alpha_n\sigma\gamma L\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} + \alpha_n\langle x^* - Gx^* + \sigma(\gamma Vx^* - \mu Fx^*), j_q(x_{n+1} - x^*)\rangle \\
&\quad = (1 - \alpha_n(\sigma(\tau - \gamma L) - L_{\delta,\lambda}))\|x_n - x^*\|\|x_{n+1} - x^*\|^{q-1} \\
&\qquad + \alpha_n\langle x^* - Gx^* + \sigma(\gamma Vx^* - \mu Fx^*), j_q(x_{n+1} - x^*)\rangle
\end{aligned}$$

$$\begin{aligned}&\leq \big(1-\alpha_n\big(\sigma(\tau-\gamma L)-L_{\delta,\lambda}\big)\big)\left[\frac{1}{q}\big\|x_n-x^*\big\|^q+\left(\frac{q-1}{q}\right)\big\|x_{n+1}-x^*\big\|^q\right]\\&\quad+\alpha_n\big\langle x^*-Gx^*+\sigma\big(\gamma Vx^*-\mu Fx^*\big),j_q\big(x_{n+1}-x^*\big)\big\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^q&\leq \big(1-\alpha_n\big(\sigma(\tau-\gamma L)-L_{\delta,\lambda}\big)\big)\big\|x_n-x^*\big\|^q\\&\quad+\frac{q\alpha_n}{1+(q-1)(\sigma(\tau-\gamma L)-L_{\delta,\lambda})}\\&\quad\times\big\langle x^*-Gx^*+\sigma\big(\gamma Vx^*-\mu Fx^*\big),j_q\big(x_{n+1}-x^*\big)\big\rangle.\end{aligned}\tag{3.22}$$

We can write (3.22) to the formula

$$\big\|x_{n+1}-x^*\big\|^q\leq(1-\tau_n)\|x_n-x\|^q+\xi_n,\tag{3.23}$$

where $\tau_n:=(\sigma(\tau-\gamma L)-L_{\delta,\lambda})\alpha_n$ and $\xi_n:=\frac{q\alpha_n}{1+(q-1)(\sigma(\tau-\gamma L)-L_{\delta,\lambda})}\langle x^*-Gx^*+\sigma(\gamma Vx^*-\mu Fx^*),j_q(x_{n+1}-x^*)\rangle$. Put $c_n=\max\{0,\xi_n\}$, from (3.21), we have $c_n\to 0$ as $n\to\infty$. Then we can rewrite (3.23) as

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^q&\leq(1-\tau_n)\big\|x_n-x^*\big\|^q+c_n\\&\leq(1-\tau_n)\big\|x_n-x^*\big\|^q+\circ(\alpha_n).\end{aligned}$$

Therefore, by Lemma 2.13, we conclude that $x_n\to x^*$ as $n\to\infty$. This completes the proof. □

4 Some applications

In this section, we will utilize Theorems 3.4 and 3.5 to study some strong convergence theorems in L_p (or ℓ_p) spaces with $1<p<\infty$. It well known that Hilbert spaces, L_p (or ℓ_p) spaces with $1<p<\infty$ and the Sobolev spaces W_m^p with $1<p<\infty$ are q-uniformly smooth, *i.e.*,

$$L_p\text{ (or }\ell_p)\text{ or }W_m^p\text{ is }\begin{cases}2\text{-uniformly smooth}, & \text{if }2\leq p<\infty,\\ p\text{-uniformly smooth}, & \text{if }1<p\leq 2.\end{cases}$$

Furthermore, we have the following properties of L_p (or ℓ_p) spaces with $1<p<\infty$ (see [36, 39]):

(1) For $2\leq p<\infty$, the spaces L_p (or ℓ_p) are 2-uniformly smooth with $C_q=C_2=p-1$.
(2) For $1<p\leq 2$, the spaces L_p (or ℓ_p) are p-uniformly smooth with $C_q=C_p=(1+t_p^{p-1})(1+t_p)^{1-p}$, where t_p is the unique solution of the equation

$$(p-2)t^{p-1}+(p-1)t^{p-2}-1=0,\quad 0<t<1.$$

(3) Every Hilbert spaces are 2-uniformly smooth with $C_q=C_2=1$.
(4) Every L_p (or ℓ_p) spaces with $1<p<\infty$ are q-uniformly smooth and uniformly convex.

(5) Every ℓ_p spaces with $1 < p < \infty$ have weakly sequentially continuous generalized duality mappings, but L_p spaces ($1 < p < \infty$, $p \neq 2$) do not have weakly sequentially continuous generalized duality mappings.

Lemma 4.1 *Let $X := L_p$ (or ℓ_p) with $1 < p \leq 2$. Let C be a nonempty, closed, and convex subset of X. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$. Let $0 < \mu < (\frac{p\eta}{D_p\kappa^p})^{\frac{1}{p-1}}$ and $\tau = \mu(\eta - \frac{D_p\mu^{p-1}\kappa^p}{p})$. Then for $t \in (0, \min\{1, \frac{1}{p\tau}\})$, the mapping $S : C \to X$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

Lemma 4.2 *Let $X := L_p$ (or ℓ_p) with $2 \leq p < \infty$. Let C be a nonempty, closed, and convex subset of X. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$. Let $0 < \mu < \frac{2\eta}{(p-1)\kappa^2}$ and $\tau = \mu(\eta - \frac{(p-1)\mu\kappa^2}{2})$. Then for $t \in (0, \min\{1, \frac{1}{2\tau}\})$, the mapping $S : C \to X$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

Lemma 4.3 *Let $X := H$ be a real Hilbert space. Let C be a nonempty, closed, and convex subset of X. Let $F : C \to X$ be a κ-Lipschitzian and η-strongly accretive operator with constants $\kappa, \eta > 0$. Let $0 < \mu < \frac{2\eta}{\kappa^2}$ and $\tau = \mu(\eta - \frac{\mu\kappa^2}{2})$. Then for $t \in (0, \min\{1, \frac{1}{2\tau}\})$, the mapping $S : C \to X$ defined by $S := I - t\mu F$ is a contraction with constant $1 - t\tau$.*

4.1 Implicit iteration schemes

Theorem 4.4 *Let C be a nonempty, closed, and convex subset of an ℓ_p space for $1 < p \leq 2$. Let Q_C, F, G, V, and T be the same as in Theorem* 3.4. *Assume that $0 < \mu < (\frac{p\eta}{D_p\kappa^p})^{\frac{1}{p-1}}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \frac{D_p\mu^{p-1}\kappa^p}{p})$. For $\sigma \in (\frac{L_{\delta,\lambda}}{\tau - \gamma L}, \min\{1, \frac{1}{p\tau}, \frac{1+L_{\delta,\lambda}}{\tau - \gamma L}\})$ and $t \in (0,1)$, the sequence $\{x_t\}$ defined by* (3.2) *converges strongly to $x^* \in \operatorname{Fix}(T)$ as $t \to 0$, where x^* is the unique solution of the variational inequality* (3.3).

Theorem 4.5 *Let C be a nonempty, closed, and convex subset of an ℓ_p space for $2 \leq p < \infty$. Let Q_C, F, G, V, and T be the same as in Theorem* 3.4. *Assume that $0 < \mu < \frac{2\eta}{(p-1)\kappa^2}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \frac{(p-1)\mu\kappa^2}{2})$. For $\sigma \in (\frac{L_{\delta,\lambda}}{\tau - \gamma L}, \min\{1, \frac{1}{2\tau}, \frac{1+L_{\delta,\lambda}}{\tau - \gamma L}\})$ and $t \in (0,1)$, the sequence $\{x_t\}$ defined by* (3.2) *converges strongly to $x^* \in \operatorname{Fix}(T)$ as $t \to 0$, where x^* is the unique solution of the variational inequality* (3.3).

Remark 4.6 If the spaces L_p has a weakly sequentially continuous generalized duality mappings, then we obtain Theorems 4.4 and 4.5 hold for L_p spaces with $1 < p < \infty$, $p \neq 2$.

4.2 Explicit iteration schemes

Theorem 4.7 *Let C be a nonempty, closed, and convex subset of an ℓ_p space for $1 < p \leq 2$. Let Q_C, F, G, V, and W_n be the same as in Theorem* 3.5. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $(0,1)$ *which satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.5 *and $\{\theta_{n,k}\}$ satisfies* (H1)-(H3). *Then the sequence $\{x_n\}$ defined by* (3.14) *converges strongly to $x^* \in \mathcal{F}$ as $n \to \infty$, where x^* is the unique solution of the variational inequality* (3.15).

Theorem 4.8 *Let C be a nonempty, closed, and convex subset of an ℓ_p space for $2 \leq p < \infty$. Let Q_C, F, G, V, and W_n be the same as in Theorem* 3.5. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in* $(0,1)$ *which satisfy the conditions* (C1) *and* (C2) *in Theorem* 3.5 *and $\{\theta_{n,k}\}$ satisfies* (H1)-(H3). *Then the sequence $\{x_n\}$ defined by* (3.14) *converges strongly to $x^* \in \mathcal{F}$ as $n \to \infty$, where x^* is the unique solution of the variational inequality* (3.15).

Remark 4.9 If the spaces L_p has a weakly sequentially continuous generalized duality mappings, then we obtain Theorems 4.7 and 4.8 hold for L_p spaces with $1 < p < \infty$, $p \neq 2$.

5 Numerical examples

In this section, we give a simple example and some numerical experiment result to explain the convergence of the sequence (3.14) as follows:

Example 5.1 Let $X = \mathbb{R}$ and $C = [0, \frac{1}{2}]$. Let $q = 2$ and $j_q = I$. We define a mapping Q_C as follows:

$$Q_C x = \begin{cases} \frac{x}{|x|}, & x \in (-\infty, 0) \cup (\frac{1}{2}, \infty), \\ x, & x \in [0, \frac{1}{2}]. \end{cases}$$

In terms of Theorem 3.5, set $\sigma = \mu = \gamma = 1$ and $\alpha_n = \frac{1}{n}$. Then we see that $\alpha_n = \frac{1}{n}$ satisfies (C1) and (C2) with $\sigma_n = \frac{1}{n^2}$. Moreover, we define the mappings F, G, and V as follows:

$$Fx = \frac{1}{3}\bigl(x^2 + 2x\bigr), \qquad Gx = x \quad \text{and} \quad Vx = x^2.$$

It is easy to observe that F is 1-Lipschitzian and $\frac{2}{3}$-strongly accretive, G is 1-strongly accretive and λ-strictly pseudo-contraction for $\lambda > 0$ and V is 1-Lipschitzian. For each $n \in \mathbb{N}$, set $S_n = I$. We show that $W_n = I$. Since $T_{n,k} = \theta_{n,k}S_k + (1 - \theta_{n,k})I$, where S_k is a λ_k-strictly pseudo-contractive mapping and $\{\theta_{n,k}\}$ satisfies (H1)-(H3). It is observe that $T_{n,k}$ is a nonexpansive mapping. From (2.5), we have

$$\begin{aligned} W_1 &= U_{1,1} = t_1 T_{1,1} U_{1,2} + (1 - t_1)I, \\ W_2 &= U_{2,1} = t_1 T_{2,1} U_{2,2} + (1 - t_1)I \\ &= t_1 T_{2,1}\bigl(t_2 T_{2,2} U_{2,3} + (1 - t_2)I\bigr) + (1 - t_1)I \\ &= t_1 t_2 T_{2,1} T_{2,2} U_{2,3} + t_1(1 - t_2)T_{2,1} + (1 - t_1)I, \\ W_3 &= U_{3,1} = t_1 T_{3,1} U_{3,2} + (1 - t_1)I \\ &= t_1 T_{3,1}\bigl(t_2 T_{3,2} U_{3,3} + (1 - t_2)I\bigr) + (1 - t_1)I \\ &= t_1 t_2 T_{3,1} T_{3,2} U_{3,3} + t_1(1 - t_2)T_{3,1} + (1 - t_1)I \\ &= t_1 t_2 T_{3,1} T_{3,2}\bigl(t_3 T_{3,3} U_{3,4} + (1 - t_3)I\bigr) + t_1(1 - t_2)T_{3,1} + (1 - t_1)I \\ &= t_1 t_2 t_3 T_{3,1} T_{3,2} T_{3,3} + t_1 t_2(1 - t_3)T_{3,1} T_{3,2} + t_1(1 - t_2)T_{3,1} + (1 - t_1)I \end{aligned}$$

and we compute (2.5) in a similar way to above, we obtain

$$\begin{aligned} W_n &= U_{n,1} \\ &= t_1 t_2 \cdots t_n T_{n,1} T_{n,2} \cdots T_{n,n} + t_1 t_2 \cdots t_{n-1}(1 - t_n) T_{n,1} T_{n,2} \cdots T_{n,n-1} \\ &\quad + t_1 t_2 \cdots t_{n-2}(1 - t_{n-1}) T_{n,1} T_{n,2} \cdots T_{n,n-2} + \cdots + t_1(1 - t_2)T_{n,1} + (1 - t_1)I. \end{aligned}$$

Since $S_n = I$ and $t_n = \alpha$, for all $n \in \mathbb{N}$, we have

$$W_n = \bigl[\alpha^n + \alpha^{n-1}(1 - \alpha) + \cdots + \alpha(1 - \alpha) + (1 - \alpha)\bigr] = I.$$

Under the above assumptions, (3.14) is simplified as follows:

$$\begin{cases} x_1 \in C := [0, \frac{1}{2}], \\ x_{n+1} = (1 - \frac{2}{3n})x_n + \frac{2}{3n}x_n^2. \end{cases} \tag{5.1}$$

Since the assumptions of Theorem 3.5 are satisfied in Example 5.1, the sequence (5.1) converges to $x^* = 0$, which is the unique fixed point of S_n.

Next, we show the numerical results by using MATLAB 7.11.0. We presented numerical comparisons for two cases of iteration process with different initial values, which show the convergence of the sequence (5.1).

When we choose $x_1 = 0.05$ and $x_1 = 0.1$, we see that the iteration process of sequence $\{x_n\}$ converges to $x^* = 0$ at $n = 8{,}615$ and $n = 28{,}946$, respectively, as shown in Table 1 and Figures 1 and 2.

Table 1 The value of sequence $\{x_n\}$ with iteration values $x_1 = 0.05$ and $x_1 = 0.1$

Iteration step (n)	Sequence value (x_n)	Error	Sequence value (x_n)	Error
1	0.0500	5×10^{-2}	0.1000	1×10^{-1}
2	0.0183	1.83×10^{-2}	0.0400	4×10^{-2}
3	0.0123	1.23×10^{-2}	0.0272	2.72×10^{-2}
4	0.0096	9.6×10^{-3}	0.0213	2.13×10^{-2}
5	0.0080	8×10^{-3}	0.0178	1.78×10^{-2}
⋮	⋮	⋮	⋮	⋮
1,658	0.0002	2×10^{-4}	0.00321	3.21×10^{-3}
⋮	⋮	⋮	⋮	⋮
5,570	0.0002	2×10^{-4}	0.00217	2.17×10^{-3}
⋮	⋮	⋮	⋮	⋮
8,614	0.0001	1×10^{-4}	0.00184	1.84×10^{-3}
8,615	0.0000	1×10^{-4}	0.00184	1.84×10^{-3}
⋮	⋮	⋮	⋮	⋮
28,945	0.0000	1×10^{-4}	0.0001	1×10^{-4}
28,946	0.0000	1×10^{-4}	0.0000	0

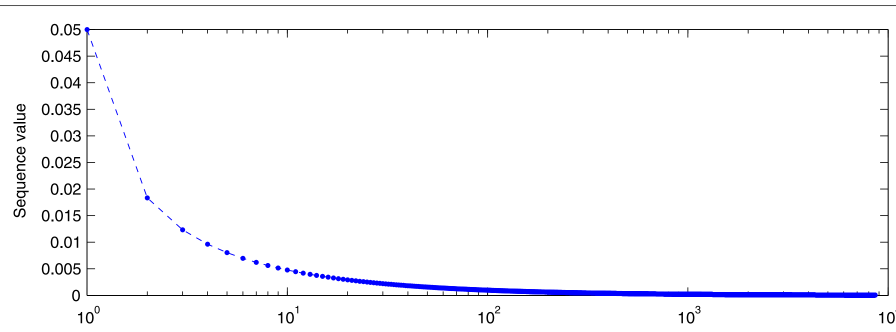

Figure 1 The iteration process with initial value $x_1 = 0.05$.

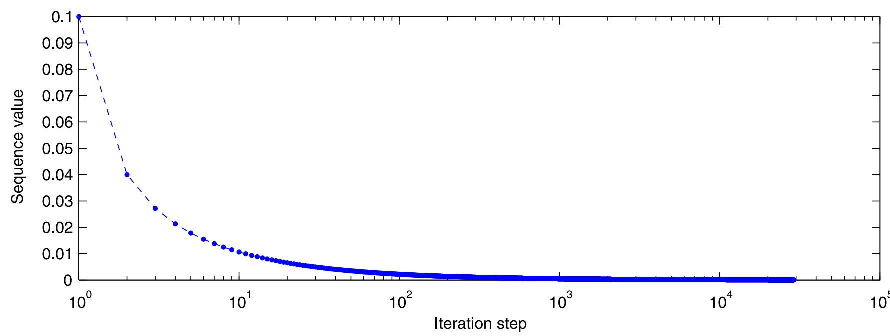

Figure 2 The iteration process with initial value $x_1 = 0.1$.

From the figures, we can see that $\{x_n\}$ is a monotone decreasing sequence and converges to 0, but an iterative process with initial value $x_1 = 0.05$ is converges faster than an iterative process with initial value $x_1 = 0.1$.

Remark 5.2 Note that Lemma 3.1 and Lemma 3.2 play an important role in the proof of Theorems 3.4 and 3.5. These are proved in the framework of the more general q-uniformly smooth Banach space.

Remark 5.3 Our main result extends the main result of Ceng *et al.* [28] in the following respects:

(1) An iterative process (1.10) is to extend to a general iterative process defined over the set of fixed points of an infinite family of strict pseudo-contractions in a more general q-uniformly smooth Banach space.

(2) The self contraction mapping $f : H \to H$ in [28, Theorem 3.2] is extended to the case of a nonself Lipschitzian mapping $V : C \to X$ on a nonempty, closed, and convex subset C of a real q-uniformly smooth Banach space X.

(3) The control condition (C3) in [28, Theorem 3.2] is removed by weaker than control condition $|\alpha_{n+1} - \alpha_n| \leq \circ(\alpha_n) + \sigma_n$ with $\sum_{n=1}^{\infty} \sigma_n < \infty$.

Furthermore, our method is extended to develop a new iterative method and method of proof is very different from that in Ceng *et al.* [28] because our method involves the sunny nonexpansive retraction and the infinite family of strict pseudo-contractions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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